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# PARAMETRIZATION AND GEOMETRIC ANALYSIS OF COORDINATION CONTROLLERS FOR MULTI–AGENT SYSTEMS

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In this paper, we address distributed control structures for multi-agent systems with linear controlled agent dynamics. We consider the parametrization and related geometric structures of the coordination controllers for multi-agent systems with fixed topologies. Necessary and sufficient conditions to characterize stabilizing consensus controllers are obtained. Then we consider the consensus for the multi-agent systems with switching interaction topologies based on control parametrization.

*Keywords:* multi-agent systems, parametrization, geometric structures, coordination control

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## 1. INTRODUCTION

Cooperation control of multi-agent systems becomes a very active research area from the beginning of this century. In fact, the distributed approach to large-scale networked systems shows more feasibility and greater operational capability than conventional centralized control and recent years have witnessed the rapid development of distributed control protocols via interconnected communication to achieve the collective tasks.

Consensus, formation, and swarming are important problems of multi-agent coordination, since in reality it is usually required that all the agent vehicles achieve the desired relative position and the same velocity. Various consensus or formation problems were formulated and considered from different viewpoints ([3, 13, 14, 15]). A general formulation of vehicle formation was shown and some problems such as how the topology of the information flow affects the stability and performance of the system coordination were discussed in [3]. The local controllers were proposed to achieve a consensus among a group of autonomous mobile agents with second-order dynamics with switching interconnection topologies in [5], while linear stabilizing feedback was studied when the directed graph associated with the considered multi-agent systems containing a spanning tree in [8].

Controller parametrization is a very fundamental problem in control theory. It provides an elegant and efficient way to solve the stabilizing and related design problems, with which all the stabilizing controllers are characterized in unified forms ([2, 7, 10, 12]). The parametrization of all asymptotically stabilizing controllers was investigated for linear and nonlinear systems in many papers (see [11, 12] and [7] and the references therein), respectively, where the parameterized controllers well showed the structures of stabilizing controllers of the considered systems. There were many results on parametrization structures of linear (dynamical) systems, but these results could not be applied straightforward to multi-agent systems due to the complexities of multi-agent interactions. On the other hand, although multi-agent consensus or formation is a problem related to synchronization or stabilization of a low-dimensional manifold (not stabilization of equilibria and limited cycles), there are some relationships between the coordination control and simultaneous stabilization of control systems ([2, 9]). In fact, linear-matrix-inequality-based formulas for this problem and related techniques on simultaneous control are inspirative in the studies of parametrization of multi-agent control design.

The objective of this paper is to study the parametrization of the coordination controllers of multi-agent systems. In other words, we aim to give a general expression of all the coordination controllers of the considered multi-agent systems. Motivated by the existing results on the necessary and sufficient conditions for the formation stabilizing controllers (see [3, 8]), we show the geometric structures of coordination controllers with necessary and sufficient conditions for agent dynamics in general linear forms. Also, we find the set of coordination controllers is diffeomorphic to the Cartesian product of the set of positive matrices and the set of skew symmetric matrices satisfying certain algebraic conditions. Here, we mainly analyze the local control make the system consensus with fixed graph topology, and the interconnection topology between agents are not to be influenced by local controllers as assumed in related papers like [3] and [8].

The paper is organized as follows. In Section 2, necessary preliminaries are given for our analysis. Then, in Section 3, we study the parametrization and related geometric structures of the coordination controllers for multi-agent systems with necessary and sufficient conditions, and moreover, in Section 4, based on parametrization, coordination problems are further investigated for multi-agent systems with switching topologies. Finally, the concluding remarks are given in Section 5.

## 2. PRELIMINARIES

In this section, we will provide some preliminary knowledge for the discussion in the following sections.

As usual, the interaction topology of multi-agent systems can be described by graphs (see [4] for the details of graph theory). Let $\mathcal{G} = (\mathcal{V},\ \mathcal{E}, A^*)$ be a weighted directed graph (or digraph) of order $N$ with the set of nodes $\mathcal{V} = \{\pi_1, \pi_2, \dots, \pi_N\}$, set of edges $\mathcal{E} \in \mathcal{V} \times \mathcal{V}$, and a weighted adjacency matrix $A^* = [a_{ij}] \in \mathbb{R}^{N\times N}$ with nonnegative adjacency elements. An edge $(\pi_i, \pi_j)$ in a weighted digraph denotes that node $\pi_j$ can obtain information from node $\pi_i$, but not necessary vice versa. In this way, we call $\pi_i$ is the parent node and $\pi_j$ is the child node. The weighted adjacency matrix $A^*$ of a weighted digraph is defined such that $a_{ij}$ is positive if $(\pi_j, \pi_i) \in \mathcal{E}$, while $a_{ij} = 0$ if $(\pi_j, \pi_i) \notin \mathcal{E}$, and moreover, $a_{ii} = 0$ because self edges are not allowed. The graph $\mathcal{G}$ is said to be undirected if $(\pi_i, \pi_j) \in \mathcal{E} \Leftrightarrow (\pi_j, \pi_i) \in \mathcal{E}$, and let, $a_{ij} = a_{ji}$, $\forall j \neq i$. The set of neighbors of node $\pi_i$ at time $t$ is denoted by $\pi^{(i)}(t) = \{\pi_j \in \mathcal{V} : (\pi_i, \pi_j) \in \mathcal{E},\ j = 1, \ldots, N\}$. In this paper, we assume that $\pi^{(i)}(t) \neq \emptyset$, $i = 1, \ldots, N$, meaning each agent can sense at least one other agent. $L = \triangle - A^*$ is the Laplacian matrix of $\mathcal{G}$ and $\triangle$ is the degree matrix of $\mathcal{G}$ with diagonal elements $d_i = \sum_{j=1}^{N} a_{ij}$, $i = 1, \ldots, N$. A directed path is a sequence of edges in the directed graph $\mathcal{G}$ of the form $(\pi_{i1}, \pi_{i2}), (\pi_{i2}, \pi_{i3}), \ldots$. An undirected path in an undirected graph is defined analogously. A tree is a digraph, in which every node has exactly one parent except for one node, called the root, which has no parent, and the root has a directed path to every other node. A spanning tree of a digraph is a tree formed by the graph edges that connect all the nodes of the graph; namely, there exists at least one node having a directed path to all of the other nodes. An undirected graph is said to be connected if it contains a spanning tree.

