KYBERNETIKA — VOLUME 45 (2009), NUMBER 5, PAGES 825 – 840

ON DELAY–DEPENDENT ROBUST STABILITY UNDER MODEL TRANSFORMATION OF SOME NEUTRAL SYSTEMS

Salvador A. Rodríguez, Luc Dugard, Jean-Michel Dion and Jesús De León

This paper focuses on the delay-dependent robust stability of linear neutral delay systems. The systems under consideration are described by functional differential equations, with norm bounded time varying nonlinear uncertainties in the "state" and norm bounded time varying quasi-linear uncertainties in the delayed "state" and in the difference operator. The stability analysis is performed via the Lyapunov–Krasovskii functional approach. Sufficient delay dependent conditions for robust stability are given in terms of the existence of positive definite solutions of LMIs.

Keywords: time-delay systems, neutral system, stability

AMS Subject Classification: 93D09, 34K20

1. INTRODUCTION

A great variety of systems can be modelled by time-delay systems see [1, 6, 15] and [24] i. e. the "future"states depend not only on the "present" states, but also on the "delayed" states. Indeed, the delay naturally occurs in the dynamical behavior of systems in many fields: mechanics, physics, etc. Even if the systems themselves do not have internal delays, closed loop systems may involve delay phenomena, because of actuators, sensors and computation time.

Among time delay systems, the class of neutral systems is characterized by the fact that the delay argument occurs in the "state" and also in the derivative of the difference operator applied to the state variable $D(t)x_t$. Some examples of such neutral systems are given in [1, 15, 19, 24].

Several works have been concerned with the stability analysis of neutral systems either in the time domain approach, see for example: [11, 27, 31] or in the frequency domain approach, see for example: [3, 20, 26, 31]. In these studies, the attention was mainly focused on delay independent stability conditions, which are rather conservative. It is then of interest to consider delay-dependent stability analysis, see [3, 13, 23, 28].

Sufficient results have been obtained by the authors [30] concerning the delay dependent robust stability of neutral systems; in these works, the proposed model transformation includes an additional dynamics that may be unstable [9]. The model transformation is a common practice [7]; for example, previous results ([17] and [23]) for neutral systems use the Leibnitz's rule to transform a system with pointwise delays to a system with distributed delay (see [15] for the terminology). Here, another model transformation is used: the technique of integration over one delay interval [16]; this model transformation is a fixed one and does not induce any additional dynamics as it happens when the Leibnitz's rule is used.

The objective of this paper is to study the stability analysis of linear neutral systems in a delay-dependent framework incorporating robustness issues. One obtains sufficient delay-dependent stability conditions via the Lyapunov–Krasovskii functional approach. The main difference w.r.t. the result given in [30] is that additional dynamics are avoided for the kind of proposed uncertainties, leading to better results; the stability results are expressed in terms of a checkable LMI.

This paper is organized as follows. Section 2 gives some preliminaries and states the problem. The model transformation is discussed in Section 3 and the main stability results (two theorems on robust stability) are given in Section 4. In Section 5 an example considering four cases for a scalar linear neutral system is presented. Some final remarks end the paper.

Notation. I_n, 0_n are respectively the identity and the zero matrices of dimensions $n \times n$. $x \in \mathbb{R}^n$, $\|\cdot\|$ denotes the Euclidean norm of x. For a real number $r > 0$, $\mathcal{C}([-r,0],\mathbb{R}^n)$ will be the Banach space of continuous vector functions $\varphi : [-r,0] \to \mathbb{R}^n$ with the supremum norm $\|\varphi\|_{\mathcal{C}} = \sup_{-r\leq t\leq 0}\|\varphi(t)\|$. $B(\mathcal{C},\mathbb{R}^n)$ is the Banach space of bounded linear mappings from $\mathcal{C}$ to $\mathbb{R}^n$ with the operator topology. The function x_t denotes the restriction of x to the interval $[t-r,t]$ so that x_t is an element of $\mathcal{C}$ defined by $x_t(\theta) = x(t+\theta)$ for $-r \leq \theta \leq 0$.

2. PROBLEM STATEMENT

In order to define a general class of neutral systems, one needs the definition of the notion of atomicity ; let us recall this approach:

Let $\mathcal{H} \subset [\tau,\infty) \times \mathcal{C}([-r,0],\mathbb{R}^n)$ be an open, $(t,\varphi) \in \mathcal{H}$, $L(t) \in B(\mathcal{C},\mathbb{R}^n)$; then, the Riesz representation theorem implies that there is an $n \times n$ matrix function μ on $[-r,0]$ of bounded variation such that

$$L(t)\varphi = \int_{-r}^{0} [\,\mathrm{d}_\theta \mu(t,\theta)]\,\varphi(\theta)\,.$$

Here we regard μ as extended to $\mathbb{R}$ so that $\mu(t,\theta) = \mu(t,-r)$ for $\theta \leq -r$, $\mu(t,\theta) = \mu(t,0)$ for $\theta \geq 0$ (see [19]).

Definition 1. (Hale and Verduyn Lunel [11]) Let $\mu(t,\beta^+) := \lim_{\varepsilon\to 0}\mu(t,\beta+\varepsilon)$ and $\mu(t,\beta^-) = \lim_{\varepsilon\to 0}\mu(t,\beta-\varepsilon)$ for some $\varepsilon > 0$. If there exists $\beta \in \mathbb{R}$ such that the matrix

$$A(t,\beta,L(t)) = \mu\left(t,\beta^+\right) - \mu\left(t,\beta^-\right) \tag{1}$$

is nonsingular at $t = t_0 \in [\tau, \infty)$, $L(t)$ is said to be atomic at β at t_0. If $A(t, \beta, L)$ is non singular on $O \subset \mathcal{H}$, then $L(t)$ is atomic at β on O.

