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COMPONENTWISE CONCAVE COPULAS AND THEIR ASYMMETRY

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The class of componentwise concave copulas is considered, with particular emphasis on its closure under some constructions of copulas (e.g., ordinal sum) and its relations with other classes of copulas characterized by some notions of concavity and/or convexity. Then, a sharp upper bound is given for the L^∞-measure of non-exchangeability for copulas belonging to this class.

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1. INTRODUCTION

A (bivariate) copula is a distribution function on $\mathbb{I}^2 = [0,1]^2$ whose univariate marginals are uniformly distributed. Copulas have received a great popularity in the recent literature due to the celebrated *Sklar's Theorem*, stating that every bivariate distribution function can be represented by means of some suitable copula and its marginal distribution functions (see [30, 31]). For an introduction to copula theory and some of its applications we refer to [1, 16, 24, 26, 28, 29].

Recently, investigations on various notions of concavity/convexity for copulas have been considered, especially because of their potential use in the construction of asymmetric stochastic models: see, for example, [1, 26] and the recent papers [2, 3, 6, 10, 11, 13]. Here, we consider copulas that are *componentwise concave*, i.e. they are concave in each argument when the other is held fixed. Such kind of concavity has an important probabilistic interpretation in terms of a positive dependence property, called *stochastic increasingness* or *positive regression dependence*, of bivariate random pairs. In fact, if C is a componentwise concave copula associated with an absolutely continuous random pair (X,Y), then for all $s \in \mathbb{R}$ the functions $t \mapsto \mathrm{P}(Y > s \mid X = t)$ and $t \mapsto \mathrm{P}(X > s \mid Y = t)$ are increasing. Intuitively, this means that Y (respectively, X) is more likely to take on larger values as X (respectively, Y) increases (see [16, 26]).

In this paper, we aim at considering in detail the class of componentwise concave copulas, with particular emphasis on its closure under some constructions of copulas

(e. g., ordinal sum) and its relations with other classes of copulas characterized by some notions of concavity and/or convexity (Section 2). Then, we investigate the asymmetry of such a kind of copulas (Section 3) by obtaining a sharp upper bound for the L^∞-measure of non-exchangeability in this class.

2. THE CLASS OF COMPONENTWISE CONCAVE COPULAS

First, we introduce some notations that will be useful in the sequel.

A function $C\colon \mathbb{I}^2 \to \mathbb{I}$ is a *copula* if it satisfies the following properties:

(C1) C is increasing in each variable,

(C2) $C(x,1) = C(1,x) = x$ for every $x \in \mathbb{I}$,

(C3) C is 2-increasing, that is, for every $x_1, y_1, x_2, y_2 \in \mathbb{I}$, $x_1 \le x_2$ and $y_1 \le y_2$, it satisfies

$$V_C([x_1,x_2]\times[y_1,y_2]) := C(x_1,y_1) + C(x_2,y_2) - C(x_1,y_2) - C(x_2,y_1) \ge 0.$$

The symbol $\mathcal{C}$ denotes the class of all copulas.

A copula C is called *componentwise concave* if, for every $z_0 \in \mathbb{I}$, the functions $x \mapsto C(x,z_0)$ and $y \mapsto C(z_0,y)$ are concave (in the classical sense), viz. for all x, y and λ in $\mathbb{I}$,

$$\begin{aligned} C(\lambda x + (1-\lambda)y, z_0) &\ge \lambda C(x,z_0) + (1-\lambda)C(y,z_0), \\ C(z_0, \lambda x + (1-\lambda)y) &\ge \lambda C(z_0,x) + (1-\lambda)C(z_0,y). \end{aligned}$$

A copula C is called *componentwise convex* if, for every z_0 in $\mathbb{I}$, the functions $x \mapsto C(x,z_0)$ and $y \mapsto C(z_0,y)$ are convex.

We will denote by $\mathcal{C}_{\text{cc}}$ and $\mathcal{C}_{\text{cx}}$, respectively, the class of componentwise concave and componentwise convex copulas. The copula $W(x,y) = \max(x+y-1,0)$ is componentwise convex, the copula $M(x,y) = \min(x,y)$ is componentwise concave, the copula $\Pi(x,y) = xy$ is both componentwise convex and concave.