The next lemma is well known (referring to [4]).

**Lemma 2.1.** 0 is an eigenvalue of Laplacian matrix $L$ with $\mathbf{1}_N = (1, \ldots, 1)^T$ as the corresponding eigenvector. A digraph has a spanning tree if and only if 0 is a simple eigenvalue of $L$.

In what follows, the digraph is assumed to have a spanning tree.

In this paper, we consider a set of $N$ agents, whose dynamics are identical as follows:

$$\dot{x}_i = Ax_i + Bu_i, \quad i = 1, \ldots, N, \tag{1}$$

where $(A,\ B)$ is stabilizable, $B$ is of full column rank, $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, are the states and controls of the agent $i$, $i = 1, \ldots, N$. In the multi-agent network, each agent receives the external state measurements relative to its neighbors as follows:

$$z_i = \sum_{j=1}^{N} a_{ij}(x_i - x_j), \quad i = 1, \ldots, N,$$

and the control law is expressed as:

$$u_i = Kz_i, \quad i = 1, \ldots, N. \tag{2}$$

The distributed control law $u_i$ based on the relative information $z_i$ of the system cannot make the system converge to any given position. Instead, it may achieve the system consensus (or formation); namely, make the following hold

$$\lim_{t \to \infty} [x_i(t) - x_j(t)] = 0, \quad i, j = 1, \ldots, N.$$

In this scenario, the graph topology is usually unchanged with control (2).

In practice, the interactions among the agents are time-varying and therefore, the interaction topologies of the considered multi-agent systems change over time. Suppose that there is an infinite sequence of bounded, non-overlapping, continuous time-intervals $[t_i, t_{i+1}),\ i = 0, 1, \ldots,$ starting at $t_0 = 0$.

Denote the set of the interconnection digraphs with spanning trees as $\mathcal{G} = \{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{\widetilde{N}}\}$ with $\mathcal{P} = \{1, 2, \ldots, \widetilde{N}\}$ as its index set. The special case is the connected undirected graphs. As usual, we assume there is a dwell time $\tau$ such that $t_{i+1} - t_i \geq \tau,\ i = 0, 1, \cdots$.

To describe the variable interconnection topologies, we define a switching signal $\sigma : [0, \infty) \to \mathcal{P}$, which is piecewise constant. Therefore, $\mathcal{G}_i$ and the connection weight $a_{ij}$ $(i, j = 1, \ldots, N)$ are time-varying, and moreover, Laplacian matrix $L_p$ $(p \in \mathcal{P})$ associated with the switching interconnection graph is also time-varying (switching at $t_i,\ i = 0, 1, \ldots$), though it is a time-invariant matrix in any interval $[t_i, t_{i+1})$. Let $x = (x_1^T, \ldots, x_N^T)^T$, then the dynamics considered under the switching interaction topologies can be rewritten in a compact form:

$$\dot{x} = [I_N \otimes A + (I_N \otimes BK)\,(L_\sigma \otimes I_n)]x, \tag{3}$$

where $\otimes$ denotes the Kronecker product. In many cases, the interaction topologies may be fixed, and then the multi-agent system becomes a special form of (3):

$$\dot{x} = [I_N \otimes A + (I_N \otimes BK)\,(L \otimes I_n)]\,x, \tag{4}$$

where $L$ still represents the Laplacian matrix of a digraph with a spanning tree.

The consensus of system (3) is achieved under control (2), if, for any initial condition $(x_1(0), \ldots, x_n(0))$, the manifold $\Omega = \{x \in \mathbb{R}^{nN} :\ x_1 = \ldots = x_N\}$ is set attractive, or equivalently,

$$\lim_{t \to +\infty} \|x_i(t) - x_j(t)\| = 0, \quad i, j = 1, \ldots, N.$$

Then (2) is called a coordination controller.

Finally, we introduce basic results of generalized inverse matrices (see [1] for details). Let $B^\dagger \in \mathbb{R}^{m\times n}$ be a generalized inverse matrix of $B$. Then

1) Both $BB^\dagger$ and $I - BB^\dagger$ are symmetric matrices. Furthermore,
$$BB^\dagger B = B,\ B^\dagger BB^\dagger = B^\dagger,\ B^T BB^\dagger = B^T,$$
2) $BB^\dagger$ is an orthogonal projection matrix to $\operatorname{Im} B$, and $I - BB^\dagger$ is the orthogonal projection matrix to orthogonal complement of $\operatorname{Im} B$.

The next lemma is important in the following analysis.

