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UNIVARIATE CONDITIONING OF COPULAS

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The univariate conditioning of copulas is studied, yielding a construction method for copulas based on an a priori given copula. Based on the gluing method, g-ordinal sum of copulas is introduced and a representation of copulas by means of g-ordinal sums is given. Though different right conditionings commute, this is not the case of right and left conditioning, with a special exception of Archimedean copulas. Several interesting examples are given. Especially, any Ali–Mikhail–Haq copula with a given parameter $\lambda > 0$ allows to construct via conditioning any Ali–Mikhail–Haq copula with parameter $\mu \in [0, \lambda]$.

Keywords: conditioning, gluing, g-ordinal sum, construction of copulas
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1. INTRODUCTION

Bivariate truncation of copulas was introduced and studied by Charpentier and Juri [3] in the framework of bivariate conditioning of (bivariate) copulas, see also [8, 9] showing a prominent role of the strict members of Clayton family of copulas. Recently, n-ary extensions of truncation were discussed in [1]. In this paper, we discuss the univariate conditioning of bivariate copulas. Though our approach is based on the representation of 2-increasing aggregation functions by means of copulas given in [5, 6], it turns out that the formula for univariate conditioning is a special case of Charpentier–Juri truncation. However, observe that while our formula can be applied to any copula, the approach introduced in [3] deals with copulas having strictly increasing horizontal and vertical sections only. Observe, that these restrictions can be relaxed, see [4]. When studying copulas invariant with respect to univariate conditioning, also some nonstrict Archimedean copulas should be considered, especially the Fréchet–Hoeffding bound W. Note that the class of copulas invariant with respect to univariate conditioning is larger than the class of bivariate truncation-invariant copulas.

The paper is organized as follow. In the next section, the univariate conditioning is introduced and some examples are given. Section 3 is devoted to the representation of conditional copulas by means of a generalization of gluing construction recently introduced [13]. In Section 4, the relations among left and right conditioning are considered, including the case of Archimedean copulas conditioning. Finally, some concluding remarks are given.

2. UNIVARIATE CONDITIONING OF COPULAS

Recall that a (bivariate) copula $C : [0,1]^2 \to [0,1]$ is a function with annihilator 0 ($C(x,0) = C(0,x) = 0$ for all $x \in [0,1]$), neutral element 1 ($C(x,1) = C(1,x) = x$ for all $x \in [0,1]$) satisfying the 2-increasing property (supermodularity) $C(\boldsymbol{x} \vee \boldsymbol{y}) + C(\boldsymbol{x} \wedge \boldsymbol{y}) \geq C(\boldsymbol{x}) + C(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^2$, where $\vee$ and $\wedge$ are the standard lattice operations on $[0,1]^2$, see [14, 12]. An aggregation function $A : [0,1]^2 \to [0,1]$ is a nondecreasing function such that $A(1,1) = 1$ and $A(0,0) = 0$, see [2]. Hence copulas are 2-increasing aggregation functions with neutral element 1.

Note that copulas can be understood as bivariate distribution function of a random vector $\boldsymbol{Z} = (X,Y)$ with marginals uniformly distributed over $[0,1]$, $C(x,y) = \Pr(X \leq x, Y \leq y)$. Let $g : [0,1] \to [0,1]$ be a continuous nondecreasing function satisfying $g(0) = 0$, $g(1) > 0$.

For a given copula C, evidently the function $A_{C,g} : [0,1]^2 \to [0,1]$ given by

$$A_{C,g}(x,y) = \frac{C(x, g(y))}{g(1)} \tag{1}$$

is a 2-increasing aggregation function with annihilator 0, and with continuous margins $\varphi, \eta : [0,1] \to [0,1]$, $\varphi(x) = \frac{C(x,g(1))}{g(1)}$ and $\eta(y) = \frac{g(y)}{g(1)}$. Due to [6], there is a copula D such that

$$A_{C,g}(x,y) = D(\varphi(x), \eta(y)).$$

Consequently, $D(u,v) = A_{C,g}(\varphi^{(-1)}(u), \eta^{(-1)}(v))$. Here the pseudo-inverse $\varphi^{(-1)} : [0,1] \to [0,1]$ is given by $\varphi^{(-1)}(x) = \sup\{t \in [0,1] \mid \varphi(t) < x\}$, and similarly $\eta^{(-1)} : [0,1] \to [0,1]$ is given by $\eta^{(-1)}(x) = \sup\{t \in [0,1] \mid \eta(t) < x\}$, see [10]. Observe that φ depends on C and $g(1) = \alpha \in]0,1]$ only and that $g(\eta^{(-1)}(v)) = g(g^{(-1)}(g(1)v)) = g(1)v$, and thus

$$D(u,v) = \frac{C(\varphi^{(-1)}(u), g(\eta^{(-1)}(g(1)v)))}{g(1)} = \frac{C(\varphi^{(-1)}(u), \alpha v)}{\alpha}.$$

The previous formula shows that the copula D depends on the a priori given copula C and the constant $\alpha \in]0,1[$ only, and formally it can be therefore seen as the copula of the conditional distribution of (X,Y) given that $Y \leq \alpha$, see [3]. Evidently, if $\alpha = 1$, then $D = C$.

Definition 1. Let $C : [0,1]^2 \to [0,1]$ be a copula and $\alpha \in]0,1[$. The copula $C_{(\alpha)} : [0,1]^2 \to [0,1]$ given by

$$C_{(\alpha)}(u,v) = \frac{C(\varphi^{(-1)}(u), \alpha\, v)}{\alpha},$$

where $\varphi^{(-1)}(u) = \sup\{t \in [0,1] \mid C(t,\alpha) < \alpha\, u\}$ is called right α-conditional copula of C. Similarly, the copula

$$C_{[\alpha]}(u,v) = \frac{C(\alpha\, u, \eta^{(-1)}(v))}{\alpha},$$

where $\eta^{(-1)}(v) = \sup\{t \in [0,1] | C(\alpha,\, t) < \alpha\, v\}$, is called left α-conditional copula of C.