In the following, we will concentrate our attention on the class $\mathcal{C}_{\text{cc}}$. In fact, componentwise convexity for copulas has been already analysed in the literature, also in connection with the notion of *directional convexity* (see [9, 15, 21] and the references therein).

Note that every $C \in \mathcal{C}_{\text{cc}}$ is greater than Π in the concordance order, that is $C(x,y) \ge \Pi(x,y)$ for every $(x,y) \in \mathbb{I}^2$. In other words, componentwise concave copulas are *positive quadrant dependent* (see [16, 26]). Moreover, the following result can be given.

Proposition 2.1. Let C be in $\mathcal{C}_{\text{cc}}$ and let $(x_0,y_0) \in]0,1[^2$ such that $C(x_0,y_0) = x_0 y_0$. Then $C = \Pi$.

Proof. For $(x_0,y_0) \in \mathbb{I}^2$ such that $C(x_0,y_0) = x_0y_0$, let us consider the function $h_{y_0}\colon \mathbb{I} \to [0,y_0]$ given by $h_{y_0}(x) = C(x,y_0)$. From the Frechét–Hoeffding bounds for copulas (see, for example, [26]), it follows that, for every $x \in \mathbb{I}$, $xy_0 \le h_{y_0}(x) \le$

$\min(x, y_0)$. Due to the fact that $C \in \mathcal{C}_{\mathrm{cc}}$ and, hence, h_{y_0} is concave, $h_{y_0}(x_0) = x_0 y_0$ implies that $h_{y_0}(x) = xy_0$. By repeating the same arguments, it also follows that, for every $x \in \mathbb{I}$, $v_x \colon \mathbb{I} \to [0, x]$ given by $v_x(y) = C(x, y)$ is a linear mapping. Thus, $C = \Pi$. □

Example 2.2. Let C_α be a member of the *Farlie–Gumbel–Morgenstern* family of copulas defined, for all α in $[-1, 1]$ by

$$C_\alpha(x, y) = xy(1 + \alpha(1 - x)(1 - y)).$$

Then C_α is componentwise convex if $\alpha \in [-1, 0]$ and it is componentwise concave if $\alpha \in \mathbb{I}$.

Example 2.3. Let $C_{\alpha,\beta}$ be a member of the *Marshall–Olkin* family of copulas defined, for all $\alpha, \beta \in \mathbb{I}$ by

$$C_{\alpha,\beta}(x, y) = \begin{cases} x^{1-\alpha} y, & x^\alpha \geq y^\beta, \\ xy^{1-\beta}, & x^\alpha < y^\beta. \end{cases}$$

Then it can be easily proved that $C_{\alpha,\beta}$ is componentwise concave. Note that this copula is not symmetric for $\alpha \neq \beta$.

Example 2.4. Let C_ψ be an *Archimedean copula*, i. e. there exists $\psi : \mathbb{I} \to [0, +\infty]$ such that

$$C_\psi(x, y) = \psi(\psi^{[-1]}(x) + \psi^{[-1]}(y)),$$

where $\psi^{[-1]}$ is the pseudo-inverse of ψ (see [23, 26]). If ψ is differentiable everywhere, it was proved that $C_\psi \in \mathcal{C}_{\mathrm{cc}}$ if, and only if, $\log(-\psi')$ is convex (see [5]). An alternative characterization is also given in [18].

Since it can be easily seen that $\mathcal{C}_{\mathrm{cc}}$ is a convex subset of $\mathcal{C}$, several other examples of copulas of this type can be easily constructed. Furthermore, the class $\mathcal{C}_{\mathrm{cc}}$ is also closed with respect to the following transformation.