**Lemma 2.2.** [1] Let $A_1 \in \mathbb{R}^{m\times n},\ A_2 \in \mathbb{R}^{p\times q}$ and $A_3 \in \mathbb{R}^{m\times q}$ be given. The linear matrix equation $A_1 X A_2 = A_3$ can be solved if and only if

$$A_1 A_1^\dagger A_3 A_2^\dagger A_2 = A_3. \tag{5}$$

Furthermore, if (5) is satisfied, all the solutions can be given by

$$X = A_1^{\dagger} A_3 A_2^{\dagger} - Z + A_1^{\dagger} A_1 Z A_2 A_2^{\dagger},$$

where $Z \in \mathbb{R}^{n \times p}$ is an arbitrary matrix.

For convenience, denote the boundary of a set $S$ as $\partial S$, the set of $n \times n$ positive definite matrices as $PD(n)$, the set of $n \times n$ skew symmetric matrices as $Skew(n)$, the set of $n \times n$ stable matrices as $\varphi(n)$, and $S_*^m = \underbrace{S_* \times \ldots \times S_*}_{m}$ for any set $S_*$.

## 3. CONTROLLER PARAMETRIZATION

In this section, we focus on the parametrization of coordination controllers for system (4).

Note that the number of the nonzero eigenvalues of the Laplacian matrix associated with the digraph having a spanning tree is $N-1$ according to Lemma 2.1. Then, we state with the following lemma.

**Lemma 3.1.** For given $Q_i \in PD(n),\ i = 1, \ldots, N-1$, if there is $K$ to satisfy

$$(A + \lambda_i BK)P_i + P_i(A + \lambda_i BK)^T + Q_i = 0, \quad P_i \in PD(n),\ i = 1, \ldots, N-1, \tag{6}$$

where $\lambda_i,\ i = 1, \ldots, N-1$ are the nonzero eigenvalues of Laplacian matrix associated with the digraph having a spanning tree, then $P_i,\ i = 1, \ldots, N-1$, satisfy

$$(I - BB^{\dagger})\,(AP_i + P_iA^T + Q_i)\,(I - BB^{\dagger}) = 0,\ i = 1, \ldots, N-1, \tag{7}$$

$$\begin{aligned} &\lambda_j [B^{\dagger}(AP_i + P_iA^T + Q_i)\left(I - \frac{1}{2}BB^{\dagger}\right)P_i^{-1} + \lambda_i B^{\dagger} S P_i^{-1}] \qquad (8)\\ =&\lambda_i \left[B^{\dagger}(AP_j + P_jA^T + Q_i)\left(I - \frac{1}{2}BB^{\dagger}\right)P_j^{-1} + \lambda_j B^{\dagger} S P_j^{-1}\right],\ j = 1, \ldots, N-1 \end{aligned}$$

and $K$ is taken the form of

$$K = -\frac{1}{\lambda_i}[B^{\dagger}(AP_i + P_iA^T + Q_i)\left(I - \frac{1}{2}BB^{\dagger}\right)P_i^{-1}] - B^{\dagger} S P_i^{-1},\ \lambda_i \neq 0, \tag{9}$$

with

$$S = BB^{\dagger}S \in Skew(n) \Leftrightarrow S = BB^{\dagger}SBB^{\dagger} \in Skew(n). \tag{10}$$

Proof. From (6), we have

$$AP_i + P_iA^T + Q_i = -(\lambda_i BKP_i + \lambda_i P_iK^TB^T). \tag{11}$$

Now pre-multiply and post-multiply $I - BB^\dagger$ in both sides of (11), then we obtain (7). Setting

$$W_1 = BB^\dagger(AP_i + P_iA^T + Q_i)BB^\dagger,$$
$$W_2 = BB^\dagger(AP_i + P_iA^T + Q_i)\,(I - BB^\dagger),$$

we get

$$AP_i + P_iA^T + Q_i = W_1 + W_2 + W_2^T$$

from (7).

Choose a matrix $S \in Skew(n)$ with $S = BB^\dagger S$ and denote $S_i = \lambda_i S$. Since

$$\begin{aligned} & BB^\dagger(AP_i + P_iA^T + Q_i)\left(I - \frac{1}{2}BB^\dagger\right) + BB^\dagger S_i \\ = \; & BB^\dagger(AP_i + P_iA^T + Q_i)\left(I - \frac{1}{2}BB^\dagger\right) + S_i \\ = \; & \frac{1}{2}W_1 + W_2 + S_i, \end{aligned}$$

by Lemma 2.2, there exists a matrix $K$ satisfying the following linear matrix equation:

$$-\lambda_i BKP_i = \frac{1}{2}W_1 + W_2 + S_i, \tag{12}$$

and all the possible solutions can be given by

$$\lambda_i K = -B^\dagger(AP_i + P_iA^T + Q_i)\left(I - \frac{1}{2}BB^\dagger\right)P_i^{-1} - B^\dagger S_i P_i^{-1} + (I - B^\dagger B)Z,\ \forall Z \in \mathbb{R}^{m\times n}.$$

Since $B$ is of full column rank matrix, the last term of the above equation is 0.

Note that the considered digraph has a spanning tree and therefore, the 0 eigenvalue of the Laplacian matrix is simple. Since $K$ satisfies $N-1$ equations in (6), $P_i,\ P_j,\ i,\ j = 1, \ldots, N-1$ satisfy (8), and $K$ takes the form of (9). □

The following theorem is about the parametrization of the multi-agent coordination controllers.

**Theorem 3.2.** Assume the interconnection digraph of the multi-agent system (4) has a spanning tree. Then the following conditions are equivalent:

1) The consensus of system (4) is achieved.
2) Any coordination controller $K$ makes $A+\lambda_i BK$ stable (Hurwitz), where $\lambda_i,\ i = 1, \ldots, N-1$ are the nonzero eigenvalues of Laplacian matrix $L$.
3) For given $Q_i \in PD(n),\ i = 1, \ldots, N-1$, $K$ satisfies (6).
4) $P_i \in PD(n),\ i = 1, \ldots, N-1$ satisfy (7), (8).
5) $K$ can be written in the form of (9) with $P_i \in PD(n),\ i = 1, \ldots, N-1$ satisfying (7), (8).

