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THE RISK–SENSITIVE POISSON EQUATION FOR A COMMUNICATING MARKOV CHAIN ON A DENUMERABLE STATE SPACE

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Dedicated to Professor O. Hernández-Lerma, on the occasion of his sixtieth birthday.

This work concerns a discrete-time Markov chain with time-invariant transition mechanism and denumerable state space, which is endowed with a nonnegative cost function with finite support. The performance of the chain is measured by the (long-run) risk-sensitive average cost and, assuming that the state space is communicating, the existence of a solution to the risk-sensitive Poisson equation is established, a result that holds even for transient chains. Also, a sufficient criterion ensuring that the functional part of a solution is uniquely determined up to an additive constant is provided, and an example is given to show that the uniqueness result may fail when that criterion is not satisfied.

Keywords: possibly transient Markov chains, discounted approach, first return time, uniqueness of solutions to the multiplicative Poisson equation

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1. INTRODUCTION

This note concerns a discrete-time Markov chain $\{X_n\}$ evolving on a denumerable state space S in accordance with a time-invariant transition matrix $P = [p_{x\,y}]$. The system is endowed with a cost function $C\colon S \to [0,\infty)$, so that a cost $C(X_t)$ is incurred at each time $t = 1, 2, 3, \ldots$ and, assuming for the sake of simplicity that the observer of the chain has unitary risk aversion coefficient[1], the overall performance of the system when the initial state is $X_0 = x$ is measured by the (long-run) risk-sensitive average cost $J_C(x)$, which is given by

$$J_C(x) = \limsup_{n\to\infty} \frac{1}{n} J_{C,n}(x) \tag{1.1}$$

where, letting $E_x[\cdot]$ be the expectation operator given $X_0 = x$,

$$J_{C,n}(x) = \log\left(E_x\left[e^{\sum_{t=0}^{n-1} C(X_t)}\right]\right) \tag{1.2}$$

[1] it is not difficult to see that the same analysis can be performed for any positive value of the risk aversion coefficient

is the risk-sensitive (expected) total cost incurred before time n. The characterization of the average cost function $J_C(\cdot)$ is usually based on the following Poisson equation associated with C:

$$e^{g+h(x)} = e^{C(x)} \sum_{y \in S} p_{x,y} e^{h(y)}, \quad x \in S, \tag{1.3}$$

where g is a real number and $h(\cdot)$ is a real-valued function defined on S. Under appropriate conditions on the function $h(\cdot)$, which will be precisely stated in the following section, if (1.3) holds then $J_C(\cdot) = g$; in particular, this occurs when h is a bounded function ([11, 13] and [21]).

The *main objective* of the paper can be stated as follows:

To establish the existence of a pair $(g, h(\cdot))$ satisfying the Poisson equation (1.3) as well as the suitable verification criterion to ensure that $J_C(\cdot) = g$.

This problem is analyzed under the following two structural conditions (a) and (b):

(a) the cost function C has finite support, and

(b) the state space is communicating, in that if $x, y \in S$ are arbitrary, then with positive probability the system visits state y when the initial state is x; notice that, under this requirement, $\{X_n\}$ may be a *transient* chain.

Within this framework, the main results of this work can be described as follows:

(i) It is proved that the Poisson equation (1.3) admits a solution satisfying the verification criterion to ensure that $J_C(\cdot) = g$; see Theorem 2.1 below.

(ii) For a transient chain $\{X_n\}$ it shown that $J_C(\cdot)$ may be positive, a fact that signals a deep difference between the risk-sensitive and risk-neutral average criteria since, for a transient system, this latter index is always null when the cost function $C(\cdot)$ has finite support.

(iii) A sufficient condition is formulated so that if $(g, h(\cdot))$ satisfies (1.3) as well as the verification condition in Lemma 2.1 below, then the function h is uniquely determined up to an additive constant.

The study of stochastic systems endowed with the risk-sensitive criterion (1.1) can be traced back, at least, to the seminal papers by Howard and Matheson [13], Jacobson [14] and Jaquette [15, 16]. Particularly, in [13] controlled Markov chains with *finite* state and action spaces were considered and, assuming that the system is communicating, the existence of a pair $(g, h(\cdot))$ satisfying an optimality equation similar to (1.3) was established. The approach in that paper is based on the Perron–Frobenius theory of nonnegative matrices, and it follows that when (1.3) holds, then e^g is the largest eigenvalue of the matrix $[e^{C(x)} p_{x\,y}]$ and $(e^{h(x)}, x \in S)$ is a corresponding eigenvector, so that $h(\cdot)$ is unique up to an additive constant (Seneta [20]). Extending these ideas, the value iteration approximation method and the policy improvement algorithm were studied in Sladký and Montes-de-Oca [22]) and Sladký [21]). A different approach to the existence of solutions for the Poisson equation on a finite state space was presented in Cavazos-Cadena and Fernández-Gaucherand [4], were the analysis is based on the risk-sensitive total cost criterion. Recently, there has been an intensive work on (controlled) stochastic system endowed with the risk-sensitive average criterion; see, for instance, Flemming and McEneany [10], Di Masi and Stettner [9], Cavazos-Cadena and Hernández-Hernández [5], Jaśkiewicz [17] and the references there in. On the other hand, a pair $(g, h(\cdot))$ satisfying (1.3) may not exist even under strong recurrency conditions, as the Doeblin condition (Cavazos-Cadena and Fernández-Gaucherand [8]), a fact that establishes a deep difference with the risk-neutral average criterion, which is constant and is determined via a risk-neutral Poisson equation under diverse variants of the Doeblin condition (Arapostathis et al. [1]); for other important differences between the risk-sensitive and risk-neutral indexes see Brau-Rojas et al. [2]. Finally, necessary and sufficient criteria for the solvability of the above Poisson equation when the state space is finite are given in Cavazos-Cadena and Hernández-Hernández [7] and in Sladký [21], dealing with the uncontrolled and controlled cases, respectively.

The approach of this work relies on a family $\{T_\alpha \mid \alpha \in (0,1)\}$ of contractive (discounted) operators whose fixed points $\{V_\alpha\}$ allow to obtain approximate solutions to (1.3); this classical idea has been widely used to study (controlled) Markov chains with the risk-neutral average criterion (Hernández-Lerma [12], Araposthatis et al. [1], Puterman[19]), and for the risk-sensitive criterion (1.1) similar ideas have been recently employed, for instance, in Cavazos-Cadena and Hernández-Hernández [5] and Cavazos-Cadena [6] to analyze models with finite state space, and in Jaśkiewicz [17] to study systems on Borel spaces.

The organization of the paper is as follows: First, in Section 2 the necessary and sufficient criterion to ensure that if (1.3) holds then g is the average cost at each state x is formally established in Lemma 2.1, and the main result on the solvability of the Poisson equation is stated as Theorem 2.1. Next, in Section 3 the family $\{T_\alpha\}$ of contractive operators on the space of bounded functions on S is introduced, the existence and location of maximizers of the corresponding fixed points V_α are analyzed, and the results in this direction are used in Section 4 to prove the main theorem. Then, in Section 5 the case of a transient Markov chain is studied and it is shown that, under the assumptions in the paper, the average cost $J_C(\cdot)$ may be positive, establishing an interesting contrast with the risk-neutral average index. Finally, in Section 6 an explicit example is given to shown that, if $(g, h(\cdot))$ is as in (1.3) and the criterion in Lemma 2.1 is satisfied then, the function $h(\cdot)$ is not generally determined in a unique way up to an additive constant; after this example, a new stochastic matrix Q on the state space S is introduced, and a criterion to ensure the uniqueness of $h(\cdot)$ modulo an additive constant is given in terms of the matrix Q.