Proposition 2.5. Let $(C_i)_{i \in \mathcal{I}}$ be a set of copulas, $C_i \in \mathcal{C}_{\mathrm{cc}}$, indexed by the (at most) countable set $\mathcal{I}$. Let $(]a_i, b_i[)_{i \in \mathcal{I}}$ be a family of pairwise disjoint subintervals of $\mathbb{I}$ indexed by the same set $\mathcal{I}$. Then the *ordinal sum* of $(C_i)_{i \in \mathcal{I}}$ with respect to $(]a_i, b_i[)_{i \in \mathcal{I}}$, that is the copula $C \colon \mathbb{I}^2 \to \mathbb{I}$ given by

$$C(x, y) = \begin{cases} a_i + (b_i - a_i) C_i \left(\frac{x - a_i}{b_i - a_i}, \frac{y - a_i}{b_i - a_i} \right), & (x, y) \in]a_i, b_i[^2, \\ \min(x, y), & \text{otherwise}, \end{cases}$$

is also an element of $\mathcal{C}_{\mathrm{cc}}$.

Proof. First, we prove that, given $y \in \mathbb{I}$, $h_y \colon \mathbb{I} \to [0, y]$ defined by $h_y(x) = C(x, y)$ is concave. Suppose that there exists $i \in \mathcal{I}$ such that $y \in]a_i, b_i[$ (otherwise, the proof is trivial). Then, we have that

$$h_y(x) = \begin{cases} x, & x \in [0, a_i], \\ a_i + (b_i - a_i) C_i \left(\frac{x - a_i}{b_i - a_i}, \frac{y - a_i}{b_i - a_i} \right), & x \in]a_i, b_i[, \\ y, & x \in [b_i, 1]. \end{cases}$$

Since $C_i \in \mathcal{C}_{\mathrm{cc}}$, h_y is a piecewise concave function. Moreover,

$$\begin{aligned}(h'_y)^-(a_i) = 1 \quad &\geq \quad (h'_y)^+(a_i) = \left[\partial_x^+ C_i\left(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}\right)\right]_{x=a_i} \\ &\geq \quad (h'_y)^-(b_i) = \left[\partial_x^- C_i\left(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}\right)\right]_{x=b_i} \geq 0 = (h'_y)^+(b_i),\end{aligned}$$

where the first and last inequality follow from the fact that the partial derivatives of a copula take values on $\mathbb{I}$ (see, for example, [26]). Thus h_y is concave on $\mathbb{I}$. Analogously, it can be proved that, for every $x \in \mathbb{I}$, $v_x\colon \mathbb{I} \to [0,x]$ given by $v_x(y) = C(x,y)$ is concave, which concludes the proof. □

The class $\mathcal{C}_{\mathrm{cc}}$ is also closed with respect to the operation sending a copula to the associated survival copula. We recall that, given a copula C, the copula $\widehat{C}$ defined, for every $(x,y) \in \mathbb{I}^2$, by

$$\widehat{C}(x,y) = x + y - 1 + C(1-x, 1-y)$$

is called *survival copula* related to C.

Proposition 2.6. If $C \in \mathcal{C}_{\mathrm{cc}}$, then $\widehat{C} \in \mathcal{C}_{\mathrm{cc}}$ as well.

Proof. For a fixed $y_0 \in \mathbb{I}$ and for $\lambda \in \mathbb{I}$ with $\overline{\lambda} = 1 - \lambda$, let x_1, x_2 be in $\mathbb{I}$. Since C is componentwise concave we have that

$$\begin{aligned}\widehat{C}(\lambda x_1 + \overline{\lambda} x_2, y_0) &= \lambda x_1 + \overline{\lambda} x_2 + y_0 - 1 + C(\lambda(1-x_1) + \overline{\lambda}(1-x_2), 1-y_0) \\ &\geq \lambda x_1 + \overline{\lambda} x_2 + y_0 - 1 + \lambda C(1-x_1, 1-y_0) + \overline{\lambda} C(1-x_2, 1-y_0) \\ &= \lambda \widehat{C}(x_1, y_0) + \overline{\lambda} \widehat{C}(x_2, y_0),\end{aligned}$$

that is C is concave in the first argument being the other fixed. The same procedure can be applied to show that C is concave in the second argument being the first fixed. □