P r o o f. 1) $\Leftrightarrow$ 2): Since the digraph has a spanning tree, it follows from Lemma 2.1 that, $\Omega = \{x \in \mathbb{R}^{nN} : x_1 = \ldots = x_N\} = \{x \in \mathbb{R}^{nN} : (L \otimes I_n)x = 0\}$. First we prove that

$$[I_N \otimes A + (I_N \otimes BK)(L \otimes I_n)]\Omega \subseteq \Omega.$$

For each $x \in \Omega,\ \exists \alpha \in \mathbb{R}^n,\ x = \mathbf{1}_N \otimes \alpha$, then we have

$$[I_N \otimes A + (I_N \otimes BK)(L \otimes I_n)](\mathbf{1}_N \otimes \alpha) = (I_N \otimes A)(\mathbf{1}_N \otimes \alpha) = \mathbf{1}_N \otimes A\alpha \in \Omega,$$

which also implies that the matrix with the transformation restriction of $I_N \otimes A + (I_N \otimes BK)(L \otimes I_n)$ on $\Omega$ is exactly $I_N \otimes A$. Thus, the matrix $I_N \otimes A + (I_N \otimes BK)(L \otimes I_n)$ induces a linear transformation on the quotient space $\mathbb{R}^{nN}/\Omega$, whose eigenvalues are those of $A + \lambda BK$ with $\lambda$ a nonzero eigenvalue of $L$. We conclude that the quotient dynamics are globally asymptotically stable if and only if $A+\lambda BK$ is stable for each $\lambda \neq 0$. Moreover, asymptotical stability of the quotient dynamics is equivalent to $x(t) + \Omega \to \Omega$, where $x(t)$ is the solution of system (4). From the definition of $\Omega$, this means that the set $\Omega$ is attractive, namely, the consensus of system (4) is achieved.

2) $\Leftrightarrow$ 3): It is obvious.

3) $\Rightarrow$ 4) and 3) $\Rightarrow$ 5): They are obtained by Lemma 3.1.

4) $\Rightarrow$ 3): From the proof of Lemma 3.1, we can see that, if $P_i,\ i = 1, \ldots, N-1$ satisfy (7) and (8), $K$ can be given in (9), and then we have the equation (12) with $W_1,\ W_2$ defined in the proof of Lemma 3.1. Noting that $S_i \in Skew(n)$, we have

$$-(\lambda_i BKP_i + \lambda_i P_i K^T B^T) = W_1 + W_2 + W_2^T.$$

With (7), $AP_i + P_iA^T + Q_i = W_1 + W_2 + W_2^T$. Then we obtain

$$-(\lambda_i BKP_i + \lambda_i P_i K^T B^T) = AP_i + P_iA^T + Q_i,$$

which is exactly (6).

The above proof also implies 5) $\Rightarrow$ 3). Thus, Theorem 3.2 is proved. □

**Remark 3.3.** One of the main result in [3] showed that the controllers stabilize the $N$ identical agents if and only if they stabilize a single one with the same dynamics modified by only a scalar taking values according to the eigenvalues of the interconnected Laplacian matrix, while, in Theorem 3.2, we only deal with the nonzero eigenvalues of the Laplacian matrix since we consider multi-agent consensus (or set stability) instead of conventional stability. Additionally, in paper [8], the authors made discussions on how to choose local feedback for the agent dynamics with system matrices in some specific forms, while our system (1) takes a more general form. In fact, we propose the coordination controllers with parametrization approaches, different from those given in [3] and [8].

It is known that the controller parametrization of multi-agent coordination is very important in practical design since it transforms the construction of the distributed controllers to the change of certain parameters in the controllers according to control tasks or systems constraints, which may largely simplify the design problem in some situations. Also, it is useful to study the geometric structures of the coordination controllers.

For convenience, set

(1) $PD_{N-1}(n; A, B, Q_1, \ldots, Q_{N-1}) = \{P = (P_1, \ldots, P_{N-1}) : P_i \in PD(n),\ i = 1, \ldots, N-1$ satisfy (7), (8)$\}$.

(2) $Skew(n; B)$ denotes the set of $n \times n$ skew symmetric matrices which satisfy (10).

(3) $\mathcal{K}_s(A, B)$ denotes the set of coordination controllers for (4), that is to say the system (4) achieves consensus under any coordination controller in $\mathcal{K}_s(A, B)$.

**Remark 3.4.** When $N = 1$, the multi-agent problem becomes a "single-agent" problem, the conventional case discussed in some papers including [12]. In fact, it is not hard to see that

1) For $P = (P_1, \ldots, P_{N-1}) \in PD_{N-1}(n; A, B, Q_1, \ldots, Q_{N-1})$, $P_i$ is a subset of $\frac{m(2n-m+1)}{2}$-dimensional linear space $\{P \in PD(n) : (A + B\bar{K})P + P(A + B\bar{K})^T + Q = 0,\ Q \in PD(n)\}$;

2) $Skew(n; B)$ is a subset of $\frac{m(m-1)}{2}$-dimensional linear subspace of $Skew(n)$ (note that if $m = 1$, $Skew(n; B)$ is empty);

3) $PD_{N-1}(n; A, B, Q_1, \ldots, Q_{N-1})$, $Skew(n; B)$ are unbounded and convex.