Notation. Throughout the remainder the state space S is endowed with the discrete topology and $\mathcal{B}(S^\infty)$ stands for the Borel σ-field of the Cartesian product $S \times S \times S \times \cdots =: S^\infty$; the distribution of the Markov chain $\{X_n\}$ when $X_0 = x$ is denoted by $P_x[\cdot]$ and, without explicit reference, all relations involving conditional expectations are assumed to hold almost surely with respect to the underlying measure. On the other hand, $B(S)$ denotes the space of all real-valued and bounded functions defined on S, that is, $D\colon S \to \mathbb{R}$ belongs to S if and only if $\|D\| < \infty$, where

$$\|D\| := \sup_{x\in S} |D(x)|$$

is the supremum norm of $D(\cdot)$. Finally, for an event A the corresponding indicator (Bernoulli) variable corresponding to A is denoted by $I[A]$.

2. VERIFICATION CRITERION AND MAIN RESULT

In this section the result concerning the existence of solutions of the Poisson equation (1.3) is stated as Theorem 2.1 below. To begin with, it is convenient to discuss the verification criterion ensuring that, if (1.3) holds, then g is the average cost at each state x. Given $x \in S$ and $n = 1, 2, 3, \ldots$, define the probability measure $\nu_{C,x,n}$ on $\mathcal{B}(S^\infty)$ as follows: For each $A \in \mathcal{B}(S^\infty)$,

$$\nu_{C,x,n}(A) = \frac{1}{E_x\left[e^{\sum_{t=0}^{n-1} C(X_t)}\right]} E_x\left[e^{\sum_{t=0}^{n-1} C(X_t)} I[(X_1, X_2, X_3, \ldots) \in A]\right]. \tag{2.1}$$

Now, let $E_{\nu_{C,x,n}}[\cdot]$ be the expectation operator associated with this measure, and notice if (1.3) holds then an induction argument yields that

$$\begin{aligned} e^{ng+h(x)} &= E_x\left[e^{\sum_{t=0}^{n-1} C(X_t)+h(X_n)}\right] \\ &= E_x\left[e^{\sum_{t=0}^{n-1} C(X_t)}\right] E_{\nu_{C,x,n}}\left[e^{h(X_n)}\right] \\ &= e^{J_{C,n}(x)} E_{\nu_{C,x,n}}\left[e^{h(X_n)}\right], \quad x \in S, \quad n = 1, 2, 3, \ldots, \end{aligned} \tag{2.2}$$

where (1.2) was used in the last step. Thus, the equality

$$g + \frac{h(x)}{n} = \frac{1}{n} J_{C,n}(x) + \frac{1}{n} \log\left(E_{\nu_{C,x,n}}\left[e^{h(X_n)}\right]\right)$$

always holds, a fact that combined with the specification of $J_C(\cdot)$ in (1.1) immediately leads to the following conclusion.

Lemma 2.1. [Verification] Assume that $g \in \mathbb{R}$ and $h\colon S \to \mathbb{R}$ are such that (1.3) holds. In this case the equality $g = J_C(\cdot)$ is valid if and only if

$$\liminf_{n\to\infty} \left(E_{\nu_{C,x,n}}\left[e^{h(X_n)}\right]\right)^{1/n} = 1, \quad x \in S, \tag{2.3}$$

where $\nu_{C,x,n}$ is the measure defied in (2.1).

For the sake of future reference, the structural assumptions described in Section 1 are formally stated below.

Assumption 2.1. The transition matrix $P = [p_{x,y}]_{x\, y\in S}$ is communicating, i. e., for each $x, y \in S$ there exists a positive integer $n = n(x, y)$ such that $P_x[X_n = y] > 0$.

Assumption 2.2. (i) The cost function C is nonnegative, and

(ii) C has finite support, that is, there exists a finite set $K \subset S$ such that

$$C(x) = 0, \quad x \in S \setminus K. \tag{2.4}$$

The following is the main result of this work.

Theorem 2.1. Under Assumptions 2.1 and 2.2, there exist $g \in \mathbb{R}$ and $h\colon S \to (-\infty, 0]$ such that the following assertions (i) and (ii) are valid:

(i) The Poisson equation (1.3) as well as the criterion (2.3) are satisfied, so that $g = J_C(\cdot)$;

Moreover,

(ii) The limit superior in (1.1) can be replaced by limit, that is, for each $x \in S$,

$$g = \lim_{n\to\infty} \frac{1}{n} J_{C,n}(x),$$

and then

$$\lim_{n\to\infty} \left(E_{\nu_{C,x,n}} \left[e^{h(X_n)} \right] \right)^{1/n} = 1.$$

The proof of this result will be presented after the preliminaries established in the following section.

3. DISCOUNTED APPROACH

The technical tools that will be used in the proof of Theorem 2.1 are collected in this section. For each $\alpha \in (0,1)$ define the operator $T_\alpha\colon B(S) \to B(S)$ by

$$T_\alpha[W](x)\colon = \log\left(E_x \left[e^{C(X_0)+\alpha W(X_1)} \right] \right), \quad W \in B(S), \quad x \in S. \tag{3.1}$$

In this case it is not difficult to see that T_α satisfies the following monotonicity and contractive properties (Cavazos-Cadena [6]): For each $W, V \in B(S)$,

$$T_\alpha[W] \geq T_\alpha[V] \quad \text{if } W \geq V, \tag{3.2}$$

and

$$\|T_\alpha[W] - T_\alpha[V]\| \leq \alpha \|W - V\|. \tag{3.3}$$

Since $B(S)$ endowed with the supremum norm is a Banach space, the contraction property yields that there exists a unique function $V_\alpha \in B(S)$ satisfying $T_\alpha[V_\alpha] = V_\alpha$, that is,

$$e^{V_\alpha(x)} = E_x \left[e^{C(X_0)+\alpha V_\alpha(X_1)} \right] = e^{C(x)} \sum_{y\in S} p_{x\,y} e^{\alpha V_\alpha(y)}, \quad x \in S. \tag{3.4}$$

Moreover, setting $T_\alpha^n[W] = T_\alpha(T_\alpha^{n-1}[W])$ for $n \geq 2$, it is not difficult to see that (3.3) yields that $\|V_\alpha - T_\alpha^n[0]\| = \|T_\alpha^n[V_\alpha] - T_\alpha^n[0]\| \leq \alpha^n \|V_\alpha\| \to 0$ as $n \to \infty$; since $T_\alpha[0] = C \geq 0$ (see (3.1) and Assumption 2.2), from the monotonicity property (3.2) it follows that $T_\alpha^n[0] \geq 0$ for every n, so that

$$0 \leq V_\alpha. \tag{3.5}$$

Observe now that (3.3) and the equalities $T_\alpha[V_\alpha] = V_\alpha$ and $T_\alpha[0] = C$ together yield that

$$\begin{aligned}\|V_\alpha\| - \|C\| &\leq \|V_\alpha - C\| \\ &\leq \|T_\alpha[V_\alpha] - T_\alpha[0]\| \leq \alpha\|V_\alpha - 0\| = \alpha\|V_\alpha\|\end{aligned}$$

and then

$$\|(1-\alpha)V_\alpha\| \leq \|C\|. \tag{3.6}$$

In the remainder of the section the existence and location of a maximizer x_α of the function V_α, as well as the limit behavior of $V_\alpha(\cdot) - V_\alpha(x_\alpha)$, are analyzed. The argument involves the simple properties in (3.5) and (3.6), and uses the idea of first return time to a subset of S, which is now introduced.

