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ON THE COMPLEXITY OF THE SHAPLEY–SCARF ECONOMY WITH SEVERAL TYPES OF GOODS

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In the Shapley–Scarf economy each agent is endowed with one unit of an indivisible good (house) and wants to exchange it for another, possibly the most preferred one among the houses in the market. In this economy, core is always nonempty and a core allocation can be found by the famous Top Trading Cycles algorithm. Recently, a modification of this economy, containing $Q \geq 2$ types of goods (say, houses and cars for $Q = 2$) has been introduced. We show that if the number of agents is 2, a complete description of the core can be found efficiently. However, when the number of agents is not restricted, the problem to decide the nonemptyness of the core becomes NP-hard already in the case of two types of goods. We also show that even the problem to decide whether an allocation exists in which each agent strictly improves compared to his endowment, is NP-complete.

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1. INTRODUCTION

In the seminal paper by Shapley and Scarf [10] a special economy with indivisible goods (the so-called housing market) was introduced. In this economy each agent owns one unit of an indivisible unique good (house) that is specific for him and wants to end up again with just one unit of good. The preference relation of an agent is simply a linear ordering (possibly with ties) of a subset of houses. Under such assumptions, a core allocation always exists, which can be proved constructively by the Top Trading Cycles (TTC for short) algorithm due to Gale.

There are many studies of the housing market in the literature, not only because it is interesting matematically, but also for its ability to model many real markets: large-scale exchange of government subsidized housing in China [11], matching graduates of the United States Naval Academy to their first posts as Naval Officers [7], assigning students to schools [1], exchange of incompatible donors of kidneys for transplantation [8] etc. Notice that a detailed mathematical analysis helped to change an inefficient mechanism for matching students in Boston to places at public schools to another mechanism with better properties [1], in the other case it helped to find suitable donors for patients on the waiting lists [8]. As in real markets the number of participants is usually very high, efficient algorithms are very important for any results to be implemented. The housing market, due to its special structure, admits such algorithms, but how important is this structure for tractability?

Konishi, Quint and Wako [6] considered a modification of the Shapley–Scarf economy with $Q \geq 2$ types of indivisible goods (if $Q = 2$, the types may be say houses and cars). Each agent originally owns one unit of each type of good (say one house and one car) and wants to exchange them so as he again ends up with one unit of each type. Preferences of agents are given as strict linear orders of Q-tuples and in [6] they are supposed to be separable. Now, the core may be empty already in the case of just two types of goods. For additively separable preferences Konishi, Quint and Wako [6] proved that the core is always nonempty if the number of agents is 3 and $Q = 2$. In the proof they transformed the economy to the associated NTU game and used the famous Scarf's theorem [9]. We show that in general it is an NP-hard problem to decide the nonemptyness of the core, even in the case $Q = 2$.

Pareto optimality in the housing market has also been studied. In [2] a polynomial algorithm for finding a Pareto optimal allocation has been proposed and some structural results for the set of Pareto optimal allocations have been derived. In the view of the negative results in Section 4 of this paper, a similar achievement for the case with several types of goods seems to be improbable.

The organization of the paper is as follows. In Section 2 we introduce the used notions. Then in Section 3 we deal with the case of just two agents and show that the core (which is here equivalent with the set of allocations that are simultaneously Pareto optimal and individually rational) is easy to describe. The hardness proofs are contained in Section 4 and finally Conclusion contains some open problems and directions for further research.

2. THE DESCRIPTION OF THE MODEL

The set of agents is denoted by A, their number by N. In the economy there are Q types of indivisible goods and each agent $a \in A$ is endowed with one, for him specific unit of each type of good, i. e. with a Q-tuple $\boldsymbol{g}(a) = (g_1(a), g_2(a), \ldots, g_Q(a))$. We shall denote by G_i the set of goods of type i in the economy, i. e. $G_i = \bigcup_{a \in A} g_i(a)$ and $G = G_1 \times G_2 \times \cdots \times G_Q$. Each agent wishes to end up with a Q-tuple from G; such Q-tuples will be called *bundles* and denoted by lowercase bold letters. Each agent $a \in A$ has linear preferences over bundles, i. e. a transitive and reflexive binary relation $P(a)$ on a subset $G(a)$ of the set G, the set of *acceptable bundles*. Notation $\boldsymbol{x} \succeq_a \boldsymbol{y}$ means that agent a prefers bundle $\boldsymbol{x}$ to bundle $\boldsymbol{y}$. If $\boldsymbol{x} \succeq_a \boldsymbol{y}$ and simultaneously $\boldsymbol{y} \succeq_a \boldsymbol{x}$ agent a is indifferent between bundles $\boldsymbol{x}$ and $\boldsymbol{y}$; if $\boldsymbol{x} \succeq_a \boldsymbol{y}$ but not $\boldsymbol{y} \succeq_a \boldsymbol{x}$ then agent a prefers bundle $\boldsymbol{x}$ to bundle $\boldsymbol{y}$ strictly. In what follows, we shall suppose that there are no indifferences and we shall represent agent's preferences by a list of his acceptable bundles in the order from the most preferred one to the least preferred one. We shall also suppose that $\boldsymbol{g}(a) \in G(a)$ for each agent $a \in A$. The N-tuple of preferences $(P(a), a \in A)$ will be denoted by $\mathcal{P}$ and called the *preference profile*. The set of all possible preference profiles will be denoted by Π. An economy is a pair $\mathcal{E} = (A, \mathcal{P})$.