Now consider the following class of uncertain neutral systems, written in the form proposed by [10], see also [11] and [15]:

$$\frac{\mathrm{d}}{\mathrm{d}t}[D(t)\,x_t] = Ax(t) + Bx(t-r_2) + \Delta_A(t, x_t(0)) + \Delta_B(t-r_2, x_t(-r_2)),\ t \ge \sigma, \tag{2}$$

$$D(t)\,\varphi := \varphi(0) - C\varphi(-r_1) + \Delta_C(t-r_1, \varphi(-r_1)), \tag{3}$$

with the initial condition
$$x_\sigma = \phi,\ \{\phi, \varphi\} \in \mathcal{C}, \tag{4}$$

where $\tau \in \mathbb{R}$, $\{\sigma, t\} \in [\tau, \infty)$, the state x_t is a functional in $\mathcal{C}$, delays $r_1 \ge 0$, $r_2 \ge 0$ are assumed to be constants and unknowns, $r := \max\{r_1, r_2\}$, i.e. linear neutral differential equations that include continuous uncertainties. Assume that A, B, and C are constant known matrices, the nonlinear mapping $\Delta_A : \mathcal{H} \to \mathbb{R}^n$, and the linear mappings w.r.t. the second argument $\Delta_B, \Delta_C : \mathcal{H} \to \mathbb{R}^n$ are continuous, uniformly bounded and take closed bounded sets into bounded sets. The mappings $\Delta_A, \Delta_B, \Delta_C$ also satisfy the following properties for all $t \in [\tau, \infty)$ and $\varphi \in \Omega_\nu = \{\varphi : \|\varphi\| \le \nu, \nu > 0\}$:

$$\begin{aligned}
&\Delta_A(t, \varphi(0)) := E_A \delta_A(t, \varphi(0)),\\
&\delta_A^\top(t+\theta, \varphi(\theta))\,\delta_A(t+\theta, \varphi(\theta)) \le \varphi^\top(\theta)\, W_A^\top W_A \varphi(\theta),\\
&\Delta_B(t-r_2, \varphi(-r_2)) := \delta_B(t-r_2)\,\varphi(-r_2),\\
&W_B + \delta_B(t+\theta) \ge 0,\ W_B - \delta_B(t+\theta) \ge 0,\\
&\Delta_C(t-r_1, \varphi(-r_1)) := \delta_C(t-r_1)\,\varphi(-r_1),\\
&W_C + \delta_C(t+\theta) \ge 0,\ W_C - \delta_C(t+\theta) \ge 0,\ \forall (t, \varphi) \in \mathbb{R}^+ \times \Omega_\nu,\ \theta \in [-r, 0],
\end{aligned} \tag{5}$$

where E_A is a known matrix and W_A, W_B and W_C are given weighting matrices. The unknown mapping δ_A satisfies $\delta_A(t, 0) \equiv 0$, so that $x = 0$ is a solution of the neutral differential equation (2)–(5).

Notice that the uncertainty Δ_A is structured by E_A, while Δ_B, Δ_C are unstructured. On the one hand, this property will be used to verify the main results, Theorems 6 and 8 by means of LMIs. On the other hand, the structure of the system under consideration is motivated by applications, e.g., the lossless transmission line circuit presented in [15, 24, 29] could present parameter uncertainties (distributed parameters like resistance, capacity and inductance are not precisely known). As a matter in fact the structure of the difference operator (3), $D(t)\,\varphi := \varphi(0) - C\varphi(-r_1) + \Delta_C(0 - r_1, \varphi(-r_1))$, with Δ_C time invariant is a class of operator often arising in practice [6]. Let us introduce now the following proposition.

Proposition 2. Let W_J and δ_J be square real matrices, such that $W_J \pm \delta_J \ge 0$ (i.e. $W_J + \delta_J \ge 0$ and $W_J - \delta_J \ge 0$) for $J \in \{B, C\}$. Then:

i) $W_J \geq 0$,

ii) $\delta^\top \delta \leq W_J^\top W_J$ and

iii) $\delta^\top S \delta \leq W_J^\top S W_J$ for all $S \geq 0$ of appropriate dimensions.

Proof. Since $x^\top (W_J + \delta_J) x + x^\top (W_J - \delta_J) x \geq 0$, hence $W_J \geq 0$ is satisfied. Next, since $W_J \pm \delta_J \geq 0$, it follows that

$$x^\top \left(W_J^\top + \delta_J^\top\right)(W_J - \delta_J) x \geq 0$$

and $\delta^\top \delta \leq W_J^\top W_J$. Finally, for all $S \geq 0$

$$x^\top \left(W_J^\top + \delta_J^\top\right) S (W_J - \delta_J) x \geq 0$$

and
$$\delta_J^\top S \delta_J \leq W_J^\top S W_J, \text{ for all } S > 0 \,. \tag{6}$$

□

Next, one considers the following problem:

Delay-dependent robust stability problem: find a bound r_2^*, if it exists, on r_2, and conditions to ensure the asymptotic stability of the neutral system (2)–(4), for $r_2 \leq r_2^*$ and for any Δ_A, Δ_B, Δ_C satisfying (5).

In this paper, we are interested in the sensitivity of stability to variations in the parameters and in the delays. Notice that the stability of a neutral system requires as a necessary condition the stability of the associated difference operator [11]. For some of our results, we deal with the difference operator given by

$$\widehat{D}\varphi = \varphi(0) - C\varphi(\widehat{r}), \widehat{r} > 0.$$

In this case, the stability of $\widehat{D}$ is ensured if $\rho(C) < 1$, where $\rho(\cdot)$ denotes the spectral radius. In a more general case we refer to [10, 11, 15]. In the next section, the simultaneous study of robust stability and delay-dependent stability will be considered.

3. MODEL TRANSFORMATION

The stability of neutral systems has been studied using model transformation, let us remind two ideas in a time approach (see other transformations in [29]). First the stability of the transformed model should imply the stability of the original model. Then stability of the transformed model is studied by means of Lyapunov–Krasovskii functionals. Linear neutral systems with pointwise delays can be transformed into systems with distributed delays by using the Leibnitz's rule in the difference operator, $D(x_t - x_{t-r_2}) = \int_{-r_2}^{0} \mathrm{d}_\theta [D x_{t+\theta}]$ (see [29]) or according to the classical Leibnitz'rule in the present state $x(t)$ (see, [13, 31]). The structure of the transformed system allows to study its stability. However the transformed model introduces additional dynamics to the original system [14].

Other transformation is the integration over one delay interval [18]. In order to show the main idea, consider the following linear time invariant neutral system

$$\dot{x}(t) = Ax(t) + Bx(t-r) + C\dot{x}(t-r), \; r > 0. \tag{7}$$

Since at least $x(t)$ is differentiable in (7), it is easy to check that every solution of (7) is also solution of [4]

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t) - Cx(t-r)\right] = Ax(t) + Bx(t-r), \tag{8}$$

where $x(t)$ does not need to be differentiable, only the difference $x(t) - Cx(t-r)$. Then the stability of (8) implies the stability of (7). In the same way, each solution $x(t)$ of (8) is solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t) - Cx(t-r) + \int_{-r}^{0} Bx(t+\theta)\,\mathrm{d}\theta\right] = (A+B)\,x(t). \tag{9}$$

The principal advantage of the technique of integration w.r.t. which uses the Leibnitz's rule, is that the technique of integration does not introduce additional dynamics to the original system (see more details in [18, 29]). On the contrary, the stability analysis of the associated difference operator for the transformed system (via the integration over one delay interval), is more difficult to study than for untransformed.