The next result further shows the geometric structures of the coordination controllers.

**Theorem 3.5.** The set $\mathcal{K}_s(A, B)$ is diffeomorphic to

$$PD_{N-1}(n; A, B, Q_1, \ldots, Q_{N-1}) \times Skew(n; B).$$

Proof. We show that (9) defines a bijective mapping

$$\psi_{(Q_1, \ldots, Q_{N-1})} : PD_{N-1}(n; A, B, Q_1, \ldots, Q_{N-1}) \times Skew(n; B) \to \mathcal{K}_s(A, B).$$

Clearly, it follows from Lemma 3.1 that $K = \psi_{(Q_1, \ldots, Q_{N-1})}(P, S)$ belongs to $\mathcal{K}_s(A, B)$, for any $(P, S) \in PD_{N-1}(n; A, B, Q_1, \ldots, Q_{N-1}) \times Skew(n; B)$. To see $\psi_{(Q_1, \ldots, Q_{N-1})}$ is bijective, we need to show that, for any $K \in \mathcal{K}_s(A, B)$, there exists a unique positive definite solution of equation (6):

$$P_i = \int_0^\infty \exp((A + \lambda_i BK)t) Q_i \exp((A + \lambda_i BK)^T t)\, dt.$$

According to (12),

$$S_i = \lambda_i S = -\left(\lambda_i BKP_i + \frac{1}{2}W_1 + W_2\right).$$

When $\lambda_i \neq 0$, we can take

$$S = -\frac{1}{\lambda_i}\left(\lambda_i BKP_i + \frac{1}{2}W_1 + W_2\right).$$

By Lemma 3.1, $P_i,\ i = 1, .., N-1$ satisfy (7), (8) and $K$ is the form of (9). The uniqueness of $S$ follows easily from the contradiction to the existence of such two pairs. Therefore, there is unique $(P_1, \dots, P_{N-1}) \in PD_{N-1}(n; A, B, Q_1, \dots, Q_{N-1})$, $S \in Skew(n; B)$. Thus, there is the inverse of mapping $\psi_{(Q_1,\dots,Q_{N-1})}$. Moreover, both $\psi_{(Q_1,\dots,Q_{N-1})}$ and $\psi^{-1}_{(Q_1,\dots,Q_{N-1})}$ are of $C^\infty$ class since $\psi_{(Q_1,\dots,Q_{N-1})}$ and $\psi^{-1}_{(Q_1,\dots,Q_{N-1})}$ are polynomial functions. Thus, $\psi_{(Q_1,\dots,Q_{N-1})}$ is diffeomorphism (bijective and differentiable mapping). $\square$

The above theorem tells us that $\mathcal{K}_s(A, B)$ and $PD_{N-1}(n; A, B, Q_1, \dots, Q_{N-1}) \times Skew(n; B)$ share the same geometric structures. Then we go further to show the relationship between

$$PD_{N-1}(n; A, B, Q_1, \dots, Q_{N-1}) \times Skew(n; B)$$

and

$$(PD(n) \times Skew(n))^{N-1} = \underbrace{(PD(n) \times Skew(n)) \times \ldots \times (PD(n) \times Skew(n))}_{N-1}.$$

In fact, for any

$$(P, S) \in PD_{N-1}(n; A, B, Q_1, \dots, Q_{N-1}) \times Skew(n; B),\ \ P = (P_1, \dots, P_{N-1}),$$

$K$ can be written as the form of (9), and then

$$\begin{aligned} A + \lambda_i BK &= A - BB^\dagger(AP_i + P_iA^T + Q_i)\left(I - \frac{1}{2}BB^\dagger\right)P_i^{-1} - \lambda_i SP_i^{-1} \\ &= -\frac{1}{2}Q_iP_i^{-1} + (S_0(P_i) - \lambda_i S)P_i^{-1}, \end{aligned}$$

where

$$S_0(P_i) = AP_i - BB^\dagger(AP_i + P_iA^T + Q_i)\left(I - \frac{1}{2}BB^\dagger\right) + \frac{1}{2}Q_i. \tag{13}$$

Therefore, $P \in PD_{N-1}(n; A, B, Q_1, \dots, Q_{N-1})$ yields

$$S_0(P_i) + S_0(P_i)^T = (I - BB^\dagger)\,(AP_i + P_iA^T + Q_i)\,(I - BB^\dagger) = 0.$$

Thus, $S_0(P_i) - \lambda_i S \in Skew(n)$.

Denote $\varphi_i(n; A, B) = \{A + \lambda_i BK : K \in \mathcal{K}_s(A, B)\}$, $i = 1, \ldots, N-1$. Obviously, a linear mapping

$$\chi : \mathcal{K}_s(A, B) \ni K \to (A + \lambda_1 BK, \ldots, A + \lambda_{N-1} BK) \in (\varphi_1(n; A, B), \ldots, \varphi_{N-1}(n; A, B)),$$

induce an imbedding $f^{-1}_{(Q_1,\ldots,Q_{N-1})} \circ \chi \circ \psi_{(Q_1,\ldots,Q_{N-1})}$ :

$$PD_{N-1}(n; A, B, Q_1, \ldots, Q_{N-1}) \times Skew(n; B) \to (PD(n) \times Skew(n))^{N-1},$$

where

$$\begin{aligned} & f^{-1}_{(Q_1,\ldots,Q_{N-1})}(A + \lambda_1 BK, \ldots, A + \lambda_{N-1} BK) \\ = \; & ((P_1, S_0(P_1) - \lambda_1 S), \ldots, (P_{N-1}, S_0(P_{N-1}) - \lambda_{N-1} S)) \, . \end{aligned}$$

Therefore, we can easily obtain

**Corollary 3.6.** There is an imbedding mapping from

$$PD_{N-1}(n; A, B, Q_1, \ldots, Q_{N-1}) \times Skew(n; B) \quad \text{to} \quad (PD(n) \times Skew(n))^{N-1}.$$

Note that when $N = 1$, the results in Theorem 3.5 and Corollary 3.6 are consistent with those obtained in [12].