Definition 3.1. If $F \subset S$, the first return time to set F is defined by

$$T_F := \min\{n \geq 1 \,|\, X_n \in F\}$$

where, by convention, the minimum of the empty set is ∞; if $F = \{z\} \subset S$ is a singleton,

$$T_z \equiv T_{\{z\}}.$$

The maximization of the function $V_\alpha(\cdot)$ is studied in the following lemma.

Lemma 3.1. Suppose that Assumptions 2.1 and 2.2 hold, and let the finite set $K \subset S$ be such that (2.4) holds. In this case, for each $\alpha \in (0,1)$, the function $V_\alpha \in B(S)$ in (3.4) attains its maximum at a point in K, that is, there exists a state x_α satisfying

$$x_\alpha \in K \quad \text{and} \quad V_\alpha(x) \leq V_\alpha(x_\alpha), \quad x \in S. \tag{3.7}$$

Proof. Given $\alpha \in (0,1)$ it will be shown that

$$\begin{aligned}&\text{For each } n = 1,2,3,\ldots \quad \text{and} \quad x \in S \setminus K \\ &e^{V_\alpha(x)} \leq \sum_{r=1}^{n} E_x\left[e^{\alpha V_\alpha(X_{T_K})} I[T_K = r]\right] + E_x\left[e^{\alpha V_\alpha(X_n)} I[T_K > n]\right].\end{aligned} \tag{3.8}$$

Assuming that this relation holds, the conclusion can be achieved following the steps (i) – (iv) below:

(i) Since K is finite, there exists a state x_α such that

$$x_\alpha \in K \quad \text{and} \quad V_\alpha(x_\alpha) \geq V_\alpha(x) \quad \text{for all } x \in K. \tag{3.9}$$

(ii) Observing that $X_{T_K} \in K$ when $T_K < \infty$ (see Definition 3.1), from (3.8) and step (i) above it follows that, for every positive integer n,

$$e^{V_\alpha(x)} \leq e^{\alpha V_\alpha(x_\alpha)} P_x[T_K \leq n] + E_x\left[e^{\alpha V_\alpha(X_n)} I[T_K > n]\right], \quad x \in S \setminus K. \tag{3.10}$$

Now set

$$\sigma \colon = \sup_{x \in S\setminus K} V_\alpha(x). \tag{3.11}$$

(iii) Since $\sigma \geq 0$, by (3.5), from the inclusion $\alpha \in (0,1)$ it follows that there exists a point $x^* \in S \setminus K$ such that $V_\alpha(x^*) \geq \alpha\sigma$. Also, observing that $X_n \in S \setminus K$ on the event $[T_K > n]$, inequality (3.10) with x^* instead of x yields that, for every positive integer n, $e^{\alpha\sigma} \leq e^{\alpha V_\alpha(x_\alpha)} P_{x^*}[T_K \leq n] + e^{\alpha\sigma} P_{x^*}[T_K > n]$, that is,

$$e^{\alpha\sigma} P_{x^*}[T_K \leq n] \leq e^{\alpha V_\alpha(x_\alpha)} P_{x^*}[T_K \leq n].$$

(iv) By Assumption 2.1 there exists an integer $n^* > 0$ such that $P_{x^*}[X_{n^*} = x_\alpha] > 0$. Since $x_\alpha \in K$ it follows that $P_{x^*}[T_K \leq n^*] \geq P_{x^*}[X_{n^*} \in K] \geq P_{x^*}[X_{n^*} = x_\alpha] > 0$, and then the above display with n^* instead of n yields that $e^{\alpha\sigma} \leq e^{\alpha V_\alpha(x_\alpha)}$, that is,

$$\sigma \leq V_\alpha(x_\alpha).$$

Combining this inequality with (3.9) and (3.11) it follows that x_α satisfies the desired conclusion (3.7). To complete the argument, (3.8) will be proved by induction. Given $x \in S \setminus K$, so that $C(x) = 0$, by (2.4), from (3.4) and Definition 3.1 it follows that

$$\begin{aligned} e^{V_\alpha(x)} &= E_x\left[e^{\alpha V_\alpha(X_1)}\right] \\ &= E_x\left[e^{\alpha V_\alpha(X_{T_K})} I[T_K = 1]\right] + E_x\left[e^{\alpha V_\alpha(X_1)} I[T_K > 1]\right] \end{aligned}$$

establishing the case $n = 1$ of (3.8). Assume now that (3.8) holds for $n = m$. Using that $V_\alpha(\cdot) \geq 0$ and $\alpha \in (0,1)$, equation (3.4) implies that

$$\begin{aligned} e^{\alpha V_\alpha(X_m)} I[T_K > m] &\leq e^{V_\alpha(X_m)} I[T_K > m] \\ &= I[T_K > m] e^{C(X_m)} \sum_{y \in S} p_{X_m\, y} e^{\alpha V_\alpha(y)} \\ &= I[T_K > m] e^{C(X_m)} E_x\left[e^{\alpha V_\alpha(X_{m+1})} \,\middle|\, X_0, \ldots, X_m\right] \\ &= I[T_K > m] E_x\left[e^{\alpha V_\alpha(X_{m+1})} \,\middle|\, X_0, \ldots, X_m\right] \\ &= E_x\left[e^{\alpha V_\alpha(X_{m+1})} I[T_K > m] \,\middle|\, X_0, \ldots, X_m\right] \end{aligned}$$

where the second equality is due to the Markov property, the third one used that on the event $[T_K > m]$ the inclusion $X_m \in S \setminus K$ holds, so that $C(X_m) = 0$, and the equality $I[T_K > m] = I[X_i \in S \setminus K, i = 1, 2, \ldots, m]$ was used to move the indicator function into the conditional expectation in the last step. Therefore,

$$\begin{aligned} &E_x\left[e^{\alpha V_\alpha(X_m)} I[T_K > m]\right] \\ &\quad \le E_x\left[e^{\alpha V_\alpha(X_{m+1})} I[T_K > m]\right] \\ &\quad = E_x\left[e^{\alpha V_\alpha(X_{m+1})} I[T_K = m+1]\right] + E_x\left[e^{\alpha V_\alpha(X_{m+1})} I[T_K > m+1]\right], \end{aligned}$$

and combining this relation with the case $n = m$ of (3.8), which is valid by the induction hypothesis, it follows that (3.8) is also satisfied for $n = m+1$, completing the proof. □

Before going any further it is convenient to introduce the following notation.