Now, the technique of integration above mentioned is applied over the delay interval $[0, r_2]$ to the neutral system (2)–(5), the system can be rewritten in terms of the new variable $\xi(t)$ as (see [14, 16])

$$\frac{\mathrm{d}\widetilde{D}(t)\xi_t}{\mathrm{d}t} = (A+B)\,\xi(t) + \Delta_A(t, \xi(t)) + \Delta_B(t, \xi(t)), \tag{10}$$

where

$$\widetilde{D}(t)\,\varphi := D(t)\,\varphi + \int_{-r_2}^{0} \left[B\varphi(\theta) + \Delta_{Bt}(\theta)\right]\mathrm{d}\theta, \tag{11}$$

is the difference operator and $D(t)$ is defined in (3).

It is not difficult to verify that every solution of the neutral system (2)–(5), is also solution of the equation (10)–(11), then the stability of (10)–(11) implies the stability of (2)–(5), see [14].

Remark 3. The proposed model transformation does not induce any additional dynamics in the characteristic equation for the system without uncertainties, but the stability of the difference operator $\widetilde{D}(t)$ is required.

The transformed model (10)–(11) is a neutral system in the sense of the definition proposed by [18]. The next proposition establishes that, if the operator $D(t)$, defined in (3), is atomic at zero, then $\widetilde{D}(t)$ in (11) is also atomic at zero, i. e. if the original system (2)–(5) is neutral in the sense of [10], then the transformed model (10)–(11) is also of neutral type in the sense of [10]. This fact allows to simplify the proof of stability in the next section.

Proposition 4. Let T be the transformation that transforms (2)–(5) into (10)–(11), with the difference operators $D(t)$ and $\widetilde{D}(t)$ respectively. Then, if $D(t)$ is atomic at zero, then $\widetilde{D}(t)$ is also atomic at zero, and the initial condition for both systems belongs to $\mathcal{C}([-r,0],\mathbb{R}^n)$.

P r o o f. First, it is shown that the system (10)–(11) is a neutral system, i.e. the difference operator (11) is atomic at zero. Then assume that $D(t) \in B(\mathcal{C},\mathbb{R}^n)$ is atomic at zero.

Now, rewriting (11) as a Riemann-Stieltjes integral

$$\widetilde{D}(t)\varphi = \int_{-r}^{0} [\mathrm{d}_\theta \mu(t,\theta)]\,\varphi(\theta),\ \theta \in [-r,0],$$

with $r := \max\{r_1, r_2\}$ and

$$\mu(t,\theta) := \mu_0(\theta) + \mu_1(\theta) + \mu_2(t,\theta) + \mu_3(t,\theta) + \mu_4(t,\theta),$$

where μ_i are functions of bounded variation and defined as

$$\mu_0(\theta) := \begin{cases} I_n & \text{if} \quad \theta = 0, \\ 0_n & \text{if} \quad \theta < 0, \end{cases}, \qquad \mu_1(\theta) := \begin{cases} -C & \text{if} \quad \theta > -r_1, \\ 0_n & \text{if} \quad \theta = -r_1, \end{cases}$$

$$\mu_2(t,\theta) := \begin{cases} -\delta_C(t-r_1) & \text{if} \quad \theta > -r_1, \\ 0_n & \text{if} \quad \theta = -r_1, \end{cases}$$

$$\mu_3(t,\theta) := \begin{cases} \theta B + r_2 B & \text{if} \quad -r_2 < \theta \le 0, \\ 0_n & \text{if} \quad \theta = -r_2, \end{cases}$$

$$\mu_4(t,\theta) := \begin{cases} \int_{-r_2}^{\theta} \delta_B(t+\vartheta)\,\mathrm{d}\vartheta & \text{if} \quad -r_2 < \theta \le 0, \\ 0_n & \text{if} \quad \theta = -r_2, \end{cases}$$

furthermore $\mu(t,\theta) := \mu(t,-r)$ for $\theta \le -r$ and $\mu(t,\theta) := \mu(t,0)$ for $\theta \ge 0$.

Since B, C are real matrices and δ_B, δ_C are continuous and bounded, $\forall\ t \ge \sigma$, $\theta \in [-r,0]$, then μ_i are of bounded variation. Hence μ is of bounded variation and $\widetilde{D}(t) \in B(\mathcal{C},\mathbb{R}^n)$.

From (1) and replacing $L(t) = \widetilde{D}(t)$ where $\widetilde{D}(t)$ is given by (11) at $\beta = \theta = 0$ it follows

$$\begin{aligned} A\left(t,0,\widetilde{D}(t)\right) = {} & \mu_0(0^+) - \mu_0(0^-) + \mu_1(0^+) - \mu_1(0^-) + \mu_2(t,0^+) - \mu_2(t,0^-) \\ & + \mu_3(t,\theta^+) - \mu_3(t,\theta^-) + \mu_4(t,\theta^+) - \mu_4(t,\theta^-). \end{aligned}$$

Since $\mu_3(t,\theta)$ and $\mu_4(t,\theta)$ are continuous functions then $\mu_3(t,\theta^+) - \mu_3(t,\theta^-) = 0$, $\mu_4(t,\theta^+) - \mu_4(t,\theta^-) = 0$ and

$$A\left(t,0,\widetilde{D}(t)\right) = I - C - \delta_C(t-r_1). \tag{12}$$

In the same way, it is easy to verify that when the operator $L(t)$ in (1) is replaced by $D(t)$ given in (3) at $\beta = \theta = 0$, it is obtained

$$A(t,0,D(t)) = I - C - \delta_C(t-r_1) = A\left(t,0,\widetilde{D}(t)\right). \tag{13}$$

Since $D(t)$ is atomic at zero, the matrix $A(t,0,D(t)) \neq 0$, $A(t,0,\widetilde{D}(t))$ is non singular for all $t \in [\tau,\infty)$ and it follows that the difference operator (11), $\widetilde{D}(t)$, is atomic at zero. It implies that (10) – (11) is a neutral system by Definition 1, [11].