Two examples are given for illustrating the above results.

**Example 1.** Here we consider a group of 3 mobile agents, described by connected undirected graphs. Obviously, the number of all the possible Laplacian matrices is 4. Here we omit their exact expressions. We only consider one of the Laplacian matrix of $\mathcal{G}$ as following:

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix},$$

the eigenvalues of it are $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 3$. In this example, set

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

$B = (1 \;\; 0)^T$, then $B^\dagger = (1 \;\; 0)$. Represent $P_i \in PD(2)$, $i = 1, 2, 3$ as

$$P_i = \begin{pmatrix} \eta_{i1} & \eta_{i2} \\ \eta_{i2} & \eta_{i3} \end{pmatrix}, \; \eta_{i1} > 0, \; \eta_{i1}\eta_{i3} > \eta_{i2}^2,$$

$$S = \begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix},$$

$$Q_i = I_2, \; i = 1, 2, 3.$$

From (10), we have $S = (0)_{2\times 2}$.

Since $m = 1$, this can also be easily obtained (see Remark 3.4). Using (7), we get $\eta_{i2} = -\frac{1}{2}$. Let

$$P_2 = \begin{pmatrix} \eta_1 & -\frac{1}{2} \\ -\frac{1}{2} & \eta_3 \end{pmatrix}, \ \eta_1 > 0, \ \eta_1\eta_3 > \frac{1}{4}.$$

We can calculate $P_3$ from (8). Based on (9), we have

$$K = \left(-\eta_1 \quad \frac{1}{2} - \eta_1 - \eta_3\right) P_2^{-1}.$$

To show the imbedding map, we calculate $S_0(P_2)$ (based on (13)), $A + BK$,

$$S_0(P_2) = \begin{pmatrix} 0 & -\eta_1 \\ \eta_1 & 0 \end{pmatrix}, \quad A + BK = S_0(P_2)P_2^{-1}.$$

The next example will show the relationship between our results and some existing coordination controllers.

**Example 2.** Consider a group of $N$ mobile agents, described by digraphs with a spanning tree.

If we set the system matrices $A = 0, \ B = 1$ and let $\lambda$ be a nonzero eigenvalue of the Laplacian matrix, then we can have $S = 0, \ P = \eta, \ \eta > 0$ with the same step as in Example 1. Then we can obtain the coordination controllers:

$$u_i = -\sum_{j=1}^{N} \frac{1}{2\lambda\eta} a_{ij}(x_i - x_j), \ i = 1, \dots, N,$$

from (9), which is consistent with the controllers given in some papers (like [13]).

For the second-order integrator system $\ddot{x}_i^* = u_i, \ i = 1, \dots, N$, we have

$$\dot{x}_i = Ax_i + Bu_i \quad \text{with} \quad x_i = (x_i^* \ \dot{x}_i^*)^T,$$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and $B = (0 \ \ 1)^T$ (and then $B^\dagger = (0 \ \ 1)$). Let $\lambda$ be the nonzero eigenvalue of the Laplacian matrix, with the same step as in Example 1, we can have $S = 0$,

$$P = \begin{pmatrix} \eta_1 & -\frac{1}{2} \\ -\frac{1}{2} & \eta_3 \end{pmatrix}, \ \eta_1 > 0, \ \eta_1\eta_3 > \frac{1}{4}.$$

Based on (9), we have

$$K = -\frac{2}{\lambda(4\eta_1\eta_3 - 1)} \left(2\eta_3^2 + \frac{1}{2} \ \eta_1 + \eta_3\right).$$

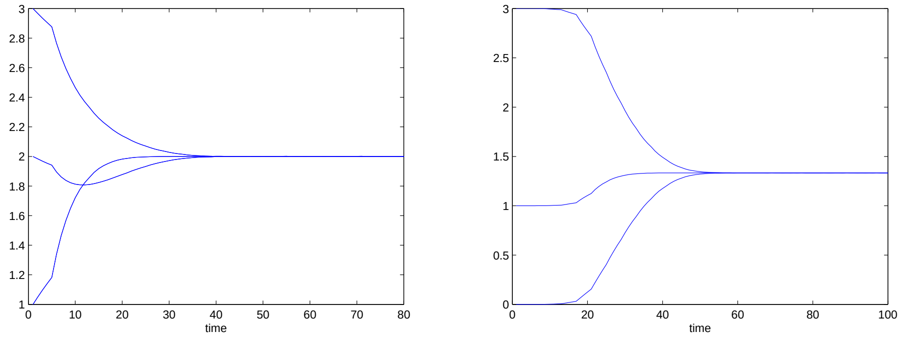


**Figure.** The positions and velocities of the three agents.

Take $\eta_1 = 2\eta_3^2 + \frac{1}{2} - \eta_3$, and then we can construct the coordination controllers:

$$u_i = -\frac{2(\eta_1 + \eta_3)}{\lambda(4\eta_1\eta_3 - 1)} \sum_{j=1}^{N} a_{ij}(x_i^* - x_j^* + \dot{x}_i^* - \dot{x}_j^*),\ i = 1, \ldots, N,$$

with the same form as given in the literature (such as [14]).