Remark 2.7. Note that $\mathcal{C}_{\mathrm{cc}}$ is not closed under isomorphic transformations of its elements by means of a concave bijection, as studied, for example, in [4, 14, 20, 25]. In fact, consider the function $\varphi\colon \mathbb{I} \to \mathbb{I}$ given by

$$\varphi(x) = \begin{cases} 2x, & x \in \left[0, \frac{1}{3}\right], \\ \frac{x+1}{2}, & x \in \left]\frac{1}{3}, 1\right]. \end{cases}$$

Let us consider the copula Π that is in $\mathcal{C}_{\mathrm{cc}}$ and its transformation $\Pi_\varphi(x,y) = \varphi^{-1}(\varphi(x)\varphi(y))$. Then, after some calculations, we get $\Pi_\varphi\left(\frac{3}{5}, \frac{3}{5}\right) = \frac{8}{25} < \frac{9}{25}$. Thus, Π_φ is not positive quadrant dependent and, hence, does not belong to $\mathcal{C}_{\mathrm{cc}}$.

An interesting geometric property can be described for any copula in $\mathcal{C}_{cc}$. We recall that, given $C \in \mathcal{C}$ and $a, b > 0$, the *decreasing affine section* at $(-a, b)$ is the function $\varphi_{C,-a,b} \colon \Omega \to \mathbb{I}$ defined by $\varphi_{C,-a,b}(x) = C(x, -ax+b)$, where Ω is a suitable set, depending on a, b, such that φ is well defined, viz. $\Omega = \{t \in \mathbb{I} \mid (-at + b) \in \mathbb{I}\}$ (see [17]).

Proposition 2.8. Every decreasing affine section of $C \in \mathcal{C}_{cc}$ is concave.

Proof. Let $a, b > 0$ and let φ be the decreasing affine section at $(-a, b)$ defined on Ω. Let x_1, x_2 be in Ω and $\lambda \in \mathbb{I}$ with $\overline{\lambda} = 1 - \lambda$. From the componentwise concavity of C we get

$$\begin{aligned}
&\varphi(\lambda x_1 + \overline{\lambda} x_2) = C(\lambda x_1 + \overline{\lambda} x_2, -a\lambda x_1 - a\overline{\lambda} x_2 + b)\\
\geq\ & \lambda C(x_1, -a\lambda x_1 - a\overline{\lambda} x_2 + b) + \overline{\lambda} C(x_2, -a\lambda x_1 - a\overline{\lambda} x_2 + b)\\
\geq\ & \lambda^2 C(x_1, -ax_1 + b) + \lambda\overline{\lambda} C(x_1, -ax_2 + b)\\
& +\lambda\overline{\lambda} C(x_2, -ax_1 + b) + \overline{\lambda}^2 C(x_2, -ax_2 + b)\\
=\ & \lambda^2 \big(C(x_1, -ax_1 + b) - C(x_1, -ax_2 + b) - C(x_2, -ax_1 + b) + C(x_2, -ax_2 + b)\big)\\
& +\lambda\big(C(x_1, -ax_2 + b) + C(x_2, -ax_1 + b) - 2C(x_2, -ax_2 + b)\big) + C(x_2, -ax_2 + b).
\end{aligned}$$

Since C satisfies the 2-increasing property (C3), we can minorize the first summand of the last expression by putting λ instead of λ^2. Then, after some little algebra, we obtain that:

$$\varphi(\lambda x_1 + \overline{\lambda} x_2) \geq \lambda C(x_1, -ax_1 + b) + \overline{\lambda} C(x_2, -ax_2 + b) = \lambda\varphi(x_1) + \overline{\lambda}\varphi(x_2),$$

which is the desired assertion. □

Note that the increasing affine sections of $C \in \mathcal{C}_{cc}$ may not be concave: consider, for example, the copula Π and its section along the main diagonal of $\mathbb{I}^2$.

The notion of componentwise concavity for copulas is connected with the notion of quasi-concavity, as the following result shows.