Definition 3.2. Suppose that Assumptions 2.1 and 2.2 hold, and for each $\alpha \in (0,1)$ let $x_\alpha \in K$ be a maximizer of $V_\alpha(\cdot)$. In this case, $g_\alpha \in \mathbb{R}$ and $h_\alpha \colon S \to \mathbb{R}$ are given by

$$g_\alpha \colon = (1-\alpha) V_\alpha(x_\alpha), \qquad h_\alpha(x) \colon = V_\alpha(x) - V_\alpha(x_\alpha), \quad x \in S.$$

Notice that (3.5), (3.6) and (3.7) together yield

$$0 \le g_\alpha \le \|C\|, \qquad -\frac{\|C\|}{1-\alpha} \le h_\alpha(\cdot) \le 0, \quad \alpha \in (0,1), \tag{3.12}$$

whereas multiplying both sides of (3.4) by $e^{-\alpha V_\alpha(x_\alpha)}$ direct rearrangements lead to

$$e^{g_\alpha + h_\alpha(x)} = e^{C(x)} \sum_{y \in S} p_{x\,y} e^{\alpha h_\alpha(y)}, \quad x \in S. \tag{3.13}$$

Lemma 3.2. Suppose that Assumptions 2.1 and 2.2 hold. In this case

$$\liminf_{\alpha \nearrow 1} h_\alpha(x) > -\infty, \quad x \in S.$$

P r o o f. The argument is by contradiction. Suppose that

$$\liminf_{\alpha \nearrow 1} h_\alpha(x_0) = -\infty \quad \text{for some } x_0 \in S. \tag{3.14}$$

In this case there exists a sequence $\{\alpha_k\} \subset (0,1)$ such that

$$\alpha_k \nearrow 1 \quad \text{and} \quad \liminf_{k \to \infty} h_{\alpha_k}(x_0) = -\infty. \tag{3.15}$$

Since the set K is finite and the inclusion $x_{\alpha_k} \in K$ always holds, taking a subsequence if necessary it can be assumed that

$$x_{\alpha_k} = x^* \in K, \quad k = 1, 2, 3, \ldots, \tag{3.16}$$

so that

$$h_{\alpha_k}(x^*) = 0, \quad k = 1, 2, 3, \ldots; \tag{3.17}$$

see Definition 3.2. Now, define the set $\mathcal{L} \subset S$ by

$$\mathcal{L} \colon = \{x \in S \mid \liminf_{k\to\infty} h_{\alpha_k}(x) = -\infty\},$$

and notice that $x_0 \in \mathcal{L}$, by (3.15). It will be shown that $\mathcal{L}$ is P-closed, i. e.,

$$x \in \mathcal{L} \quad \text{and} \quad p_{x\,y} > 0 \implies y \in \mathcal{L}. \tag{3.18}$$

To establish this fact notice that the first relation in (3.12) and (3.13) together yield that the inequalities $e^{2\|C\|+h_{\alpha_k}(x)} \geq \sum_{w\in S} p_{x\,w} e^{\alpha_k h_{\alpha_k}(w)} \geq p_{x\,y} e^{\alpha_k h_{\alpha_k}(y)}$ always hold and, recalling that $\alpha_k \nearrow 1$, it follows that for every $x, y \in S$

$$e^{2\|C\|+\liminf_{k\to\infty} h_{\alpha_k}(x)} \geq p_{x\,y} e^{\liminf_{k\to\infty} h_{\alpha_k}(y)}.$$

Now let $x \in \mathcal{L}$ be arbitrary. In this case the left-hand side of this inequality is null and it follows that $p_{x\,y} e^{\liminf_{k\to\infty} h_{\alpha_k}(y)} = 0$, so that $p_{x\,y} > 0$ leads to $e^{\liminf_{k\to\infty} h_{\alpha_k}(y)} = 0$, that is, $\liminf_{k\to\infty} h_{\alpha_k}(y) = -\infty$, and then $y \in \mathcal{L}$. This establishes (3.18) and an induction argument allows to obtain

$$x \in \mathcal{L} \quad \text{and} \quad P_x[X_n = y] > 0 \quad \text{for some integer } n \implies y \in \mathcal{L}.$$

Recall now that $x_0 \in \mathcal{L}$ and let $x^* \in K$ be as in (3.16). By Assumption 2.1, there exists n^* such that $P_{x_0}[X_{n^*} = x^*] > 0$ and then the above display yields that $x^* \in \mathcal{L}$, i. e., $\liminf_{k\to\infty} h_{\alpha_k}(x^*) = -\infty$, contradicting (3.17). Thus, the starting point in this argument, namely, assertion (3.14), does not hold and it follows that $\liminf_{\alpha\nearrow 1} h_\alpha(x) > -\infty$ for every $x \in S$. □

4. PROOF OF THE MAIN RESULT

In the section the above preliminary results will be used to establish the main conclusion of this note, namely, Theorem 2.1.

Proof. Let $\{\alpha_k\} \subset (0, 1)$ be a fixed sequence increasing to 1, and let g_{α_k} and $h_{\alpha_k}(\cdot)$ be as in Definition 3.2. Combining (3.12) with Lemma 3.2 it follows that there exists a finite function $L \colon S \to (-\infty, 0]$ such that

$$(g_{\alpha_k}; h_{\alpha_k}(x), x \in S) \in [0, \|C\|] \times \prod_{x\in S} [L(x), 0];$$

since the right-hand side of this inclusion is a compact metric space, after taking a subsequence if necessary it can be assumed that the following limits exist:

$$\lim_{k\to\infty} g_{\alpha_k} =: g \in [0, \|C\|], \qquad \lim_{k\to\infty} h_{\alpha_k}(x) =: h(x) \in [L(x), 0], \quad x \in S. \tag{4.1}$$

It will be shown that the desired conclusions are satisfied by g and $h(\cdot)$ in this display.

(i) Since $\alpha_k \nearrow 1$ and $h_{\alpha_k}(\cdot) \le 0$, (4.1) and the bounded convergence theorem together yield that

$$\lim_{k\to\infty} \sum_{y\in S} p_{x\,y} e^{\alpha_k h_{\alpha_k}(y)} = \sum_{y\in S} p_{x\,y} e^{h(y)}.$$

Replacing α by α_k in (3.13) and taking limit as k goes to ∞ in both sides of the resulting equality, the above display and (4.1) together imply that the Poisson equation (1.3) holds.

(ii) Let $\alpha \in (0,1)$ be arbitrary but fixed, and recall that the (bounded) function $h_\alpha(\cdot)$ is non positive, so that $\alpha h_\alpha(\cdot) \ge h_\alpha(\cdot)$, an inequality that combined with (3.13) yields that, for each state $x \in S$, $e^{g_\alpha + h_\alpha(x)} \ge e^{C(x)} \sum_{y\in S} p_{x\,y} e^{h_\alpha(y)} = E_x\left[e^{C(X_0)+h_\alpha(X_1)}\right]$. From this point, an induction argument using the Markov property allows to obtain that, for each $n = 1, 2, 3, \ldots$ and $x \in S$

$$\begin{aligned} e^{ng_\alpha + h_\alpha(x)} &\ge E_x\left[e^{\sum_{t=0}^{n-1} C(X_t) + h_\alpha(X_n)}\right] \\ &\ge E_x\left[e^{\sum_{t=0}^{n-1} C(X_t)}\right] e^{-\|h_\alpha\|} \\ &\ge e^{J_{C,n}(x) - \|h_\alpha\|}; \end{aligned}$$

see (1.2). It follows that $g_\alpha + (h_\alpha(x) + \|h_\alpha\|)/n \ge J_{C,n}(x)/n$ so that, for each sate x,

$$g_\alpha \ge \limsup_{n\to\infty} \frac{1}{n} J_{C,n}(x).$$

Since this inequality holds for each $\alpha \in (0,1)$, via the first convergence in (4.1) it follows that

$$g \ge \limsup_{n\to\infty} \frac{1}{n} J_{C,n}(x), \quad x \in S. \tag{4.2}$$

On the other hand, since the Poisson equation (1.3) holds, an induction argument yields that for every integer $n > 0$ and $x \in S$

$$e^{ng + h(x)} = E_x\left[e^{\sum_{t=0}^{n-1} C(X_t)} e^{h(X_n)}\right] \le E_x\left[e^{\sum_{t=0}^{n-1} C(X_t)}\right] = e^{J_{C,n}(x)},$$

where the inequality is due to the relation $h(\cdot) \le 0$, and (1.2) was used in the last step. Thus, $g + h(x)/n \le J_{C,n}(x)/n$ and then

$$g \le \liminf_{n\to\infty} \frac{1}{n} J_{C,n}(x), \quad x \in S;$$

via (4.2), it follows that $g = \lim_{n\to\infty} \frac{1}{n} J_{C,n}(x)$ for every state x, a fact that, using the factorization equality (2.2), is equivalent to $\lim_{n\to\infty} \left(E_{\nu_{C,x,n}}\left[e^{h(X_n)}\right]\right)^{1/n} = 1$, completing the proof. □