The initial conditions are randomly selected as follows:

$$x_1(0) = 1,\ x_2(0) = 3,\ x_3(0) = 2, v_1(0) = 1,\ v_2(0) = 0,\ v_3(0) = 3.$$

## 4. SWITCHING TOPOLOGIES

In this section, we study the multi-agent coordination with switching interaction topologies based on controller parametrization.

Let us consider the system (3) with switching topologies. As defined before, there are at most $\widetilde{N}$ digraphs with spanning trees. For each digraph with a spanning tree, there are $N-1$ nonzero eigenvalues of its Laplacian matrix. Therefore, for all the $\widetilde{N}$ digraphs, we will have $\widetilde{N}(N-1)$ nonzero eigenvalues. In fact, since some eigenvalues may be the same, the number of all different nonzero eigenvalues of the $\widetilde{N}$ Laplacian matrices, whose graphs have spanning trees, is finite, denoted as $\widetilde{M}$, which is no larger than $\widetilde{N}(N-1)$. Denote the set of all the different nonzero eigenvalues as $\Lambda = \{\lambda_i \mid \lambda_i,\ i = 1, \ldots, \widetilde{M}\}$ for convenience.

The following result gives conditions on the consensus for system (3).

**Theorem 4.1.** Suppose the digraphs to describe the switching interconnection topologies have spanning trees and $\widetilde{M}$ is the number of all different nonzero eigenvalues of all the Laplacian matrices, whose graphs have spanning trees. Given $Q_i \in PD(n)(i = 1, \ldots, \widetilde{M})$, if for any $0 < \varepsilon$, there exist $P_i \in PD(n)(i = 1, \ldots, \widetilde{M})$ satisfying

$$(I - BB^\dagger)\,(AP_i + P_iA^T + Q_i)\,(I - BB^\dagger) = 0,\ i = 1, \ldots, \widetilde{M}, \tag{14}$$

$$\begin{aligned}&\lambda_j[B^\dagger(AP_i+P_iA^T+Q_j)\left(I-\frac{1}{2}BB^\dagger\right)P_i^{-1}+\lambda_iB^\dagger SP_i^{-1}] \\ =\ &\lambda_i[B^\dagger(AP_j+P_jA^T+Q_i)\left(I-\frac{1}{2}BB^\dagger\right)P_j^{-1}+\lambda_jB^\dagger SP_j^{-1}],\ j=1,\dots,\widetilde{M}\end{aligned} \tag{15}$$

and $\lambda_{\min}\{Q_iP_i^{-1}\}\geq -2\frac{\ln\varepsilon}{\tau}$ ($\tau$ is the dwell time), $i=1,\dots,\widetilde{M}$, then the consensus is achieved for system (3) under the distributed controller $K$ of form (9).

P r o o f. Denote

$$\lambda^*_{\min}=\min_{\lambda_i\in\Lambda}\{\lambda_i\}>0,\ \lambda^*_{\max}=\max_{\lambda_i\in\Lambda}\{\lambda_i\}.$$

For any $p\in\mathcal{P}$, there exists a $T_p$ making $U_p=T_pL_pT_P^{-1}$ upper triangular with diagonal elements $0,\lambda_2^p,\dots,\lambda_N^p$. Set

$$\nu=\max_{p\in\mathcal{P}}\{\|T_p\|\|T_p^{-1}\|\},\ \varepsilon=\frac{1}{2\nu}. \tag{16}$$

We claim

$$-\frac{1}{2}\lambda_{\max}\{Q_iP_i^{-1}\}\leq \mathrm{Re}(\lambda(A+\lambda_iBK))\leq -\frac{1}{2}\lambda_{\min}\{Q_iP_i^{-1}\},\ i=1,\dots,\widetilde{M}. \tag{17}$$

In fact, if there exist $P_i\in PD(n),\ i=1,\dots,\widetilde{M}$ satisfying (14), (15), with the same procedure as given in Theorem 3.2, we obtain

$$(A+\lambda_iBK)P_i+P_i(A+\lambda_iBK)^T+Q_i=0,\ i=1,\dots,\widetilde{M}. \tag{18}$$

Clearly, we have

$$P_i^{-\frac{1}{2}}(A+\lambda_iBK)P_i^{\frac{1}{2}}+P_i^{\frac{1}{2}}(A+\lambda_iBK)^TP_i^{-\frac{1}{2}}+P_i^{-\frac{1}{2}}Q_iP_i^{-\frac{1}{2}}=0.$$

Obviously,

$$\lambda(P_i^{-\frac{1}{2}}Q_iP_i^{-\frac{1}{2}})=\lambda(P_i^{\frac{1}{2}}P_i^{-\frac{1}{2}}Q_iP_i^{-\frac{1}{2}}P_i^{-\frac{1}{2}})=\lambda(Q_iP_i^{-1}). \tag{19}$$

Let $\alpha^*$ be the eigenvector of $P_i^{-\frac{1}{2}}(A+\lambda_iBK)P_i^{\frac{1}{2}}$ with respect to its eigenvalue $\lambda^*$, that is,

$$P_i^{-\frac{1}{2}}(A+\lambda_iBK)P_i^{\frac{1}{2}}\alpha^*=\lambda^*\alpha^*. \tag{20}$$

It is easy to see

$$\begin{aligned}\lambda_{\min}(-P_i^{-\frac{1}{2}}Q_iP_i^{-\frac{1}{2}})\,(\alpha^*)^T\bar{\alpha^*} &\leq (\lambda^*+\bar{\lambda^*})\,(\alpha^*)^T\bar{\alpha^*}=2\mathrm{Re}(\lambda^*)\,(\alpha^*)^T\bar{\alpha^*}\\ &\leq \lambda_{\max}(-P_i^{-\frac{1}{2}}Q_iP_i^{-\frac{1}{2}})\,(\alpha^*)^T\bar{\alpha^*},\end{aligned}$$

where $\bar{\lambda}^*,\ \bar{\alpha^*}$ are the complex conjugates of $\lambda^*,\ \alpha^*$. Based on (19), we can have

$$-\lambda_{\max}\{Q_iP_i^{-1}\} \le 2\mathrm{Re}(\lambda^*) = 2\mathrm{Re}(\lambda(A+\lambda_i BK)) \le -\lambda_{\min}\{Q_iP_i^{-1}\},$$

which implies (17).