On the other hand, since both systems do have the same delay, then the initial condition for (10) – (11) belongs to $\mathcal{C}$, i. e. the same space that for original neutral system (2) – (5). □

Remark 5. The fact that the initial condition belongs to the same space for the original (2) – (5) and transformed system (10) – (11), does not always hold, see [29], where the initial condition for the original space is $\mathcal{C}([-r,0],\mathbb{R}^n)$ while it belongs to $\mathcal{C}([-r-r_2,0],\mathbb{R}^n)$ for the transformed one.

4. ROBUST STABILITY

To prove sufficient conditions for asymptotic stability of neutral systems of the form (10) – (11), one way is to propose a degenerated Lyapunov–Krasovskii functional on $D(t)$ (see for instance [24] to the case $D(t) = D$) and check the negativity of the derivative of the functional along the solution of (10) – (11) and the stability of both operators $D(t)$ and $\widetilde{D}(t)$. In this section, a slightly different way is considered, that uses the Lyapunov–Krasovskii functional approach on the operator $\widetilde{D}(t)$, with the help of Theorem 8.1 given in [11] to prove the main results of this paper, given in the following theorems. The results are then compared with previous ones, given in the literature.

Theorem 6. Consider the Neutral System (2) – (5) and assume that the following conditions are satisfied:

i) $A_1 := A + B$ is a Hurwitz stable matrix;

ii) The difference operators $D(t)\varphi := [\varphi(0) - C\varphi(-r_1) - \Delta_C(t,\varphi)]$ and $\widetilde{D}(t)\varphi := \{D(t)\varphi + \int_{-r_2}^{0}[B\varphi(\theta) + \Delta_B(t+\theta,\varphi(\theta))]\,\mathrm{d}\theta\}$ are linear in φ, continuous and uniformly stable with respect to $\mathcal{C}([\sigma,\infty),\mathbb{R}^n)$, and $D(t)$ is atomic at 0;

iii) there exist a real positive number r_2^* and positive definite matrices P, $S_i > 0$, $i = \overline{1,6}$ such that the following LMIs hold:

$$\Gamma := \begin{pmatrix} Q(r_2^*) & \Gamma_{12} & \overline{S}(r_2^*) \\ \Gamma_{12}^\top & \Gamma_{22} & 0 \\ \overline{S}^\top(r_2^*) & 0 & R \end{pmatrix} < 0, \tag{14}$$

$$S_2 > S_3 + W_B^\top S_4 W_B + (2r_2^*+3)\,B^\top S B + (2r_2^*+3)\,W_B^\top S W_B, \tag{15}$$

where

$$S := \left(W_A^\top W_A + W_B^\top S_5 W_B + S_1 + r_2^* S_2\right), \tag{16}$$

$$Q\left(r_2^*\right) := \Gamma_{11} + \overline{S}\left(r_2^*\right) R^{-1} \overline{S}^\top \left(r_2^*\right), \tag{17}$$

$$\Gamma_{11} = 2A_1^\top P + 2S + 2r_2^* S - \overline{S}\left(r_2^*\right) R^{-1} \overline{S}^\top \left(r_2^*\right), \tag{18}$$

$$\overline{S}\left(r_2^*\right) := \begin{pmatrix} \sqrt{r_2^*} P A_1 B & \sqrt{r_2^*} P A_1 & P & P A_1^\top & P E_A \end{pmatrix}, \tag{19}$$

$$R^{-1} := \begin{pmatrix} -S_3^{-1} & 0 & 0 & 0 & 0 \\ 0 & -S_4^{-1} & 0 & 0 & 0 \\ 0 & 0 & -S_5^{-1} & 0 & 0 \\ 0 & 0 & 0 & -S_6^{-1} & 0 \\ 0 & 0 & 0 & 0 & -I \end{pmatrix}, \tag{20}$$

$$\Gamma_{12} := P A_1 C + S C, \tag{21}$$

$$\Gamma_{22} := -S_1 + W_C^\top S_6 W_C + \left(2 + r_2^*\right) C^\top S C + \left(2 + r_2^*\right) W_C^\top S W_C. \tag{22}$$

Then, the Neutral System (2) – (5) is robustly delay-dependent asymptotically stable for any $r_2 \leq r_2^*$.

Remark 7. Matrix inequalities (14) – (15) can be checked by LMI tools for every $r_2^* > 0$ fixed.

Proof. Consider the Neutral System (2) – (5) and the Lyapunov–Krasovskii functional:

$$V\left(t, \varphi\right) := V_1\left(t, \varphi\right) + V_2\left(\varphi\right) + V_3\left(\varphi\right), \tag{23}$$

where

$$V_1\left(t, \varphi\right) := \left[\widetilde{D}\left(t\right) \varphi\right]^\top P \widetilde{D}\left(t\right) \varphi, \tag{24}$$

$$V_2\left(\varphi\right) := \int_{-r_1}^{0} \varphi^\top\left(\theta\right) S_1 \varphi\left(\theta\right) \mathrm{d}\theta, \tag{25}$$

$$V_3\left(\varphi\right) := \int_{-r_2}^{0} \int_{\theta}^{0} \varphi^\top\left(\vartheta\right) S_2 \varphi\left(\vartheta\right) \mathrm{d}\vartheta \mathrm{d}\theta. \tag{26}$$

For the functional V, one can construct $\dot{V}$ along the trajectories of (10) in terms of $\widetilde{D}\left(t\right) x_t$ (see for example [30]). Let $A_1 = A + B$, then we have:

$$\begin{aligned} \dot{V}_1\left(t, x_t\right) = {} & 2\left[\widetilde{D}\left(t\right) x_t\right]^\top A_1^\top P \widetilde{D}\left(t\right) x_t + 2\left[\widetilde{D}\left(t\right) x_t\right]^\top P A_1 C x_t\left(-r_1\right) \\ & - 2\left[\widetilde{D}\left(t\right) x_t\right]^\top P A_1 B \int_{-r_2}^{0} x_t\left(\theta\right) d\theta \\ & - 2\left[\widetilde{D}\left(t\right) x_t\right]^\top P A_1 \int_{-r_2}^{0} \Delta_{Bt}\left(\theta\right) \mathrm{d}\theta \\ & + 2\Delta_{At}^\top\left(0\right) P \widetilde{D}\left(t\right) x_t + 2\Delta_{Bt}^\top\left(0\right) P \widetilde{D}\left(t\right) x_t \\ & + 2\Delta_{Ct}^\top\left(-r_1\right) A_1^\top P \widetilde{D}\left(t\right) x_t, \end{aligned} \tag{27}$$

$$\dot{V}_2(x_t) = x^\top(t) S_1 x(t) - x^\top(t-r_1) S_1 x(t-r_1), \quad (28)$$

$$\dot{V}_3(x_t) = r_2 x^\top(t) S_2 x(t) - \int_{-r_2}^{0} x_t^\top(\theta) S_2 x_t(\theta)\,\mathrm{d}\theta. \quad (29)$$