With some abuse of notation, in the case of bundles we shall usually write, say in case $Q = 4$, instead of $(g_1(a), g_2(b), g_3(c), g_4(d))$ simply (a, b, c, d), as no confusion should arise.

An *allocation* is a function $\mathcal{X} : A \to G$, i. e. $\boldsymbol{x}(a) = (x_1(a), x_2(2), \dots, x_Q(a))$, such that $x_i(a) \neq x_i(b)$ for each $i = 1, 2, \dots, Q$ and each $a, b \in A, a \neq b$. An allocation $\mathcal{X}$ is *individually rational*, if $\boldsymbol{x}(a) \in G(a)$ for each agent $a \in A$.

Definition 1. A coalition $S \subseteq A$ *blocks* an allocation $\mathcal{X}$ if there exists an allocation $\mathcal{Y}$ such that

1. $\boldsymbol{y}(a) \succ_a \boldsymbol{x}(a)$ for each agent $a \in S$, and
2. $\bigcup_{a \in S} \boldsymbol{y}(a) = \bigcup_{a \in S} \boldsymbol{g}(a)$.

Definition 2. An allocation $\mathcal{X}$ is in the *core* of economy $\mathcal{E}$ if no coalition blocks it and it is said to be *Pareto optimal* for economy $\mathcal{E}$ if A does not block it.

The set of all core allocations of economy $\mathcal{E}$ will be denoted by $Core(\mathcal{E})$.

Konishi, Quint and Wako [6] proved in Proposition 2.1 that $Core(\mathcal{E}) \neq \emptyset$ for each economy with $N = 3, Q = 2$. However, they assumed that the agents' preferences are additively separable, i. e. each agent a has Q utility functions $u_i^a : G_i \to \mathbb{R}$, $i = 1, 2, \dots, Q$ such that a prefers bundle $\boldsymbol{x} = (x_1, x_2, \dots, x_Q)$ to bundle $\boldsymbol{y} = (y_1, y_2, \dots, y_Q)$ if and only if $\sum_{i=1}^{Q} u_i^a(x_i) > \sum_{i=1}^{Q} u_i^a(y_i)$. Already with $N = 4$ the core may be empty (see Example 2.3 in [6]) even with additively separable preferences. Here we show that the assumption of additive separability in Proposition 2.1 of [6] is crucial.

Example 1. Consider economy $\mathcal{E}$ with three agents and two types of goods with the following preferences:

$$\begin{aligned} P(a) :& \quad (a, b), (c, a), (a, a) \\ P(b) :& \quad (c, b), (b, a), (b, b) \\ P(c) :& \quad (a, c), (b, c), (c, c) \end{aligned}$$

To show that no allocation $\mathcal{X} = (\boldsymbol{x}(a), \boldsymbol{x}(b), \boldsymbol{x}(c))$ can be in $Core(\mathcal{E})$, consider three cases.

(i) If $\boldsymbol{x}(a) = (a, b)$, then necessarily $\boldsymbol{x}(b) = (b, a)$. Then $\boldsymbol{x}(c) = (c, c)$ and coalition $\{b, c\}$ is blocking via $\boldsymbol{y}(b) = (c, b)$ and $\boldsymbol{y}(c) = (b, c)$.

(ii) If $\boldsymbol{x}(a) = (c, a)$, then agent b must receive bundle (b, b). But then coalition $\{a, b\}$ is blocking by assigning bundles (a, b) and (b, a) to agents a, b, respectively.

(iii) Finally, if $\boldsymbol{x}(a) = (a, a)$, then agent c cannot obtain $\boldsymbol{x}(c) = (a, c)$, so coalition $\{a, c\}$ is blocking via $\boldsymbol{y}(a) = (c, a)$ and $\boldsymbol{y}(c) = (a, c)$.

3. TWO–AGENTS ECONOMIES

Let $A = \{a, b\}$ and let Q be arbitrary. We will show that in case of two agents the complete core of the economy can be generated by a simple algorithm.

For a bundle $\boldsymbol{x} \in G$ we shall denote by $\overline{\boldsymbol{x}}$ its *complement*, i.e. $x_i = a$ implies $\overline{x}_i = b$ and $x_i = b$ implies $\overline{x}_i = a$ for all $i = 1, 2, \ldots, Q$. Suppose that the preferences of agents are of the following form

$$\begin{array}{ll} P(a): & \boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^k \\ P(b): & \boldsymbol{y}^1, \boldsymbol{y}^2, \ldots, \boldsymbol{y}^\ell \end{array}$$

and consider algorithm CoreN2 given in Figure 1.