Since $P_j$ satisfies

$$\lambda\{Q_jP_j^{-1}\} \ge -2\frac{\ln\varepsilon}{\tau},\ j=1,\ldots,\widetilde{M},$$

$\mathrm{Re}(\lambda(A+\lambda_i BK)) \le \frac{\ln\varepsilon}{\tau},\ i=1,\ldots,\widetilde{M}$. Set $\delta = \min_{\lambda_i\in\Lambda}\{-\mathrm{Re}(A+\lambda_i BK)\ \} > 0$, then $-\delta \le \frac{\ln\varepsilon}{\tau}$. Set

$$r(t) = (L\otimes I_n)x(t),\ w(t) = (T\otimes I_n)x(t),$$

Denote the Laplacian matrix as $L_k$ during the interval $[t_{k-1}, t_k),\ k=1,2,\ldots,$ for simplicity. $U_k = T_kL_kT_k^{-1}$ is upper triangular with diagonal elements $0,\lambda_2^k,\ldots,\lambda_N^k$. Then

$$r(t) = (L_k\otimes I_n)x(t),\ w(t) = (T_k\otimes I_n)x(t),$$

$$\widetilde{C}_k = (T_k\otimes I_n)[I_N\otimes A + (I_N\otimes BK)\,(L_k\otimes I_n)](T_k^{-1}\otimes I_n) = I_N\otimes A + U_k\otimes BK,$$

where $\widetilde{C}_k$ is upper triangular with diagonal blocks $0, A+\lambda_2^k BK,\ldots,A+\lambda_N^k BK$. Since

$$\dot{w}(t) = (T_k\otimes I_n)\dot{x}(t) = \widetilde{C}_k w(t),$$

$$w(t_k) = \exp(\widetilde{C}_k(t_k - t_{k-1}))w(t_{k-1}).$$

Moreover,

$$\begin{aligned} r(t_k) &= (L_k\otimes I_n)x(t_k) = (L_k\otimes I_n)\,(T_k^{-1}\otimes I_n)w(t_k) \\ &= (T_k^{-1}\otimes I_n)\,(U_k\otimes I_n)\exp(\widetilde{C}_k(t_k-t_{k-1}))w(t_{k-1}). \end{aligned}$$

Since

$$\begin{aligned} (U_k\otimes I_n)\widetilde{C}_k &= (U_k\otimes I_n)(I_N\otimes A + U_k\otimes BK) \\ &= U_k\otimes A + U_k^2\otimes BK = (I_N\otimes A + U_k\otimes BK)\,(U_k\otimes I_n) \\ &= \widetilde{C}_k(U_k\otimes I_n), \end{aligned}$$

$U_k\otimes I_n,\ \exp(\widetilde{C}_k(t_k-t_{k-1}))$ can commute. Then

$$\begin{aligned} r(t_k) &= (T_k^{-1}\otimes I_n)\exp(\widetilde{C}_k(t_k-t_{k-1}))(U_k\otimes I_n)w(t_{k-1}) \\ &= (T_k^{-1}\otimes I_n)\exp(\widetilde{C}_k(t_k-t_{k-1}))\,(T_k\otimes I_n)r(t_{k-1}). \end{aligned}$$

Since the eigenvalues of $\widetilde{C}_k$ are the same with those of $A+\lambda_i^k BK,\ \lambda_i^k\in\Lambda$, we can have $\mathrm{Re}(\lambda(\widetilde{C}_k)) \le -\delta$. It follows from [16] that

$$\|\exp(\widetilde{C}_k(t_k-t_{k-1}))\| \le \|\exp(-\delta(t_k-t_{k-1}))\| \le \exp(-\delta\tau) \le \varepsilon.$$

As a result,

$$\|r(t_k)\| \leq \nu\varepsilon\|r(t_{k-1})\| = \frac{1}{2}\|r(t_{k-1})\|,$$

where $\nu$, $\varepsilon$ are defined in (16). Thus, $\|r(t_k)\| \to 0$, as $k \to \infty$. Since the digraphs have spanning trees, according to Lemma 2.1 that $x(t_k) \to \Omega$, as $k \to \infty$. Therefore, $x(t) \to \Omega$ as $t \to \infty$, which implies the consensus is achieved for system (3). $\square$

## 5. CONCLUSIONS

The parametrization of coordination controllers is important in theoretical analysis and practical applications. In this paper, we studied the parametrization problem of multi-agent systems in a general linear form. The geometric structures of the coordination controllers for multi-agent systems were obtained with fixed topologies. Some necessary and sufficient conditions were found to characterize the consensus controllers. Also, the multi-agent consensus with switching interconnection topologies was considered with help of controller parametrization. In future, the parametrization and geometric analysis of coordination controllers for multi-agent systems with heterogeneous dynamic systems will be considered.

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