5. THE TRANSIENT CASE

In this section Theorem 2.1 will be used to point out an interesting contrast between the risk-sensitive index (1.1), and the risk-neutral average cost criterion which, for $C \in B(S)$, is given by

$$\tilde{J}_C(x) = \limsup_{n\to\infty} \frac{1}{n} E_x\left[\sum_{n=1}^{n-1} C(X_t)\right], \quad x \in S.$$

Suppose that Assumption 2.1 holds and that the transition matrix $P = [p_{x\,y}]$ is *transient*, that is,

$$P_z[T_z < \infty] < 1 \tag{5.1}$$

for some (and hence, for all) $z \in S$. In this context it is known that Assumption 2.1 yields

$$\frac{1}{n} E_x\left[\sum_{t=1}^{n-1} I[X_t = y]\right] \to 0 \quad \text{as } n \to \infty,$$

for every $x, y \in S$, (Loève [18]), so that $\tilde{J}_C(\cdot) = 0$ when C has finite support. In contrast, it will be shown in the second part of the following theorem that the risk-sensitive average cost $J_C(\cdot)$ may be *positive* under Assumptions 2.1 and 2.2 even if the transience condition (5.1) holds. Given $z \in S$ and $a \geq 0$ set

$$\begin{aligned} C_{z\,a}(x) &= 0 \quad \text{if } x \neq z \\ &= a, \quad \text{if } x = z. \end{aligned} \tag{5.2}$$

Theorem 5.1. Let $z \in S$ and $a > 0$ be arbitrary but fixed, and suppose that Assumption 2.1 and the transience condition (5.1) hold.

(i) If $a \in (0, -\log(P_z[T_z < \infty])$ set

$$L\colon = -\log\left(\frac{e^{-a} - P_z[T_z < \infty]}{P_z[T_z = \infty]}\right) \tag{5.3}$$

and define $h\colon S \to \mathbb{R}$ by

$$\begin{aligned} h(x) \;&:=\; 0 \quad \text{if } x = z \\ &:=\; \log(P_x[T_z < \infty] + e^{-L} P_x[T_z = \infty]) \quad \text{if } x \neq z. \end{aligned} \tag{5.4}$$

In this case assertions (a) – (c) below hold:

(a) $L \in (0, \infty)$ and $h(x) \in (-L, 0]$ for every $x \in S$.

(b) $h(x) < 0$ for some $x \neq z$;

(c) The pair $(0, h(\cdot))$ satisfies the Poisson equation (1.3) associated with the cost function $C_{z\,a}$:

$$e^{h(x)} = e^{Cz\,a(x)} \sum_{y\in S} p_{x\,y} e^{h(y)}, \quad x \in S. \tag{5.5}$$

Consequently,

(ii) $J_{C_{z\,a}}(\cdot) > 0$ if and only if $a > -\log(P_z[T_z < \infty])$.

The proof of this result uses the following lemma

Lemma 5.1. For each $D \in B(S)$ and $x \in S$ let $J_D(x)$ be the risk-sensitive average cost at x corresponding to D, which is obtained replacing C by D in (1.1) and (1.2). In this case, the mapping $D \mapsto J_D(\cdot)$ is convex, that is, for each $D, D_1 \in B(S)$ and $\beta \in (0,1)$,

$$J_{\beta D+(1-\beta)D_1}(x) \le \beta J_D(x) + (1-\beta)J_{D_1}(x), \quad x \in S.$$

Proof. Notice that for each positive integer n and $x \in S$

$$\begin{aligned} e^{J_{\beta D+(1-\beta)D_1,n}(x)} &= E_x\left[e^{\sum_{t=0}^{n-1}(\beta D(X_t)+(1-\beta)D_1(X_t))}\right] \\ &= E_x\left[e^{\beta\sum_{t=0}^{n-1}D(X_t)}e^{(1-\beta)\sum_{t=0}^{n-1}D_1(X_t))}\right] \end{aligned}$$

and an application of Hölder's inequality yields

$$\begin{aligned} e^{J_{\beta D+(1-\beta)D_1,n}(x)} &\le \left(E_x\left[e^{\sum_{t=0}^{n-1}D(X_t)}\right]\right)^{\beta}\left(E_x\left[e^{\sum_{t=0}^{n-1}D_1(X_t))}\right]\right)^{(1-\beta)} \\ &= e^{\beta J_{D,n}(x)}e^{(1-\beta_1)J_{D_1,n}(x)}. \end{aligned}$$

Therefore,

$$\frac{1}{n}J_{\beta D+(1-\beta)D_1,n}(x) \le \beta\frac{1}{n}J_{D,n}(x) + (1-\beta)\frac{1}{n}J_{D_1,n}(x)$$

and the conclusion follows taking the limit superior as $n \to \infty$. □

The following argument establishes Theorem 5.1.

Proof. (i) Let $a \in (0, -\log(P_z[T_z < \infty]))$ be arbitrary but fixed. In this case $e^a P_z[T_z < \infty] < 1$, so that

$$\frac{e^{-a} - P_z[T_z < \infty]}{P_z[T_z = \infty]} = \frac{1 - e^a P_z[T_z < \infty]}{e^a(1 - P_z[T_z < \infty])} \in (0,1)$$

and it follows that L in (5.3) is well-defined and is a positive finite number. On the other hand, using that $P_x[T_z < \infty] > 0$ always holds, by Assumption 2.1, it follows that

$$e^{-L} = e^{-L}P_x[T_z < \infty] + e^{-L}P_x[T_z = \infty] < P_x[T_z < \infty] + e^{-L}P_x[T_z = \infty] \le 1$$

and then the specification of $h(\cdot)$ yields that $h(\cdot) \in (-L, 0]$, establishing the part (a). To prove the part (b), notice that $P_z[T_z = \infty] > 0$, by (5.1), and using the Markov property and Definition 3.1 it follows that $0 < P_z[T_z = \infty] = \sum_{y \neq z} p_{z\,y} P_y[T_z = \infty]$, so that there exists $y \neq z$ such that $P_y[T_z = \infty] > 0$; in this case, since L is a finite positive number, $P_y[T_z < \infty] + e^{-L} P_y[T_z = \infty] < P_y[T_z < \infty] + P_y[T_z = \infty] = 1$, and then $h(y) < 0$, by (5.4). To establish (5.5), first notice that (5.3) and (5.4) together imply that, for each $x \in S$, the following equalities are valid:

$$\begin{aligned} 1 = e^{h(z)} &= e^a P_z[T_z < \infty] + e^{a-L} P_z[T_z = \infty]; \\ e^{h(x)} &= P_x[T_z < \infty] + e^{-L} P_x[T_z = \infty], \quad x \neq z, \end{aligned} \tag{5.6}$$

whereas, via the Markov property, Definition 3.1 yields that

$$\begin{aligned} P_x[T_z < \infty] &= p_{x\,z} + \sum_{y \neq z} p_{x\,y} P_y[T_z < \infty], \\ P_x[T_z = \infty] &= \sum_{y \neq z} p_{x\,y} P_y[T_z < \infty]. \end{aligned} \tag{5.7}$$