Input: Economy $\mathcal{E} = (A, \mathcal{P})$ with $|A| = 2$.
Output: Reduced preference lists $P'(a), P'(b)$.
1. **begin** denote all entries in $P(b)$ as unlabelled;
2. **for** $i = 1, \ldots, k$ **do if** $\overline{\boldsymbol{x}}^i \notin P(b)$ **then** delete $\boldsymbol{x}^i$ from $P(a)$ **else** label $\overline{\boldsymbol{x}}^i$ in $P(b)$;
3. delete all unlabelled entries from $P(b)$;
4. denote all entries in $P(b)$ as unlabelled;
5. **for** $i = 1, 2, \ldots, k$ **do if** $\boldsymbol{x}^i \in P(a)$ **then**
6. **begin** label $\overline{\boldsymbol{x}}^i$;
7. **for each** $\boldsymbol{y}^j \in P(b)$ **do**
8. **if** $\boldsymbol{y}^j$ is written after $\overline{\boldsymbol{x}}^i$ in $P(b)$ **and** unlabelled **then**
9. delete $\boldsymbol{y}^j$ from $P(b)$ and $\overline{\boldsymbol{y}}^j$ from $P(a)$
10. **end**;
11. **end**

Fig. 1. Algorithm CoreN2.

Example 2. Let $Q = 4$ and the preference lists of agents a and b be

$$\begin{array}{ll} P(a): & (a,b,a,b), (b,b,b,b), (a,a,b,b), (a,b,b,b), (a,a,a,a) \\ P(b): & (b,a,a,a), (a,b,b,a), (b,a,b,a), (a,b,b,b), (b,a,a,b), (b,b,b,b) \end{array}$$

(so $k = 5$ and $\ell = 6$). In line 2 of the algorithm, bundles $(b,b,b,b), (a,a,b,b)$ are deleted from $P(a)$ and in $P(b)$ bundles $(b,a,b,a), (b,a,a,a)$ and (b,b,b,b) are labelled. Then in line 3, bundles $(a,b,b,a), (a,b,b,b)$ and (b,a,a,b) are deleted from $P(b)$. We are left with the lists

$$\begin{array}{ll} P(a): & (a,b,a,b), (a,b,b,b), (a,a,a,a) \\ P(b): & (b,a,a,a), (b,a,b,a), (b,b,b,b) \end{array}$$

Then, for (a,b,a,b) in $P(a)$, bundle (b,a,b,a) is labelled in $P(b)$ and bundles (b,b,b,b) and (a,a,a,a) are deleted from $P(b)$ and $P(a)$, respectively. Finally, for bundle (a,b,b,b) in $P(a)$, bundle (b,a,a,a) is labelled in $P(b)$, but nothing is deleted, as the only bundle (b, a, b, a) appearing after it in $P(b)$ is labelled. Finally, $Core(\mathcal{E})$ contains two allocations

$$\mathcal{X}_1 = ((a, b, a, b), (b, a, b, a)) \quad \text{and} \quad \mathcal{X}_2 = ((a, b, b, b), (b, a, a, a)).$$

Now we argue that Algorithm CoreN2 is correct.

Theorem 1. If Algorithm CoreN2 results in both preference lists being empty then $Core(\mathcal{E}) = \emptyset$. Otherwise $Core(\mathcal{E})$ is equal to the set of all allocations of the form $(\boldsymbol{x}, \overline{\boldsymbol{x}})$ for $\boldsymbol{x} \in P'(a)$.

P r o o f . First we show that no bundle that was deleted in the course of Algorithm CoreN2 can be a part of a core allocation. This is clear for the bundles deleted in lines 2 and 3 of the algorithm, as their complements are not acceptable for the other agent. Now suppose that bundle $\boldsymbol{y}$ was deleted from $P(b)$ in line 9 of the algorithm; suppose that this happened because of $\boldsymbol{x}^i \in P(a)$. But this means that agent b prefers $\overline{\boldsymbol{x}}^i$ to $\boldsymbol{y}$. Moreover, since $\boldsymbol{y}$ was not labelled before, $\boldsymbol{y}$ is not a complement for any of the bundles $\boldsymbol{x}^j$, $j < i$ and so agent a prefers $\boldsymbol{x}^i$ to $\overline{\boldsymbol{y}}$. Thus allocation $(\overline{\boldsymbol{y}}, \boldsymbol{y})$ cannot be in the core of $\mathcal{E}$.

Conversely, if $\boldsymbol{x} \in P'(a)$, it is easy to see that $(\boldsymbol{x}, \overline{\boldsymbol{x}}) \in Core(\mathcal{E})$, as for any other allocation $(\boldsymbol{y}, \overline{\boldsymbol{y}})$ either agent a prefers $\boldsymbol{x}$ to $\boldsymbol{y}$ of agent b prefers $\overline{\boldsymbol{x}}$ to $\overline{\boldsymbol{y}}$. □

The complexity estimation of Algorithm CoreN2 can be obtained as follows. In line 2 it is necessary, for each $\boldsymbol{x}^i$ in $P(a)$, to scan the whole preference list $P(b)$. This gives $O(k\ell)$ operations. Lines 3 and 4 need $O(\ell)$ operations each and lines 5–10 again $O(k\ell)$ operations. As the length of each preference list is at most 2^Q, the complexity bound of Algoritm CoreN2 is $O(2^{2Q})$, but this is still polynomial in the size of the representation of the economy.