Proposition 2.9. **([3])** A componentwise concave copula C is quasi-concave, i. e. for all $x_1, x_2, y_1, y_2 \in \mathbb{I}$ and $\lambda \in \mathbb{I}$,

$$C(\lambda x_1 + (1 - \lambda) y_1, \lambda x_2 + (1 - \lambda) y_2) \geq \min(C(x_1, x_2), C(y_1, y_2)).$$

Notice that there are quasi-concave copulas that are not greater than Π (in the pointwise order) and, hence, cannot be componentwise concave (see [3]). Moreover, it can be derived from [3], that, given $C \in \mathcal{C}_{cc}$, the following statements are equivalent:

(a) C is symmetric,

(b) C is Schur-concave,

(c) C is weakly Schur-concave.

For the latter two notions, we refer to [13, 22] and [10], respectively.

3. ASYMMETRY FOR COMPONENTWISE CONCAVE COPULAS

Recently, a great attention has been devoted to the study of possible asymmetry of the copula of random pairs whose components are identically distributed [7, 19, 27]. In particular, it has been investigated the asymmetry of copulas characterized by some analytical and/or statistical properties, with particular emphasis on positive dependence properties (see, for example, [3, 6, 10, 11]). Among various ways for measuring this asymmetry, a special interest has been devoted to the L^∞-measure of non-exchangeability μ_∞ for random pairs (X, Y), such that X and Y are identically distributed and C is their copula, defined by

$$\mu_\infty(C) := 3\left(\max_{(x,y)\in\mathbb{I}^2} |C(x,y) - C(y,x)|\right). \tag{1}$$

Notice that this measure takes value on $\mathbb{I}$, since it has been proved in [19, 27] that

$$\max_{(x,y)\in\mathbb{I}^2} |C(x,y) - C(y,x)| = \frac{1}{3}.$$

Here we aim at finding

$$\mathcal{E}(\mathcal{C}_{\mathrm{cc}}) = \sup_{C\in\mathcal{C}_{\mathrm{cc}}} \mu_\infty(C),$$

which is the best possible upper bound for the measure of non-exchangeability of an element of $\mathcal{C}_{\mathrm{cc}}$. Now, $\mathcal{E}(\mathcal{C}_{\mathrm{cc}})$ is the supremum of the functional μ_∞, which is continuous with respect to the L^∞-norm in $\mathcal{C}$ and to the Euclidean norm in $\mathbb{I}$. Furthermore, $\mathcal{C}_{\mathrm{cc}}$ is a closed subset in the compact set $\mathcal{C}$ with respect to the L^∞-norm. Then, it follows that there exists $\widetilde{C} \in \mathcal{C}_{\mathrm{cc}}$ such that $\mathcal{E}(\mathcal{C}_{\mathrm{cc}}) = \mu_\infty(\widetilde{C})$. Copulas $\widetilde{C}$ of this type are called *maximally non-exchangeable* elements of $\mathcal{C}_{\mathrm{cc}}$.

Proposition 3.1. $\mathcal{E}(\mathcal{C}_{\mathrm{cc}}) = 3 \cdot \frac{5\sqrt{5}-11}{2} \approx 0.27.$

Proof. Let C be a maximally non-exchangeable copula in $\mathcal{C}_{\mathrm{cc}}$. We can suppose, without loss of generality, that there exist $x, y \in]0,1[$, $x < y$, such that

$$\mu_\infty(C) = 3(C(x,y) - C(y,x)).$$

Given $x \in]0,1[$, we denote by h_x and v_x the horizontal and vertical sections of C at x given, respectively, by $h_x(t) = C(t,x)$ and $v_x(t) = C(x,t)$. Since $C \in \mathcal{C}_{\mathrm{cc}}$, these sections are increasing, concave and satisfy $tx \le h_x(t) \le \min(x,t)$ and $tx \le v_x(t) \le \min(x,t)$ for any $t \in \mathbb{I}$. Let $\beta = C(x,x) \ge x^2$. Since C is maximally non-exchangeable, h_x and v_x are such that $\delta_C(x,y) = |h_x(y) - v_x(y)|$ takes its maximum value.