Using the equations and inequalities (5)–(6), and the following well known inequality

$$-2a^\top b \le \inf_{S>0} \left\{ a^\top S a + b^\top S^{-1} b \;\forall\, a, b \in \mathbb{R}^n \right\} \quad (30)$$

one can bound (28)–(29) and then get a bound of $\dot{V}$ in terms of $\widetilde{D}(t)\,x_t$:

$$\begin{aligned}
\dot{V}(t,x_t) \le\; & 2\left[\widetilde{D}(t)\,x_t\right]^\top A_1^\top P \widetilde{D}(t)\,x_t + 2x_t^\top(-r_1)\left(C^\top S + C^\top A_1^\top P\right)\widetilde{D}(t)\,x_t \\
&+ \left[\widetilde{D}(t)\,x_t\right]^\top \left[P A_1 R_5^{-1} A_1^\top P + 2S + P R_4^{-1} P + P E_A R_3^{-1} E_A^\top P\right]\widetilde{D}(t)\,x_t \\
&+ r_2\left[\widetilde{D}(t)\,x_t\right]^\top \left[2S + P A_1 B R_1^{-1} B^\top A_1^\top P + P A_1 R_2^{-1} A_1^\top P\right]\widetilde{D}(t)\,x_t \\
&+ x^\top(t-r_1)\left(-S_1 + W_C^\top R_5 W_C + 3W_C^\top S W_C + 2C^\top S C\right) x(t-r_1) \\
&+ r_2 x^\top(t-r_1)\left(2W_C^\top S W_C + 2r_2 C^\top S C\right) x(t-r_1) \\
&+ \int_{-r_2}^{0} x_t^\top(\theta)\left(-S_2 + 3B^\top S B + 3W_B^\top S W_B + R_1 + W_B^\top R_2 W_B\right) x_t(\theta)\,\mathrm{d}\theta \\
&+ r_2 \int_{-r_2}^{0} x^\top(t+\theta)\left(2B^\top S B + 2W_B^\top S W_B\right) x(t+\theta)\,\mathrm{d}\theta,
\end{aligned} \quad (31)$$

$$S := \left(W_A^\top R_3 W_A + W_B^\top R_4 W_B + S_1 + r_2 S_2\right). \quad (32)$$

Finally, one chooses $R_1 = S_3$, $R_2 = S_4$, $R_3 = I$, $R_4 = S_5$, $R_5 = S_6$ where

$$\begin{aligned} S_2 > R_1 + W_B^\top R_2 W_B + 3B^\top S B + 3W_B^\top S W_B \\ + 2r_2 B^\top S B + 2r_2 W_B^\top S W_B. \end{aligned} \quad (33)$$

Now, with all these inequalities and identities, if there exists a real positive number $r_2^* \ge r_2$ such that the LMI (14) holds, then this LMI is equivalent (via an appropriate Schur transformation [2]) to the following inequality:

$$\dot{V}(t,x_t) \le \zeta^\top \Gamma \zeta < 0, \quad (34)$$

$$\zeta = \begin{pmatrix} \widetilde{D}(t)\,x_t \\ x(t-r_1) \end{pmatrix}, \Gamma = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{12}^\top & \Gamma_{22} \end{pmatrix}$$

where r_2^* was replaced by r_2 and $(\Gamma_{ij})_{i=1,2,j=1,2}$ are defined in Theorem 6.

The inequality $\Gamma < 0$ in (34) means that $\Gamma_{11} < 0$ and that there exists some $\gamma > 0$ such that $\dot{V}(t,x_t) \le -\gamma \left\|D(t)\,x_t\right\|$ for all $t \ge \sigma$. $D(t)$ and $\widetilde{D}(t)$ are stable by Assumption ii) of Theorem 6, see [4, 11], then, the robust asymptotic stability of (2)–(5) is ensured by Theorem 2.3 of [4] or Theorem 8.1 of [11] for all delay $r_2 \le r_2^*$. □

A variation of the same result can be derived if the bound of (34) is performed in terms of $x(t)$, $x(t-r_1)$:

Theorem 8. Consider the Neutral System (2)–(5) with mapping $\Delta_A = \delta_A x(t)$, δ_A, δ_B, δ_C constant uncertain matrices and assume that the following conditions are satisfied:

i) $A_1 := A + B$ is a Hurwitz stable matrix;

ii) The difference operators $D\varphi := [\varphi(0) - (C + \delta_C)\varphi(-r_1)]$ and $\widetilde{D}\varphi := [D\varphi + \int_{-r_2}^{0} (B + \delta_B)\varphi(\theta)]$ are uniformly stable with respect to $\mathcal{C}([\sigma,\infty), \mathbb{R}^n)$, and D is atomic at 0;

iii) there exist a real positive number r_2^* and positive definite matrices P, S_i, $R_j > 0$, $i = \overline{1,2}$, $j = \overline{1,13}$, such that the following LMIs hold:

$$\Gamma := \begin{pmatrix} Q(r_2^*) & \Gamma_{12}^\top & \overline{S}(r_2^*) \\ \Gamma_{12}^\top & \Gamma_{22} & 0 \\ \overline{S}^\top(r_2^*) & 0 & R \end{pmatrix} < 0,$$

$$-S_2 + B^\top(R_8 + R_{10} + R_{12})B + W_B^\top(R_9 + R_{11} + R_{13})W_B < 0,$$

where

$$\Gamma_{12}^\top := -A_1^\top PC, \overline{S}(r_2^*) = \begin{pmatrix} \overline{S}_1 & \overline{S}_2(r_2^*) \end{pmatrix},$$

$$\overline{S}_1 = \begin{pmatrix} P & P & A_1^\top P & W_A^\top E_A^\top P & W_A^\top E_A^\top P & W_B^\top P & W_B^\top P \end{pmatrix},$$