4. NP–HARD PROBLEMS

In our transformations showing NP-completeness of some problems, we shall use a variant of Satisfiability, called R3-SAT, see e. g. [3]. In an instance of R3-SAT a boolean formula B in CNF is given, such that each clause contains exactly 3 literals and each variable appears in B exactly twice nonnegated and exactly twice negated. The question is whether B is satisfiable.

For each instance B of R3-SAT with clauses $C_1, C_2, \ldots, C_m$ and variables $v_1, v_2, \ldots,$ v_n we construct an econonomy $\mathcal{E}_B$ with a special structure. The agents will be divided into n *variable cells* $\mathcal{E}_B(v_j)$ and m *clause cells* $\mathcal{E}_B(C_i)$. In a variable cell $\mathcal{E}_B(v_j)$, agents p_j^1 and p_j^2 will correspond to the first and to the second occurrence of literal v_j, agents q_j^1 and q_j^2 to the first and to the second occurrence of literal $\overline{v}_j$. In the clause cell $\mathcal{E}_B(C_i)$ agents c_i^1, c_i^2, c_i^3 correspond to the first, second and third position in C_i.

We will also use the notation $c(a)$ for $a \in \{p_j^1, p_j^2, q_j^1, q_j^2\}$ to denote the clause agent corresponding to the position in the formula containing the associated literal, and conversely, $v(a)$ for $a \in \{c_i^1, c_i^2, c_i^3\}$ will denote the variable agent corresponding to the particular occurrence of a variable in the associated position.

For example, let

$$C_3 = v_1 + v_2 + \overline{v}_3$$

and for v_1 let this be its second occurence, and for v_2 and $\overline{v}_3$ let these be their first occurences in the formula. Then

$$v(c_3^1) = p_1^2, v(c_3^2) = p_2^1, v(c_3^3) = q_3^1, c(p_1^2) = c_3^1, c(p_2^1) = c_3^2, c(q_3^1) = c_3^3.$$

We shall say that an agent a is *linking* in allocation $\mathcal{X}$, if $\boldsymbol{x}(a)$ contains some good originally owned by an agent not belonging to the cell of agent a.

4.1. Core of the economy

Let us consider the problem

CORE EXISTENCE
Instance. An economy $\mathcal{E}$.
Question. Does $\mathcal{E}$ admit a core allocation?

Theorem 2. Problem CORE EXISTENCE is NP-hard already in the case when $Q = 2$ and agents have strict preferences over bundles.

Proof. In the polynomial transformation, econonomy $\mathcal{E}_B$ constructed for an instance B of R3-SAT with clauses $C_1, C_2, \ldots, C_m$ and variables $v_1, v_2, \ldots, v_n$ will have $N = 4m + 6n$ agents and $Q = 2$.

Variable cell $\mathcal{E}_B(x_j)$ consists of 6 agents $p_j^1, p_j^2, q_j^1, q_j^2, r_j^1$, r_j^2 and clause cell $\mathcal{E}_B(C_i)$ consists of four agents c_i^1, c_i^2, c_i^3, z_i. Preferences of agents of $\mathcal{E}_B$ are given in Figures 2 and 3.

$$\begin{array}{lllll}
P(p_j^1): & (r_j^1, p_j^2), & (p_j^1, q_j^2), & (p_j^1, c(p_j^1)), & (p_j^1, p_j^1) \\
P(p_j^2): & (r_j^2, p_j^1), & (p_j^2, q_j^1), & (p_j^2, c(p_j^2)), & (p_j^2, p_j^2) \\
P(q_j^1): & (r_j^1, q_j^2), & (q_j^1, p_j^2), & (q_j^1, c(q_j^1)), & (q_j^1, q_j^1) \\
P(q_j^2): & (r_j^2, q_j^1), & (q_j^2, p_j^1), & (q_j^2, c(q_j^2)), & (q_j^2, q_j^2) \\
P(r_j^1): & (p_j^2, r_j^2), & (q_j^2, r_j^2), & (r_j^1, r_j^1) & \\
P(r_j^2): & (q_j^1, r_j^1), & (p_j^1, r_j^1), & (r_j^2, r_j^2) &
\end{array}$$

Fig. 2. Preferences of agents of a variable cell.