By the concavity of the sections, it is easy to see that this maximum is realized when h_x and v_x assume, respectively, the smallest and biggest possible values on $[x,1]$. Thus, we should consider that h_x is affine on $[x,1]$, i. e.

$$h_x(t) = \tfrac{x-\beta}{1-x}(t-x) + \beta = \tfrac{\beta(1-t)+xt-x^2}{1-x},$$

and $v_x(t) = x$ on some suitable interval $J \subseteq [x, 1]$. Now, since $v_x(x) = C(x, x) = \beta \geq \frac{x^2}{y}$, the maximum value of δ_C is obtained when we take $\beta = \frac{x^2}{y}$.

Therefore,

$$\begin{aligned}
\max_{(x,y)\in\mathbb{I}^2} \delta_C(x, y) &= \max_{(x,y)\in\mathbb{I}^2} \left(x - \frac{\beta(1-y) + xy - x^2}{1-x} \right) \\
&= \max_{(x,y)\in\mathbb{I}^2} \frac{x(y-x)(1-y)}{(1-x)y}.
\end{aligned}$$

Let us denote $f(x, y) = \dfrac{x(y-x)(1-y)}{(1-x)y}$. In order to find the maximum of f, let us consider, for a fixed x, the function $g(y) = \frac{f(x,y)}{x}$. Now, g achieves the maximum for $y = \sqrt{x}$. Thus, the maximum of f, for a fixed x, is given by $k(x) = g(x, \sqrt{x}) = \frac{x(1-\sqrt{x})}{1+\sqrt{x}}$. After little algebra, it can be showed that the maximum of k is reached at the point $x_0 = \frac{3-\sqrt{5}}{2}$, for which $k(x_0) = \frac{5\sqrt{5}-11}{2}$. This concludes the proof. □

From the proof of the above Theorem, it is easy to construct a maximally non-exchangeable copula $\widetilde{C} \in \mathcal{C}_{\mathrm{cc}}$. Given $x_0 = \frac{3-\sqrt{5}}{2}$, $\widetilde{C}$ has horizontal section at x_0 equal to

$$h_{x_0}(t) = \begin{cases} \sqrt{x_0}\,t, & t \in [0, x_0[, \\ \frac{x_0^{3/2}(1-t)+x_0 t - x_0^2}{1-x_0}, & t \in [x_0, 1], \end{cases}$$

and vertical section at x_0 equal to

$$v_{x_0}(t) = \begin{cases} \sqrt{x_0}\,t, & t \in [0, y_0[, \\ x_0, & t \in [y_0, 1]. \end{cases}$$

Methods for constructing copulas with preassigned vertical and horizontal sections (and, hence, such $\widetilde{C}$) are discussed, for example, in [8, 12]

Thus, we have calculated sharp upper bound (approximately equal to 0.27) for the class of identically distributed random pairs that are stochastically increasing. Note that the sharp upper bound for positive quadrant dependent random variables has been obtained in [6] and it equals $3(3-2\sqrt{2})$ (≈ 0.516). Thus, we may say that pairs of continuous random variables whose copula is componentwise concave (the latter being seen as a property of positive dependence) tend to manifest a symmetric relation in the dependence structure, as illustrated by Proposition 3.1.

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REFERENCES

[1] C. Alsina, M. J. Frank, and B. Schweizer: Associative Functions. Triangular Norms and Copulas. World Scientific, Hackensack, NJ 2006.

[2] E. Alvoni, F. Durante, P.-L. Papini, and C. Sempi: Different types of convexity and concavity for copulas. In: New Dimensions in Fuzzy Logic and related Technologies – Proc. 5th EUSFLAT Conference (M. Štěpnička, V. Novák, and U. Bodenhofer, eds.), Volume 1, University of Ostrava 2007, pp. 185–189.

[3] E. Alvoni and P. L. Papini: Quasi-concave copulas, asymmetry and transformations. Comment. Math. Univ. Carolin. *48* (2007), 311–319.

[4] E. Alvoni, P.-L. Papini, and F. Spizzichino: On a class of transformations of copulas and quasi-copulas. Fuzzy Sets and Systems *160* (2009), 334–343.

[5] P. Capéràa and C. Genest: Spearman's ρ is larger than Kendall's τ for positively dependent random variables. J. Nonparametr. Statist. *2* (1993), 183–194.