Now let $x \neq z$ be arbitrary. In this case the above display and the second equality in (5.6) together lead to

$$\begin{aligned} e^{h(x)} &= P_x[T_z < \infty] + e^{-L} P_x[T_z = \infty] \\ &= p_{x\,z} + \sum_{y \neq z} p_{x\,y} P_y[T_z < \infty] + e^{-L} \sum_{y \neq z} p_{x\,y} P_y[T_z = \infty] \\ &= p_{x\,z} + \sum_{y \neq z} p_{x\,y} \left(P_y[T_z < \infty] + e^{-L} P_y[T_z = \infty] \right) \\ &= p_{x\,z} + \sum_{y \neq z} p_{x\,y} e^{h(y)} \\ &= \sum_{y \in S} p_{x\,y} e^{h(y)}, \end{aligned}$$

where the specification $h(z) = 0$ was used in the last step; since $C_{z\,a}(x) = 0$, it follows that the equality in (5.5) holds if $x \neq z$. To conclude, notice that (5.7) and the first equation in (5.6) together imply that

$$\begin{aligned} 1 = e^{h(z)} &= e^a P_z[T_z < \infty] + e^{a-L} P_z[T_z = \infty] \\ &= e^a \left(p_{z\,z} + \sum_{y \neq z} p_{z\,y} P_y[T_z < \infty] \right) + e^{a-L} \sum_{y \neq z} p_{z\,y} P_y[T_z = \infty] \\ &= e^a p_{z\,z} + e^a \sum_{y \neq z} p_{z\,y} \left(P_y[T_z < \infty] + e^{-L} P_y[T_z = \infty] \right) \\ &= e^a p_{z\,z} + e^a \sum_{y \neq z} p_{z\,y} e^{h(y)} \\ &= e^a p_{z\,z} e^{h(z)} + e^a \sum_{y \neq z} p_{z\,y} e^{h(y)} = e^a \sum_{y \in S} p_{z\,y} e^{h(y)}, \end{aligned}$$

and then, since $C_{z\,a}(z) = a$, it follows that the equation in (5.5) also holds for $x = z$, completing the proof of the part (i).

(ii) Given $a > 0$ and $z \in S$, it will be shown that

$$J_{C_{z\,a}}(\cdot) = 0 \iff a \le -\log(P_z[T_z < \infty]),$$

an assertion that is equivalent to the desired conclusion.

• *Suppose that* $J_{C_{z\,a}}(\cdot) = 0$. By Theorem 2.1 there exists a function $h(\cdot)$ such that, for every $x \in S$,

$$\begin{aligned} e^{h(x)} &= e^{C_{z\,a}(x)} \sum_{y \in S} p_{x\,y} e^{h(y)} \\ &= E_x\left[e^{C_{z\,a}(X_0)+h(X_1)}\right] \\ &= E_x\left[e^{C_{z\,a}(X_0)+h(X_1)} I[T_z = 1]\right] + E_x\left[e^{C_{z\,a}(X_0)+h(X_1)} I[T_z > 1]\right] \\ &= E_x\left[e^{C_{z\,a}(X_0)+h(z)} I[T_z = 1]\right] + E_x\left[e^{C_{z\,a}(X_0)+h(X_1)} I[T_z > 1]\right], \end{aligned}$$

where it was used that $X_{T_z} = z$ on the event $[T_z < \infty]$. From this point, an induction argument using the Markov property yields that, for each $x \in S$ and $n = 1, 2, 3, \ldots$

$$e^{h(x)} = E_x\left[e^{\sum_{t=0}^{T_z-1} C_{z\,a}(X_t)+h(z)} I[T_z \le n]\right] + E_x\left[e^{\sum_{t=0}^{n-1} C_{z\,a}(X_t)+h(X_n)} I[T_z > n]\right].$$

Therefore, $e^{h(x)} \ge E_x\left[e^{\sum_{t=0}^{T_z-1} C_{z\,a}(X_t)+h(z)} I[T_z \le n]\right]$ and via Fatou's lemma this implies that $e^{h(x)} \ge E_x\left[e^{\sum_{t=0}^{T_z-1} C_{z\,a}(X_t)+h(z)} I[T_z < \infty]\right]$; setting $x = z$ in this last inequality, it follows that

$$1 \ge E_z\left[e^{\sum_{t=0}^{T_z-1} C_{z\,a}(X_t)} I[T_z < \infty]\right].$$

Observing that $X_t \ne z$ for $1 \le t < T_z$, by Definition 3.1, it follows from the specification of $C_{z\,a}$ that $\sum_{t=0}^{T_z-1} C_{z\,a}(X_t) = C_{z\,a}(X_0) = a$ P_z-almost surely, so that the above displayed relation is equivalent to $1 \ge E_z[e^a I[T_z < \infty]] = e^a P_z[T_z < \infty]$, and then $a \le -\log(P_z[T_z < \infty])$.

• *Suppose that* $a \le -\log(P_z[T_z < \infty])$. In this case it will be shown that $J_{C_{z\,a}}(\cdot) = 0$, and to achieve this goal firstly assume that

$$a < -\log(P_z[T_z < \infty]),$$

so that part (i) yields that there exists a bounded function $h(\cdot)$ such that the pair $(0, h(\cdot))$ satisfies the Poisson equation associated with $C_{z\,a}$, that is $e^{h(x)} = e^{C_{z\,a}(x)} \sum_{y \in S} p_{x\,y} e^{h(y)}$ for each $x \in S$; from the boundedness of $h(\cdot)$ it follows that (2.3) holds, and then

$$J_{C_{z\,a}}(x) = 0, \quad x \in S, \quad 0 < a < -\log(P_z[T_z < \infty]),$$

by Lemma 2.1. On the other hand, combining Lemma 5.1 with the specification of $C_{z\,a}(\cdot)$ in (5.2), it follows that the mapping $a \mapsto J_{C_{z\,a}}(x)$ is always convex; consequently, such a mapping is continuous, and then the above display yields that $J_{C_{z\,a}}(\cdot) = 0$ if $0 < a \leq -\log(P_z[T_z < \infty])$, which is the desired conclusion. □

Example 5.1. Given a sequence $\{p_k \,|\, k = 0, 1, 2, 3, \ldots\} \subset (0, 1)$, set

$$q_k := 1 - p_k, \quad k = 0, 1, 2, 3, \ldots,$$

and on the space S of nonnegative integers define the transition matrix $P = [p_{x\,y}]$ as follows:

$$\begin{aligned} &p_{0\,0} = p_0, \quad p_{0\,1} = q_0 \\ &p_{x\,0} = p_x, \quad p_{x\,x+1} = q_x, \quad x = 1, 2, 3, \ldots \end{aligned}$$

In this case it is not difficult to see that the state space is communicating with respect to P, and that $P_0[T_0 > n] = \prod_{k=0}^{n-1} q_k$, so that

$$P_0[T_0 = \infty] = \prod_{k=0}^{\infty} q_k = \prod_{k=0}^{\infty} (1 - p_k),$$

which is positive if and only if

$$\sum_{k=0}^{\infty} p_k < \infty. \tag{5.8}$$

Under this condition Theorem 5.1 implies that $J_{C_{0,a}}(\cdot) > 0$ if $a > -\sum_{k=0}^{\infty} \log(1-p_k)$, whereas if

$$0 < a < -\sum_{k=0}^{\infty} \log(q_k) \tag{5.9}$$

then there exists a non constant and bounded function such that

$$\begin{aligned} e^{h(0)} &= e^a p_0 e^{h(0)} + e^a q_0 e^{h(1)} \\ e^{h(x)} &= p_x e^{h(0)} + q_x e^{h(x+1)}, \quad x = 1, 2, 3, \ldots; \end{aligned} \tag{5.10}$$

this latter fact will be useful in the following section.