$$\overline{S}_2(r_2^*) = \sqrt{r_2^*}\begin{pmatrix} A_1^\top P & A_1^\top P & W_A^\top E_A^\top P & W_A^\top E_A^\top P & W_B^\top P & W_B^\top P \end{pmatrix},$$

$$Q(r_2^*) := \Gamma_{11} + \overline{S}(r_2^*) R^{-1} \overline{S}^\top(r_2^*),$$

$$\Gamma_{11} = 2A_1^\top P + W_A^\top E_A^\top R_1 E_A W_A^\top + W_B^\top R_2 W_B + S_1 + r_2^* S_2 - \overline{S}(r_2^*) R^{-1} \overline{S}^\top(r_2^*),$$

$$R^{-1} := \operatorname{diag}(R_j^{-1}),\ j = \overline{1,13},$$

$$\Gamma_{22} := -S_1 + W_C^\top(R_3 + R_5 + R_7)W_C + C^\top(R_4 + R_6)C.$$

Then, the Neutral System (2)–(5) is robustly delay-dependent asymptotically stable for any $r_2 \le r_2^*$.

Proof. The proof is just sketched since it follows the basic ideas of Theorem 6. Relations (23)–(29) are obtained but terms $\widetilde{D}x_t$ are expanded and inequalities (5)–(6) and (30) are used to get

$$\dot{V}(t, x_t) \le \zeta^\top \Gamma \zeta < 0, \tag{35}$$

$$\zeta = \begin{pmatrix} x(t) \\ x(t - r_1) \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{12}^\top & \Gamma_{22} \end{pmatrix}.$$

Similar arguments to (34) are used in (35) to prove stability. □

Conditions of Theorem 6 are not difficult to check, except the difference operator stability $\widetilde{D}(t)$ given in equation (11). Now, some references are given for sufficient conditions based on robust control techniques.

For the difference operator $\widetilde{D}\varphi$ defined in Theorem 8, mapping $\Delta_A = \delta_A x(t)$ and δ_A, δ_B, δ_C constant matrices of appropriate dimensions, then the uniform asymptotic stability of (11) is equivalent to the stability of the corresponding characteristic equation, that is, all the solutions $s \in \mathbb{C}$ of the associated characteristic equation

$$\det \left(I_n - (C + \delta_C) e^{-sr_1} + (B + \delta_B) \int_{-r_2}^{0} e^{s\theta} \, \mathrm{d}\theta \right) = 0, \tag{36}$$

have negative real part, i. e. $\operatorname{Re}(s) \leq -\varepsilon < 0$, $(\varepsilon > 0)$. This equation was studied in [23] where some results are given.

Remark 9. In [23], where the uncertainties Δ_A, Δ_B, Δ_C are zero, the delay dependent stability problem is studied with the Lyapunov–Krasovskii functional (23), $V(\varphi) := V_1(\varphi) + V_2(\varphi) + V_3(\varphi)$ with V_1 redefined as

$$\begin{aligned} &V_1(\varphi) := \left[\widetilde{D}\varphi\right]^\top P\widetilde{D}\varphi, \\ &\widetilde{D}\varphi := \varphi(0) - C\varphi(-r_1) + B \textstyle\int_{-r_2}^{0} \varphi(\theta) \, \mathrm{d}\theta. \end{aligned} \tag{37}$$

However, the main result in [23] is not equivalent to the one given here by Theorems 6 and 8 when $W_A = W_B = W_C = 0$, since the bound in [23] is taken with $\dot{V}(x_t) \leq -\gamma \|Dx_t\|$ and not with $\dot{V}(x_t) \leq -\gamma \left\|\widetilde{D}(t) x_t\right\|$.

Remark 10. In [21], the neutral system (2) – (5) is considered with multiple time delays but with $C = 0$, $\Delta_C(t - r_1, \varphi) = \Delta_C(\varphi)$, $\Delta_B = \Delta_C = 0$. The Lyapunov–Krasovskii functional studied in [21] corresponds to $V(\varphi) := V_1(\varphi) + V_2(\varphi) + V_3(\varphi)$ when there are two delays and V_1 given in (37).

5. EXAMPLE

An interesting problem in control theory is to determine the total set of parameters which guarantees stability of a system. For simplicity, in this section, the delay dependent robust stability of the scalar linear neutral system (38) is studied in the parameter space (a, b, c), for a given $r > 0$.

We consider the following scalar neutral time-delay system

$$\dot{x}(t) = -ax(t) - bx(t - r) - c\dot{x}(t - r), \tag{38}$$

the covector $\begin{pmatrix} a & b & c & r \end{pmatrix}$ associated to it, and the following cases:

i) $\begin{pmatrix} a & b & 0 & r \end{pmatrix}$, a time-delay systems (TDS) of retarded type;

ii) $\begin{pmatrix} 0 & b & c & r \end{pmatrix}$, a TDS of neutral type;

iii) $\begin{pmatrix} a & 0 & c & r \end{pmatrix}$, a TDS of neutral type;

iv) $\begin{pmatrix} a & ac & c & r \end{pmatrix}$, a TDS of neutral type.

The stability regions of systems (i) – (iv) can be drawn in the 3D parameter space (a, b, c). Note that the stability region of systems (i) – (iii) are the projections of the stability region w.r.t. the parameters (a, b, c) of (38) in the planes (a, b), (b, c) and

(a, c) respectively. To check the stability regions of (i) – (iv) with numerical values of parameters a, b and c, the Theorem 8 was used. It was assumed that some $r > 0$ is given and that parameters a, b and c had uncertainties. The stability results are presented in table I and include the percentage of error in the parameters (a, b, c) different from zero, it is denoted by "error". The stability of $\widetilde{D}$ holds from the continuity argument of the roots w.r.t. r (see for example [23, 5]).

Table										
Item	a	b	c	r	W_A	W_B	W_C	% error	stability	figure
1	1	0	0	1000	0.99	0	0	99	✓	1
2	1	1/2	0	0.5	0.46	0.23	0	46	✓	1
3	1	1/2	0	1.4	0.1	0.05	0	10	✓	1
4	0	1/2	0	0.5	0	0.23	0	46	✓	1
5	0	1/2	0	1.4	0	0.05	0	10	✓	1
6	0	1/2	±1/10	0.5	0	0.175	0.035	35	✓	2
7	0	1/2	±1/10	1.2	0	0.05	0.01	10	✓	2
8	1	0	±1/2	1000	0.23	0	0.125	23	✓	3
9	1	1/3	±1/3	0.6	0.2	$0.0\overline{6}$	$0.0\overline{6}$	20	✓	4
10	1	1/3	±1/3	1	0.13	$0.04\overline{3}$	$0.04\overline{3}$	13	✓	4

The stability regions can be obtained by the study of the transfer functions of systems (i) – (iv) evaluated in $s = j\omega$.