[6] B. De Baets, H. De Meyer, and R. Mesiar: Asymmetric semilinear copulas. Kybernetika *43* (2007), 221–233.

[7] F. Durante, E. P. Klement, C. Sempi, and M. Úbeda-Flores: Measures of non-exchangeability for bivariate random vectors. Statist. Papers (2009), to appear (doi:10.1007/s00362-008-0153-0).

[8] F. Durante, A. Kolesárová, R. Mesiar, and C. Sempi: Copulas with given values on a horizontal and a vertical section. Kybernetika *43* (2007), 209–220.

[9] F. Durante, R. Mesiar, P.-L. Papini, and C. Sempi: 2-increasing binary aggregation operators. Inform. Sci. *177* (2007), 111–129.

[10] F. Durante and P.-L. Papini: A weakening of Schur-concavity for copulas. Fuzzy Sets and Systems *158* (2007), 1378–1383.

[11] F. Durante and P.-L. Papini: Non-exchangeability of negatively dependent random variables. Metrika (2009), to appear (doi:10.1007/s00184-008-0207-2).

[12] F. Durante, S. Saminger-Platz, and P. Sarkoci: Rectangular patchwork for bivariate copulas and tail dependence. Comm. Statist. Theory Methods *38* (2009), 2515–2527.

[13] F. Durante and C. Sempi: Copulæ and Schur-concavity. Internat. Math. J. *3* (2003), 893–905.

[14] F. Durante and C. Sempi: Copula and semicopula transforms. Internat. J. Math. Math. Sci. *2005* (2005), 645–655.

[15] F. Durante and C. Sempi: On the characterization of a class of binary operations on bivariate distribution functions. Publ. Math. Debrecen *69* (2006), 47–63.

[16] H. Joe: Multivariate Models and Dependence Concepts. Chapman and Hall, London 1997.

[17] E. P. Klement and A. Kolesárová: Intervals of 1-Lipschitz aggregation operators, quasi-copulas, and copulas with given affine section. Monatsh. Math. *152* (2007), 151–167.

[18] E. P. Klement, A. Kolesárová, R. Mesiar, and A. Stupňanová: Lipschitz continuity of discrete universal integrals based on copulas. Internat. J. Uncertain. Fuzziness Knowledge-Based Systems (to appear).

[19] E. P. Klement and R. Mesiar: How non-symmetric can a copula be? Comment. Math. Univ. Carolin. *47* (2006), 141–148.

[20] E. P. Klement, R. Mesiar, and E. Pap: Transformations of copulas. Kybernetika *41* (2005), 425–434.

[21] M. Marinacci and L. Montrucchio: Ultramodular functions. Math. Oper. Res. *30* (2005), 311–332.

[22] A. W. Marshall and I. Olkin: Inequalities: Theory of Majorization and its Applications. Academic Press, New York 1979.

[23] A. J. Mc Neil and J. Nešlehová: Multivariate Archimedean copulas, d-monotone functions and ℓ_1-norm symmetric distributions. Ann. Statist. *37* (2009), 3059–3097.

[24] A. J. McNeil, R. Frey, and P. Embrechts: Quantitative Risk Management. Concepts, Techniques and Tools. Princeton University Press, Princeton, NJ 2005.

[25] P. M. Morillas: A method to obtain new copulas from a given one. Metrika *61* (2005), 169–184.

[26] R. B. Nelsen: An Introduction to Copulas. Second edition. Springer, New York 2006.

[27] R. B. Nelsen: Extremes of nonexchangeability. Statist. Papers *48* (2007), 329–336.

[28] G. Salvadori, C. De Michele, N. T. Kottegoda, and R. Rosso: Extremes in Nature. An Approach Using Copulas. Springer, Dordrecht 2007.

[29] B. Schweizer and A. Sklar: Probabilistic Metric Spaces. Dover Publications, Mineola, N.Y. 2006.

[30] A. Sklar: Fonctions de répartition à n dimensions et leurs marges. Publ. Inst. Statist. Univ. Paris *8* (1959), 229–231.

[31] A. Sklar: Random variables, joint distributions, and copulas. Kybernetika *9* (1973), 449–460.

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