6. UNIQUENESS

In this section the uniqueness of a pair $(g, h(\cdot))$ satisfying the Poisson equation (1.3) and the verification criterion (2.3) is analyzed. By Lemma 2.1, if such requirements hold then g is uniquely determined as the risk-sensitive average cost corresponding to C, whereas it is not difficult to see that both conditions still hold if $h(\cdot)$ is replaced by $h(\cdot)+M$, where M is an arbitrary real number. The main objective of this section is to provide a sufficient criterion such that, if $(g, h(\cdot))$ satisfies the Poisson equation (1.3) as well as the criterion (2.3), then $h(\cdot)$ is uniquely determined up to an additive constant. As the following example shows, this latter property does not necessarily occur.

Example 6.1. Consider a sequence $\{p_k\}$ satisfying

$$\{p_k\} \subset (0,1), \quad p_0 = 2/3, \quad \text{and} \quad \sum_{k=0}^{\infty} p_k < \infty,$$

and let $a > 0$ be such that

$$a < -\sum_{k=0}^{\infty} \log(q_k) \tag{6.1}$$

where, as before, $q_k = 1 - p_k$; as it was shown in Example 5.1, in this framework there exists a bounded and non constant function $h(\cdot)$ so that (5.10) holds. Now, on the space S of nonnegative integers, define the transition matrix $P = [p_{x\,y}]$ by

$$\begin{aligned}
&p_{0\,0} = p_{0\,1} = p_{0\,2} = \frac{1}{3}\\
&p_{2x\,0} = p_{2x-1\ 0} = p_x, \quad p_{2x\ 2x+2} = p_{2x-1\,2x+1} = q_x = 1 - p_x, \quad x = 1, 2, 3, \ldots.
\end{aligned}$$

For this matrix P, the existence of bounded functions $H(\cdot)$ such that the pair $(0, H(\cdot))$ satisfies the following Poisson equation corresponding to the cost function $C_{0\,a}$ in (5.2) will be analyzed:

$$e^{0+H(x)} = e^{C_{0\,a}(x)} \sum_{y \in S} p_{x\,y} e^{H(y)}, \quad x \in S. \tag{6.2}$$

More explicitly, this can be written as

$$\begin{aligned}
e^{H(0)} &= e^a \left[\frac{1}{3} e^{H(0)} + \frac{1}{3} e^{H(1)} + \frac{1}{3} e^{H(2)}\right],\\
e^{H(2x)} &= p_x e^{H(0)} + q_x e^{H(2x+2)}, \quad x = 1, 2, 3, \ldots,\\
e^{H(2x-1)} &= p_x e^{H(0)} + q_x e^{H(2x+1)}, \quad x = 1, 2, 3, \ldots.
\end{aligned}$$

To build solutions for this system, first impose the condition $H(0) = H(1)$. In this case from the third line of the above display it is not difficult to see that $H(x) = H(0)$ if x is odd. Also, recalling that $p_0 = 2/3$, the first and second line become

$$\begin{aligned}
e^{H(0)} &= e^a \left[p_0 e^{H(0)} + q_0 e^{H(2)}\right]\\
e^{H(2x)} &= p_x e^{H(0)} + q_x e^{H(2(x+1))}, \quad x = 1, 2, 3, \ldots,
\end{aligned}$$

and a glance at (5.10) shows that this system is satisfied setting $H(2x) = h(x)$ for every x. Thus, the following function is a solution of (6.2):

$$H_1(x) = h(0) \text{ if } x \text{ is odd, and } \ H_1(x) = h(k), \ \text{ if } x = 2k, \quad k = 1, 2, 3, \ldots$$

Similarly, it can be shown that (6.2) is also satisfied by the function $H_2(\cdot)$ given by

$$H_2(x) = h(0) \ \text{ if } x \text{ is even, and } \ H_2(x) = h(k) \ \text{ if } x = 2k - 1, \quad k = 1, 2, 3, \ldots.$$

In short, each pair $(0, H_i(\cdot))$, $i = 1, 2$, is a solution to the Poisson equation associated with the cost function $C_{0\,a}$, where a satisfies (6.1). Since $H_i(\cdot)$ is bounded, Lemma 2.1 yields that $J_{C_{0\,a}}(\cdot) = 0$. However, recalling that $h(\cdot)$ takes on at least two different values, it follows that the difference $H_1 - H_2$ is not constant.

The discussion on the uniqueness properties of a pair $(g, h(\cdot))$ satisfying the Poisson equation (1.3) and the criterion (2.3) is based on the following idea.

Definition 6.1. Suppose that Assumptions 2.1 and 2.2 hold, and let $g \in \mathbb{R}$ and $h\colon S \to (-\infty, 0]$ be such that the conclusions of Theorem 2.1 hold.

(i) The matrix $Q = [q_{x\,y}]_{x,y\in S}$ corresponding to the pair $(g, h(\cdot))$ is defined by

$$q_{x\,y} := \frac{e^{C(x)-g} p_{x\,y} e^{h(y)}}{e^{h(x)}}, \quad x, y \in S;$$

notice that, since the Poisson equation (1.3) holds, this matrix Q is stochastic, that is, $1 = \sum_{y\in S} q_{x\,y}$ for each $x \in S$.

(ii) For each $x \in S$, Q_x denotes the distribution on $\mathcal{B}(S^\infty)$ induced by matrix Q when the initial state is $X_0 = x$.

In the following lemma sufficient conditions are given so that the matrix Q in the above definition is recurrent.

Lemma 6.1. Suppose that Assumptions 2.1 and 2.2 hold. In this case the following assertions (i) – (iii) hold:

(i) The state space S is Q-communicating, that is, for each $x, y \in S$ there exists a positive integer $n = n(x, y)$ such that $Q_x[X_n = y] > 0$.

(ii) For each $W \subset S$ and $n = 1, 2, 3, \ldots$

$$Q_x[T_W > n] = \frac{1}{e^{h(x)}} E_x\left[e^{\sum_{t=0}^{n-1}(C(X_t)-g)} e^{h(X_n)} I[T_W > n]\right], \quad x \in S. \tag{6.3}$$

(iii) Suppose that at least one of the following conditions (a) and (b) hold:

(a) The Markov chain $\{X_n\}$ is P-recurrent, i.e., $P_z[T_z < \infty] = 1$ for some (and hence, for all) state $z \in S$;

(b) The average cost $g = J_C(\cdot)$ is positive.

In this context the Markov chain $\{X_n\}$ is Q-recurrent, that is, the equality

$$Q_z[T_z < \infty] = 1$$

holds for every $z \in S$.

Proof.

(i) Observing that Definition 6.1 yields that $q_{x\,y} \neq 0$ if and only if $p_{x\,y} \neq 0$, the conclusion follows from Assumption 2.1.