For system (i) the delay-dependent stability domain is well known (see for instance [15, 25]), it is depicted in Figure 1 for $r = 0.5$, but Theorem 8 was verified for large numbers for example the one given in table I is $r = 1000$. For fixed $r > 0$, the upper boundary of the region of stability is given parametrically by the equations

$$a = \frac{-\omega}{\tan(\omega r)}, \quad b = \frac{\omega}{\sin(\omega r)}, \quad r = \frac{\arccos(-a/b)}{\sqrt{b^2 - a^2}} > 0, \quad \arccos(-a/b) \in (0, \pi),$$

and the lower boundary by

$$b = -a, \, a \geq -1/r.$$

The region $a > |b|$ corresponds to the region for which there is asymptotic stability for all $r > 0$.

For system (ii), the boundary of the region of stability is given by

$$b = \omega \sin(\omega r), \, c = -\cos(\omega r), \; r = \frac{\sqrt{1 - c^2}}{b} \arctan\left(\sqrt{\frac{1}{c^2} - 1}\right)$$

(see also [23] to compare). In this example the shaded area in Figure 2, was verified with the values shown in Table

For system (iii), it is easy to check that the stability region (see Figure 3) is the domain limited by

$$-1 < c < 1, a > 0$$

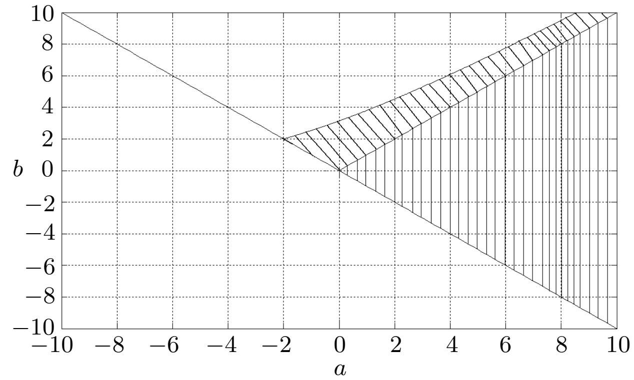

Fig. 1. Plane (a, b).

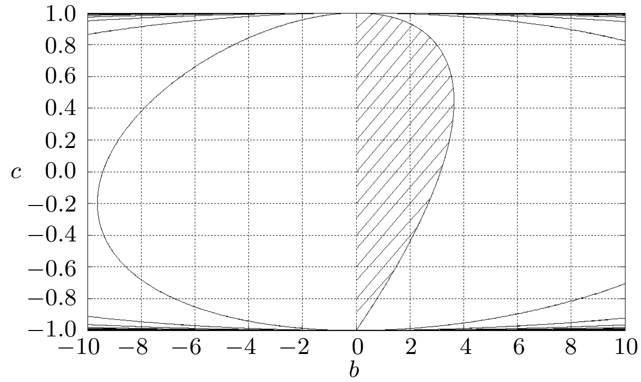

Fig. 2. Plane (b, c).

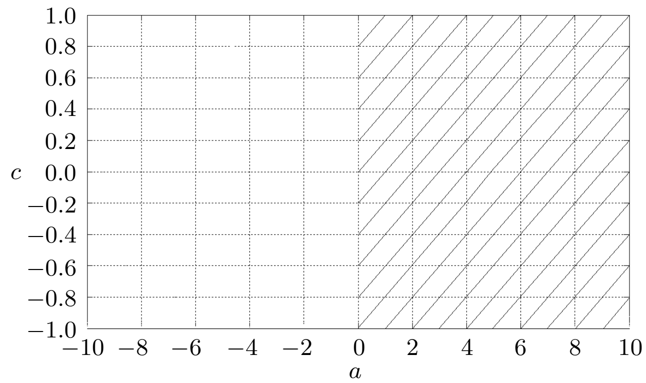

Fig. 3. Plane (a, c).

and is independent of the delay r. For the analysis of asymptotic stability of systems of the form (iii) see [12] and the references therein.

Finally, from the transfer function of system (iv),

$$(s+a)\left(1+ce^{-sr}\right)=0,$$

it is concluded that

$$-1<c<1, a>0$$

implies the delay-independent stability of system (iv), the stable surface is given in Figure 4.

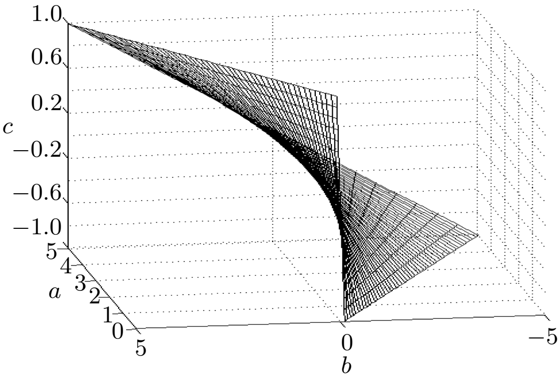

Fig. 4. Stable surface of systems (a ac c r).

6. CONCLUSIONS

In this paper, the delay-dependent robust stability of linear neutral systems was considered. For a general class of uncertain neutral systems, sufficient conditions were obtained, checkable in the LMI framework. The analysis has been performed in terms of an appropriate Lyapunov–Krasovskii functional. This work which is the continuation of [30], avoids the additional dynamics introduced in the transformation model of [30]. The proposed stability analysis extends some previous studies on the subject. Along the same lines, an interesting topic for further studies is the delay dependent robust stabilization of neutral systems.

ACKNOWLEDGEMENTS

This work was partially supported by the PROMEP México and the LAFMAA French–Mexican Laboratory in Applied Automatic Control.

(Received August 5, 2009.)

REFERENCES

[1] C. Abdallah, M. Ariola, and V. Koltchinskii: Statistical-learning control of multiple-delay systems with applications to ATM Networks. Kybernetika *37* (2001), 355–365.