Now we derive the properties of core allocations in variable and clause cells.

$$\begin{array}{llllll}
P(c_i^1): & (c_i^1, v(c_i^1)), & (z_i, c_i^1), & (z_i, c_i^2), & (c_i^1, c_i^1) & \\
P(c_i^2): & (c_i^2, v(c_i^2)), & (c_i^2, c_i^1), & (c_i^1, c_i^1), & (c_i^2, c_i^3), & (c_i^2, c_i^2) \\
P(c_i^3): & (c_i^3, v(c_i^3)), & (c_i^3, z_i), & (c_i^3, c_i^2), & (z_i, z_i), & (c_i^3, c_i^3) \\
P(z_i): & (c_i^3, z_i), & (c_i^3, c_i^3), & (c_i^2, z_i), & (z_i, z_i) &
\end{array}$$

Fig. 3. Preferences of agents of a clause cell.

Lemma 1. Let $\mathcal{X}$ be a core allocation. Then in each variable cell $\mathcal{E}_B(v_j)$, allocation $\mathcal{X}$ can behave in only one of the two ways, either

$$p_j^1 \to (r_j^1, p_j^2);\ p_j^2 \to (r_j^2, p_j^1);\ r_j^1 \to (p_j^2, r_j^2);\ r_j^2 \to (p_j^1, r_j^1)$$

(this will be called the $\mathcal{X}_j^1$ case), or

$$q_j^1 \to (r_j^1, q_j^2);\ q_j^2 \to (r_j^2, q_j^1);\ r_j^1 \to (q_j^2, r_j^2);\ r_j^2 \to (q_j^1, r_j^1)$$

which will be called the $\mathcal{X}_j^2$ case in what follows.

P r o o f. Clearly, any allocation $\mathcal{X}$ that is not blocked by any coalition must be individually rational. Now consider several cases for $\mathcal{X}$.

- *Case 1.* Suppose that $\boldsymbol{x}(r_j^1) = (r_j^1, r_j^1)$. Then necessarily $\boldsymbol{x}(r_j^2) = (r_j^2, r_j^2)$ and this implies that none of the agents $p_j^1, p_j^2, q_j^1, q_j^2$ is assigned his first choice bundle. But then $\mathcal{X}$ is blocked e. g. by coalition $\{p_j^1, p_j^2, r_j^1, r_j^2\}$ assigned bundles according to $\mathcal{X}_j^1$. (In the case $\boldsymbol{x}(r_j^2) = (r_j^2, r_j^2)$ the argument is symmetric.)
- *Case 2.* Suppose that both r-agents are assigned their second choice bundles, i. e. $\boldsymbol{x}(r_j^1) = (q_j^2, r_j^2)$ and $\boldsymbol{x}(r_j^2) = (p_j^1, r_j^1)$. Then $\boldsymbol{x}(p_j^1) = (r_j^1, p_j^2)$ and $\boldsymbol{x}(q_j^2) = (r_j^2, q_j^1)$. But then agents p_j^2 and q_j^1 cannot be assigned their first and second choice bundles and they will form a blocking coalition by helping themselves to the bundles (p_j^2, q_j^1) and (q_j^1, p_j^2), respectively.
- *Case 3.* Now suppose that both r-agents are assigned their first choice bundles, i. e. $\boldsymbol{x}(r_j^1) = (p_j^2, r_j^2)$ and $\boldsymbol{x}(r_j^2) = (q_j^1, r_j^1)$. Then we have $\boldsymbol{x}(p_j^2) = (r_j^2, p_j^1)$ and $\boldsymbol{x}(q_j^1) = (r_j^1, q_j^2)$. But then agents p_j^1 and q_j^2 cannot be assigned their first and second choice bundles and they will form a blocking coalition by exchanging their endowments to get the bundles (p_j^1, q_j^2) and (q_j^2, p_j^1), respectively.

Hence necessarily one of the r-agents has his first choice bundle and the other one his second choice bundle, so w.l.o.g. suppose $\boldsymbol{x}(r_j^1) = (p_j^2, r_j^2)$ and $\boldsymbol{x}(r_j^2) = (p_j^1, r_j^1)$. Then the only available bundles for players p_j^1 and p_j^2 are their first choice bundles, i. e. (r_j^1, p_j^2) and (r_j^2, p_j^1). This leads to the allocation $\mathcal{X}_j^1$.

Finally notice that the assumption $\boldsymbol{x}(r_j^1) = (q_j^2, r_j^2)$ and $\boldsymbol{x}(r_j^2) = (q_j^1, r_j^1)$ leads to the allocation $\mathcal{X}_j^2$. □

Lemma 2. If $\mathcal{X}$ is a core allocation, then in each clause cell $\mathcal{E}_B(C_i)$, $\boldsymbol{x}(c_i^k) = (c_i^k, v(c_i^k))$ for at least one agent c_i^k, $k \in \{1, 2, 3\}$.