(ii) Let $W \subset S$ and $x_0 \in S$ be arbitrary. Using Definitions 3.1 and 6.1 it follows that for each $n = 1, 2, 3, \ldots$

$$\begin{aligned} Q_{x_0}[T_W > n] &= Q_{x_0}[X_1 \notin W, \ldots, X_n \notin W] \\ &= \sum_{\substack{x_i \notin W \\ i=1,2,\ldots,n}} \prod_{i=1}^{n} q_{x_{i-1},\,x_i} \\ &= \sum_{\substack{x_i \notin W \\ i=1,2,\ldots,n}} \prod_{i=1}^{n} \frac{e^{C(x_{i-1})-g} e^{h(x_i)} p_{x_{i-1}\,x_i}}{e^{h(x_{i-1})}} \\ &= \frac{1}{e^{h(x_0)}} \sum_{\substack{x_i \notin W \\ i=1,2,\ldots,n}} e^{\sum_{t=0}^{n-1}(C(x_t)-g)} e^{h(x_n)} \prod_{i=1}^{n} p_{x_{i-1}\,x_i} \\ &= \frac{1}{e^{h(x_0)}} E_{x_0}\left[e^{\sum_{t=0}^{n-1}(C(X_t)-g)} e^{h(X_n)} I[X_1 \notin W, \ldots, X_n \notin W] \right] \\ &= \frac{1}{e^{h(x_0)}} E_{x_0}\left[e^{\sum_{t=0}^{n-1}(C(X_t)-g)} e^{h(X_n)} I[T_W > n] \right]. \end{aligned}$$

(iii) Using that the state space S is Q-communicating and that the set K in (2.4) is finite, it follows that the Q-recurrence of $\{X_n\}$ is equivalent to

$$\lim_{n\to\infty} Q_x[T_K > n] = 0, \quad x \in S \setminus K. \tag{6.4}$$

This property can be verified under either of the conditions (a) and (b) as follows: Firstly, recall that $X_t \notin K$ for $1 \leq t < T_K$, by Definition 3.1, and in this case $C(X_t) = 0$, since C is supported on K. Thus, if $X_0 \notin K$ it follows that $\sum_{t=0}^{n-1}(C(X_t) - g) = -ng$ on the event $[T_K > n]$, and via the previous part it follows that for each positive integer n and $x \in S \setminus K$

$$\begin{aligned} Q_x[T_K > n] &= \frac{1}{e^{h(x)}} E_x\left[e^{\sum_{t=0}^{n-1}(C(X_t)-g)} e^{h(X_n)} I[T_K > n] \right] \\ &= \frac{1}{e^{h(x)}} e^{-ng} E_x\left[e^{h(X_n)} I[T_K > n] \right], \end{aligned}$$

and recalling that $h(\cdot) \leq 0$ (see Theorem 2.1 and Definition 6.1), this leads to

$$Q_x[T_K > n] \leq \frac{1}{e^{h(x)}} e^{-ng} P_x[T_K > n], \quad x \in S \setminus K, \quad n = 1, 2, 3, \ldots \tag{6.5}$$

(a) Assume that $\{X_n\}$ is P-recurrent. In this case, it follows that $\lim_{n\to\infty} P_x[T_K > n] = 0$ for every $x \in S \setminus K$, and then, since $g \geq 0$, the above display immediately leads to (6.4).

(b) Suppose that $g > 0$. In this context (6.5) yields that, for each state $x \in S \setminus K$,

$$\lim_{n\to\infty} Q_x[T_K > n] \le e^{-h(x)} \lim_{n\to\infty} e^{-ng} = 0,$$

so that (6.4) also holds in this case.

□

The main result of this section is the following.

Theorem 6.1. Suppose that Assumptions 2.1 and 2.2 hold, let $g \in R$ and $h\colon S \to (-\infty, 0]$ be as in Theorem 2.1 and assume that

$$\{X_n\} \text{ is } Q\text{-recurrent}, \tag{6.6}$$

where Q is the matrix in Definition 6.1. In this case, if $\tilde{h}\colon S \to \mathbb{R}$ is such that

$$e^{g+\tilde{h}(x)} = e^{C(x)} \sum_{y\in S} p_{x\,y} e^{\tilde{h}(y)}, \quad x \in S, \tag{6.7}$$

then $\tilde{h}(\cdot) - h(\cdot)$ is constant.

Notice that Example 6.1 explicitly shows that if condition (6.6) does not hold, then the conclusion of this theorem does not necessarily occur.

Proof. From (6.7) and Definition 6.1 it follows that for every $x \in S$

$$\begin{aligned} e^{\tilde{h}(x)-h(x)} &= e^{C(x)-g} \sum_{y\in S} \frac{p_{x\,y}}{e^{h(x)}} e^{\tilde{h}(y)} \\ &= \sum_{y\in S} \frac{e^{C(x)-g} p_{x\,y} e^{h(y)}}{e^{h(x)}} e^{\tilde{h}(y)-h(y)} \\ &= \sum_{y\in S} q_{x\,y} e^{\tilde{h}(y)-h(y)} \end{aligned}$$

so that

$$e^{\tilde{h}(x)-h(x)} = E_x^Q\left[e^{\tilde{h}(X_1)-h(X_1)}\right], \quad x \in S,$$

where $E_x^Q[\cdot]$ stands for the expectation operator with respect to Q_x. It follows that

$$e^{\tilde{h}(x)-h(x)} = e^{\tilde{h}(z)-h(z)} Q_x[T_z = 1] + E_x^Q\left[e^{\tilde{h}(X_1)-h(X_1)} I[T_z > 1]\right]$$

for every $x, z \in S$, and an induction argument using the Markov property yields that

$$e^{\tilde{h}(x)-h(x)} = e^{\tilde{h}(z)-h(z)} Q_x[T_z \le n] + E_x^Q\left[e^{\tilde{h}(X_n)-h(X_n)} I[T_z > n]\right]$$

always holds, so that

$$e^{\tilde{h}(x)-h(x)} \ge e^{\tilde{h}(z)-h(z)} Q_x[T_z \le n], \quad x, z \in S, \quad n = 1, 2, 3, \ldots$$

To conclude notice that, since the state space is Q-communicating, (6.6) implies that

$$1 = Q_x[T_z < \infty] = \lim_{n\to\infty} Q_x[T_z \le n], \quad x, z \in S,$$

a fact that, via the previous display, yields that the inequality $e^{\tilde h(x)-h(x)} \ge e^{\tilde h(z)-h(z)}$ is always valid, and it follows that $\tilde h(\cdot) - h(\cdot)$ is constant. □

From Theorem 6.1 it follows that, if $\{X_n\}$ is Q-recurrent, then conditions (1.3) and (2.3) determine function $h(\cdot)$ up to an additive constant.

Corollary 6.1. Suppose that Assumptions 2.1 and 2.2 as well as condition (6.6) hold, where matrix Q is specified in Definition 6.1. In this framework, if $g_i \in \mathbb{R}$ and $h_i \colon S \to \mathbb{R}$, $i = 1, 2$ are such that

$$e^{g_i + h_i(x)} = e^{C(x)} \sum_{y\in S} p_{x,y} e^{h_i(y)} \tag{6.8}$$

and

$$\liminf_{n\to\infty} \left(E_{\nu_{C,x,n}} \left[e^{h_i(X_n)} \right] \right)^{1/n} = 1 \tag{6.9}$$

for every $x \in S$ and $i = 1, 2$, then $h_1(\cdot) - h_2(\cdot)$ is constant.

P r o o f. Let $g \in \mathbb{R}$ and $h(\cdot)$ be as in Theorem 2.1. By Lemma 2.1, (6.8) and (6.9) together imply that $g_i = J_C(\cdot) = g$ for each i. Then, Theorem 6.1 yields that $h_i(\cdot) - h(\cdot)$ is constant for $i = 1, 2$; see (6.7) and (6.8). Thus, $h_1(\cdot) - h_2(\cdot)$ is also constant. □

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