[2] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan: Linear Matrix Inequalities in System and Control Theory. Society for Industrial and Applied Mathematics, Philadelphia 1994.

[3] J. Chen: On computing the maximal delay intervals for stability of linear delay systems. IEEE Trans. Automat. Control *40* (1995), 1087–1093.

[4] M. A. Cruz and J. K. Hale: Stability of functional differential equations of neutral type. J. Differential Equations *7* (1970), 334–355.

[5] L. E. El'sgol'ts and S. B. Norkin: Introduction to the Theory and Applications of Differential Equations with Derivating Arguments (translated from Russian). (Mathematics in Science and Engineering, Vol. 105.) Academic Press, New York 1973.

[6] M. Fliess and H. Mounier: On a class of linear delay systems often arising in practice. Kybernetika *37* (2001), 295–308.

[7] E. Fridman: Descriptor discretized Lyapunov functional method: analysis and design. IEEE Trans. Automat. Control *51* (2006), 890–897.

[8] K. Gopalsamy: Stability and Oscillations in Delay Differential Equations of Population Dynamics (Mathematics and Its Applications Series 74). Kluwer Academic Publishers, Dordrecht 1992.

[9] K. Gu and S. Niculescu: Additional dynamics in transformed time-delay systems. Proc. IEEE Trans. Automat. Control *45* (2000), 572–575.

[10] J. K. Hale and M. A. Cruz: Existence, uniqueness and continuous dependence for hereditary systems. Ann. Mat. Pura Appl. *85* (1970), 63–82.

[11] J. K. Hale and S. M. Verduyn Lunel: Introduction to Functional Differential Equations. Springer–Verlag, Berlin 1993.

[12] J. K. Hale and M. A. Cruz: Strong stabilization of neutral functional differential equations. IMA J. Math. Control Inform. *19* (2002), 5–23.

[13] D. Ivanescu, S. Niculescu, L. Dugard, J. M. Dion, and E. I. Verriest: On delay dependent stability for linear neutral systems. Automatica *39* (2003), 255–261.

[14] V. L. Kharitonov: Robust stability analysis of time-delay systems: a survey. In: IFAC Conference System Structure and Control, Nantes 1998, pp. 1–12.

[15] V. B. Kolmanovskii and A. D. Myshkis: Introduction to The Theory and Applications of Functional Differential Equations. Kluwer Academic Publishers, Dordrecht 1999.

[16] V. B. Kolmanovskii and V. R. Nosov: On the stability of first order nonlinear equations of neutral type. Prikl. Mat. Mekh. *34* (1970), 587–594.

[17] V. B. Kolmanovskii and J. P. Richard: Stability of some systems with distributed delays. JESA, Special Issue on Analysis and Control of Time-delay Systems *31* (1997), 13–18.

[18] V. B. Kolmanovskii and J. P. Richard: Stability of some linear systems with delays. IEEE Trans. Automat. Control *44* (1999), 984–989.

[19] Y. Kuang: Delay-Differential Equation with Applications in Population Dynamics. Academic Press, New York 1993.

[20] H. Li, S.-M. Zhong, and H. B. Li: Some new simple stability criteria of linear neutral systems with a single delay. J. Comput. Appl. Math. *200* (2007), 441–447.

[21] C.-H. Lien: Asymptotic criterion for neutral systems with multiple time delays. IEE Elec. Lett. *35* (1999), 850–851.

[22] S. I. Niculescu and B. Brogliato: Force measurement time-delays and contact instability phenomenon. European J. Control *5* (1999), 279–289.

[23] S. I. Niculescu: On delay-dependent stability under model transformations of some neutral linear systems. Internat. J. Control *74* (2001), 609–617.

[24] S. I. Niculescu: On robust stability of neutral systems. Kybernetika *37* (2001), 253–263.

[25] S. I. Niculescu, E. I.Verriest, L. Dugard, and J.-M. Dion: Stability and robust stability of time–delay systems: A guided tour. In: Stability and Control of Time-Delay Systems (L. Dugard and E. I. Verriest, eds.), Springer–Verlag, Berlin 1997, pp. 1–71.

[26] P. Ngoc and B. Lee: Some sufficient conditions for exponential stability of linear neutral functional differential equations. Appl. Math. Comput. *170* (2005), 515–530.

[27] J. P. Richard: Time-delay systems: An overview of some recent advances and open problems. Automatica *39* (2003), 1667–1694.

[28] S. A. Rodriguez, J. M. Dion and L. Dugard: Robust stability analysis of neutral systems under model transformation. In: Proc. 41st IEEE Conference on Decision and Control, Las Vegas 2002, pp. 1850–1855.

[29] S.A. Rodriguez, J.M. Dion and L. Dugard: Stability of neutral time delay systems: A survey of some results. In: Advances in Automatic Control (Mihail Voicu ed.), Kluwer Academic Publishers, Boston 2003, pp. 315–336.

[30] S. A. Rodriguez, J. M. Dion, and L. Dugard: Robust delay-dependent stability analysis of neutral systems. In: Advances in Time-Delay Systems Vol. 38 (S. Niculescu and K. Gu, eds.), Springer–Verlag, Berlin 2004, pp. 269–284.

[31] E. I. Verriest and S. I. Niculescu: Delay-independent stability of LNS: A Riccati equation approach. In: Stability and Control of Time-Delay Systems (L. Dugard and E. I. Verriest, eds.), Springer–Verlag, Berlin 1997, pp. 92–100.

Salvador Antonio Rodríguez Paredes, SEPI ESIME-UA IPN, Av. de las Granjas No. 682 Col. Sta. Catarina Azcapotzalco, C.P. 02550, DF México and División Académica de Ingeniería y Arquitectura, UJAT, Km 1, Cunduacán-Jalpa de Mendez C.P. 86690, Tabasco. México.
e-mails: sarodriguezp@ipn.mx, salvador.rodriguez@daia.ujat.mx

Luc Dugard and Jean-Michel Dion, Gipsa-Lab, CNRS–INPG–UJF, Grenoble, Control Department, ENSE3 – BP 46, 38402 Saint Martin d'Heres Cedex. France.
e-mails: luc.dugard@gipsa-lab.grenoble-inp.fr, jean-michel.dion@gipsa-lab.grenoble-inp.fr

Jesús De León Morales, Facultad de Ingeniería Mecánica y Eléctrica, UANL, Av. Universidad s/n Ciudad Universitaria, C.P. 66451, Nuevo León México. México.
e-mail: drjleon@gmail.com