Proof. First suppose that $g_2(v(c_i^k))$ is unavailable for all agents in $\mathcal{E}_B(C_i)$. Then in fact, the agents of $\mathcal{E}_B(C_i)$ have to consider the reduced preference lists given in Figure 4.

$$\begin{array}{lllll} P(c_i^1): & (z_i, c_i^1), & (z_i, c_i^2), & (c_i^1, c_i^1) & \\ P(c_i^2): & (c_i^2, c_i^1), & (c_i^1, c_i^1), & (c_i^2, c_i^3), & (c_i^2, c_i^2) \\ P(c_i^3): & (c_i^3, z_i), & (c_i^3, c_i^2), & (z_i, z_i), & (c_i^3, c_i^3) \\ P(z_i): & (c_i^3, z_i), & (c_i^3, c_i^3), & (c_i^2, z_i), & (z_i, z_i) \end{array}$$

Fig. 4. Reduced preferences of agents of a clause cell.

The reduced economy $\mathcal{E}'$ of Figure 4 is in fact identical with the economy constructed by Konishi, Quint and Wako in [6, Example 2.3.], which has no core allocation. To be self contained, we repeat here the argument from [6]. There are only four individually rational allocations for $\mathcal{E}'$, namely

$$\begin{array}{ll} \mathcal{Y}_i^1: & c_i^1 \to (c_i^1, c_i^1);\ c_i^2 \to (c_i^2, c_i^2);\ c_i^3 \to (c_i^3, c_i^3);\ z_i \to (z_i, z_i) \\ \mathcal{Y}_i^2: & c_i^1 \to (c_i^1, c_i^1);\ c_i^2 \to (c_i^2, c_i^3);\ c_i^3 \to (c_i^3, c_i^2);\ z_i \to (z_i, z_i) \\ \mathcal{Y}_i^3: & c_i^1 \to (z_i, c_i^2);\ c_i^2 \to (c_i^1, c_i^1);\ c_i^3 \to (c_i^3, c_i^3);\ z_i \to (c_i^2, z_i) \\ \mathcal{Y}_i^4: & c_i^1 \to (c_i^1, c_i^1);\ c_i^2 \to (c_i^2, c_i^2);\ c_i^3 \to (z_i, z_i);\ z_i \to (c_i^3, c_i^3) \end{array}$$

Allocation $\mathcal{Y}_i^1$ is blocked by coalition $\{c_i^2, c_i^3\}$ via allocation $\mathcal{Y}_i^2$, allocation $\mathcal{Y}_i^2$ is blocked by coalition $\{c_i^1, c_i^2, z_i\}$ via allocation $\mathcal{Y}_i^3$, allocation $\mathcal{Y}_i^3$ is blocked by coalition $\{c_i^3, z_i\}$ via allocation $\mathcal{Y}_i^4$ and finally allocation $\mathcal{Y}_i^4$ is blocked by coalition $\{c_i^2, c_i^3\}$ via allocation $\mathcal{Y}_i^2$. □

Lemma 3. If at least one of the agents c_i^1, c_i^2, c_i^3 is linking and for the other c-agents of $\mathcal{E}_B(C_i)$ the good $g_2(v(c_i^k))$ is unavailable, then there exists an assignment of bundles to agents of $\mathcal{E}_B(C_i)$ such that no agent from $\mathcal{E}_B(C_i)$ can be in a blocking coalition.

Proof. It is easy to verify that the sought assignments are

- if all c-agents are linking, let $\boldsymbol{x}(z_i) = (z_i, z_i)$,
- if both c_i^1 and c_i^3 or both c_i^2 and c_i^3 are linking, assign to the remaining agents the bundles as in $\mathcal{Y}_i^1$;
- if both c_i^1 and c_i^2 or c_i^2 only are linking, assign to the remaining agents the bundles as defined by $\mathcal{Y}_i^4$;
- if only c_i^1 is linking, remaining agents will be assigned the bundles as in $\mathcal{Y}_i^2$; and finally
- if only c_i^3 is linking, remaining agents will be assigned the bundles as in $\mathcal{Y}_i^3$.□

Now suppose that formula B is satisfied by a truth assignment f. Let us create allocation $\mathcal{X}$ by assigning bundles to agents in the following way:

(i) If v_j is true in f, set $\boldsymbol{x}(p_j^1) = (p_j^1, c(p_j^1))$, $\boldsymbol{x}(p_j^2) = (p_j^2, c(p_j^2))$ and the remaining agents of $\mathcal{E}_B(v_j)$ will get the bundles according to allocation $\mathcal{X}_j^2$.

(ii) If v_j is false in f, set $\boldsymbol{x}(q_j^1) = (q_j^1, c(q_j^1))$, $\boldsymbol{x}(q_j^2) = (q_j^2, c(q_j^2))$ and the remaining agents of $\mathcal{E}_B(v_j)$ will get the bundles according to allocation $\mathcal{X}_j^1$.

(iii) Assign to each agent c_i^k corresponding to a position of a true literal in B the bundle $(c_i^k, v(c_i^k))$. Since B is true in f, in each $\mathcal{E}_B(C_i)$ at least one of the agents c_i^k gets this bundle and for the remaining agents of $\mathcal{E}_B(C_i)$ the good $g_2(v(c_i^k)))$ is not available due to (i) and (ii), so a core allocation exists due to Lemma 3.

So we have a core allocation for the constructed economy.

Conversely, let $\mathcal{X}$ be a core allocation for economy $\mathcal{E}_B$. Then $\mathcal{X}$ acts on any variable cell $\mathcal{E}_B(v_j)$ according either to allocation $\mathcal{X}_j^1$ or to allocation $\mathcal{X}_j^2$ (Lemma 1). In the former case set v_j to be false and in the latter case to be true. Further, Lemma 2 implies that in each clause cell $\mathcal{E}_B(C_i)$ at least one agent c_i^k is linking – and it is easy to see that this agent will correspond to a true literal. So B is satisfied. □

4.2. Pareto optimal allocations

Problem BETTER ALLOCATION.
Instance. An economy $\mathcal{E}$.
Question. Does $\mathcal{E}$ admit an allocation $\mathcal{Y}$ such that $\boldsymbol{y}(a) \succ_a \boldsymbol{g}(a)$ for each agent $a \in A$?

Theorem 3. Problem BETTER ALLOCATION is NP-complete even in the case $Q = 2$.

Proof. We again use a polynomial transformation from R3-SAT. Econonomy $\mathcal{E}_B$ will have $N = 6m + 5n$ agents.

Variable cell $\mathcal{E}_B(v_j)$ consists of 5 agents $p_j^1, p_j^2, q_j^1, q_j^2$, r_j and clause cell $\mathcal{E}_B(C_i)$ contains 6 agents c_i^1, c_i^2, c_i^3 and z_i, t_i^1, t_i^2.

Preferences of agents of $\mathcal{E}_B$ are given in Figures 5 and 6. In Figure 6 the symbol C_i^2 represents all the bundles (c_i^k, c_i^ℓ) for $k, \ell = 1, 2, 3$, $k \neq \ell$ in any strict order and the symbol (z_i, Y_i) represents all the bundles (z_i, c_i^k) for $k = 1, 2, 3$ in any strict order. Further, the symbol $V(c_i^k)$ is equal to the pair $(v(c_i^k), c_i^k)$ if the k^{th} position of C_i is the first occurrence of a particular literal, or equivalently if $v(c_i^k)$ is equal to p_j^1 or to q_j^1. Similarly, $V(c_i^k)$ is equal to the pair $(c_i^k, v(c_i^k))$ if the k^{th} position of C_i is the second occurrence of a particular literal, which happens if $v(c_i^k)$ is equal to p_j^2 or to q_j^2.

Now we show that $\mathcal{E}$ admits an allocation $\mathcal{Y}$ such that $\boldsymbol{y}(a) \succ_a \boldsymbol{g}(a)$ for each agent $a \in A$ if and only if B is satisfiable.

$$
\begin{array}{llll}
P(p_j^1): & (c(p_j^1), p_j^1), & (p_j^1, r_j), & (p_j^1, p_j^1) \\
P(p_j^2): & (p_j^2, c(p_j^2)), & (r_j, p_j^2), & (p_j^2, p_j^2) \\
P(q_j^1): & (c(q_j^1), q_j^1), & (q_j^1, r_j), & (q_j^1, q_j^1) \\
P(q_j^2): & (q_j^2, c(q_j^2)), & (r_j, q_j^2), & (q_j^2, q_j^2) \\
P(r_j): & (p_j^2, p_j^1), & (q_j^2, q_j^1), & (r_j, r_j)
\end{array}
$$

Fig. 5. Preferences of agents of a variable cell.

$$
\begin{array}{llllll}
P(c_i^k): & (c_i^k, z_i), & (z_i, c_i^k), & V(c_i^k), & (c_i^k, c_i^k) & \text{for } k = 1, 2, 3 \\
P(z_i): & C_i^2, & (z_i, Y_i), & (z_i, t_i^1); & (z_i, z_i) & \\
P(t_i^1): & (t_i^1, t_i^2), & (t_i^1, t_i^1) & & & \\
P(t_i^2): & (t_i^2, t_i^1), & (t_i^2, z_i), & (t_i^2, t_i^2) & &
\end{array}
$$

Fig. 6. Preferences of agents of a clause cell.

First, let B be satisfied by a truth assignment f. Let us construct allocation $\mathcal{Y}$ in the following way: If v_j is true, we set

$$
\begin{array}{rcllll}
\boldsymbol{y}(p_j^1) & = & (c(p_j^1), p_j^1); & \boldsymbol{y}(p_j^2) = (p_j^2, c(p_j^2)); & \\
\boldsymbol{y}(q_j^1) & = & (q_j^1, r_j); & \boldsymbol{y}(q_j^2) = (r_j, q_j^2); & \boldsymbol{y}(r_j) = (q_j^2, q_j^1)
\end{array}
$$

and if v_j is false, we have

$$
\begin{array}{rcllll}
\boldsymbol{y}(q_j^1) & = & (c(q_j^1), q_j^1); & \boldsymbol{y}(q_j^2) = (q_j^2, c(q_j^2)); & \\
\boldsymbol{y}(p_j^1) & = & (p_j^1, r_j); & \boldsymbol{y}(p_j^2) = (r_j, p_j^2); & \boldsymbol{y}(r_j) = (p_j^2, p_j^1)
\end{array}
$$

In each clause cell $\mathcal{E}_B(C_i)$ we assign $\boldsymbol{y}(c_i^k) = V(c_i^k)$ for each agent c_i^k that corresponds to a position containing true literal. Since each clause is satisfied by f, either one, two or all three c-agents of in $\mathcal{E}_B(C_i)$ are already assigned. For the other agents of $\mathcal{E}_B(C_i)$ do the following. If just one c-agent, say c_i^k, has not yet been assigned, put

$$\boldsymbol{y}(c_i^k) = (c_i^k, z_i);\ \boldsymbol{y}(z_i) = (z_i, c_i^k);\ \boldsymbol{y}(t_i^1) = (t_i^1, t_i^2);\ \boldsymbol{y}(t_i^2) = (t_i^2, t_i^1).$$

If two c-agents, say c_i^k and c_i^ℓ, have not yet been assigned, put

$$\boldsymbol{y}(c_i^k) = (c_i^k, z_i);\ \boldsymbol{y}(c_i^\ell) = (z_i, c_i^\ell); \boldsymbol{y}(z_i) = (c_i^\ell, c_i^k);\ \boldsymbol{y}(t_i^1) = (t_i^1, t_i^2);\ \boldsymbol{y}(t_i^2) = (t_i^2, t_i^1).$$

If all the c-agents have been assigned, put

$$\boldsymbol{y}(z_i) = (z_i, t_i^1);\ \boldsymbol{y}(t_i^1) = (t_i^1, t_i^2);\ \boldsymbol{y}(t_i^2) = (t_i^2, z_i).$$

Clearly, all agents have improved compared to their endowments.

Now suppose that there exists an allocation $\mathcal{Y}$ in which all agents $a \in A$ strictly prefer the bundle $\boldsymbol{y}(a)$ to the bundle $\boldsymbol{g}(a)$. Then since in each clause cell $\mathcal{E}_B(C_i)$, at most two c-agents can improve by getting the good of agent z_i, at least one of them, say c_i^k will be linking, i. e. receive the bundle $V(c_i^k)$. Now let us look at $\mathcal{E}_B(v_j)$. If agent p_j^1 or agent p_j^2 is linking, then agent r_j must receive the bundle (q_j^2, q_j^1), and so neither agent q_j^1 nor agent q_j^2 can be linking. Similarly, if agent q_j^1 or agent q_j^2 is linking, then agent r_j must receive the bundle (p_j^2, p_j^1), and so neither agent p_j^1 nor agent p_j^2 can be linking. Now it is clear that when setting v_j true in the former case and v_j false in the latter case, the truth assignment will be consistent and formula B satisfied. □

Problem PO-TEST
Instance. An economy $\mathcal{E}$ and an allocation $\mathcal{X}$ for $\mathcal{E}$.
Question. Is $\mathcal{X}$ not Pareto optimal for $\mathcal{E}$?

Corollary 4.1. Problem PO-TEST is NP-complete even in the case $Q = 2$.

Proof. The statement of the corollary is implied by Theorem 3, as problem PO-TEST is a special case of BETTER ALLOCATION. □

5. CONCLUSION AND OPEN PROBLEMS

The results of this paper contribute to a better understanding of the computational complexity issues arising in markets with indivisible goods. They show that the polynomiality results for Shapley–Scarf economy are very much dependent from its special structure. As a complement to hardness results of [4] implied by equivalence of goods owned by some agents we show that it is NP-hard to decide the nonemptyness of the core of the economy and Pareto optimality of a given allocation if agents are allowed to own only one unit of several types of goods.

As this is the first paper dealing with computational complexity in this model, there are still many open questions and here we suggest at least some of them.

1. The size of the description of the economy grows exponentially with the number of types of goods, since the preference list of an agent can contain up to N^Q entries. Are there some interesting cases with a succinct representation? One possibility is suggested in [6]: for a complete description of additively separable preferences, just N^2Q numbers are needed (utility values of each agent for each good). However, additively separable preferences form a relatively small class, moreover, even for them the complexity of the core existence problem is not resolved. Another possibility would be to consider a different form of separable preferences, e. g. obtained by extending linear orderings of goods of one type to preferences over bundles in some way.
2. A *competitive equilibrium* of an economy is a pair of two objects: prices for each good and an allocation such that each agent is assigned the best bundle (according to his preferences) he can afford at the current prices, when he sells his endowment. Konishi, Quint and Wako [6] constructed an example that does not admit any competetive equilibrium (Example 3.3), but the complexity of the existence problem for the competitive equilibrium remains open.

3. Even if the number of agents N as well as the number of types of goods Q are very small, the number of different ecomomies is huge: for $N = 3$ and $Q = 2$ it is equal to $(9!)^3$. The proof of core nonemptyness for this case presented in [6] (Proposition 2.1) uses a detour through NTU games and the deep Scarf's Theorem. It would be very interesting to find a purely combinatorial proof of this assertion.

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