Analogously, $C_{[\alpha]}$ can be seen as the copula of the conditional distribution of (X,Y) given that $X \leq \alpha$.

Example 1.

(i) Consider the Fréchet–Hoeffding bound W given by $W(x,y) = \max\{0, x+y-1\}$. For $\alpha \in]0,1[$,

$$\varphi^{(-1)}(u) = \sup\{t \in [0,1] | t + \alpha - 1 < \alpha\, u\} = 1 - \alpha + \alpha\, u$$

and thus

$$W_{(\alpha)}(u,v) = \frac{W(1-\alpha+\alpha\, u, \alpha\, v)}{\alpha} = \frac{\max\{0, \alpha\, u + \alpha\, v - \alpha\}}{\alpha} = W(u,v),$$

i. e., W is invariant with respect to α-conditioning for each $\alpha \in]0,1[$.

(ii) Ali–Mikhail–Haq copula $C : [0,1]^2 \to [0,1]$ with parameter $\lambda = 1$ is given by

$$C(x,y) = \frac{x\, y}{1 + (1-x)\,(1-y)}.$$

Let $\alpha \in]0,1[$, then

$$\varphi^{(-1)}(u) = \frac{u\,(2-\alpha)}{1 + u\,(1-\alpha)}$$

$$C_\alpha(u,v) = \frac{C\left(\frac{u\,(2-\alpha)}{1+u\,(1-\alpha)}, \alpha\, v\right)}{\alpha} = \frac{u\, v}{1 + \frac{\alpha}{2-\alpha}(1-u)\,(1-v)},$$

i. e., all members of Ali–Mikhail–Haq family with parameter $\mu \in [0,1]$ can be obtained by its conditioning (for the limit member one should take the pointwise limit). In general, starting from an Ali–Mikhail–Haq copula C with parameter $\lambda > 0$, any Ali–Mikhail–Haq copula with parameter $\mu \in]0,\lambda]$ can be obtained by conditioning. Due to the continuity of the Ali–Mikhail–Haq family in parameter, also the boundary case $\mu = 0$ can be obtained as $\lim_{\alpha\to 0^+} C_\alpha$.

(iii) Conditioning of a symmetric copula need not be symmetric (and vice versa). For example, for a singular copula with support on segments connecting points $\left(0,\frac{1}{2}\right)$ with $\left(\frac{1}{2},1\right)$, and $\left(\frac{1}{2},0\right)$ with $\left(1,\frac{1}{2}\right)$ (i. e., the strongest copula with diagonal section $\delta(x) = \max\{0, 2x-1\} = \delta_W(x)$), the conditional copula $C_{\left(\frac{3}{4}\right)}$ is a singular copula with support on segments connecting the points $\left(0,\frac{2}{3}\right)$ with $\left(\frac{1}{3},1\right)$, and $\left(\frac{1}{3},0\right)$ with $\left(1,\frac{2}{3}\right)$. Evidently, C is symmetric while $C_{\left(\frac{3}{4}\right)}$ not.

Note that the conditioning based on α and β commutes.

Proposition 1. For any copula C and $\alpha, \beta \in]0,1[$, $\left(C_{(\alpha)}\right)_{(\beta)} = \left(C_{(\beta)}\right)_{(\alpha)} = C_{(\alpha\beta)}$.

Proof. Denote $\varphi_\alpha(x) = \frac{C(x,\alpha)}{\alpha}$, $\varphi_{\alpha\,\beta}(x) = \frac{C(x,\alpha\,\beta)}{\alpha\,\beta}$ and $\varphi_{\alpha,\,\beta}(x) = \frac{C_\alpha(x,\beta)}{\beta}$. Then the equality $\left(C_{(\alpha)}\right)_{(\beta)} = C_{(\alpha\,\beta)}$ is equivalent to $\varphi_{\alpha,\,\beta} \circ \varphi_\alpha = \varphi_{\alpha\,\beta}$. To see the last equality, it holds

$$\varphi_{\alpha,\,\beta} \circ \varphi_\alpha(x) = \varphi_{\alpha,\,\beta}\left(\frac{C(x,\alpha)}{\alpha}\right) = \frac{C_\alpha\left(\frac{C(x,\alpha)}{\alpha},\beta\right)}{\beta} = \frac{C\left(\varphi_\alpha\left(\frac{C(x,\alpha)}{\alpha}\right),\alpha\,\beta\right)}{\alpha\,\beta}$$

$$= \frac{C\left(\sup\left\{t\in[0,1]\,|\,\frac{C(t,\alpha)}{\alpha} < \frac{C(x,\alpha)}{\alpha}\right\},\alpha\,\beta\right)}{\alpha\,\beta} = \frac{C(x,\alpha\,\beta)}{\alpha\,\beta} = \varphi_{\alpha\,\beta}(x).$$

Similarly, the equality $\left(C_{(\beta)}\right)_{(\alpha)} = C_{(\alpha\,\beta)}$ can be shown. □

Remark 1. There is also a probabilistic proof of Proposition 1. Indeed, the copula $C_{(\alpha)}$ is linked to joint distribution function $F_{(\alpha)}$, $F_{(\alpha)}(x,y) = \frac{C(x,\alpha y)}{\alpha}$ (compare (1)), while the copula $(C_{(\alpha)})_{(\beta)}$ is linked to joint distribution function $(F_{(\alpha)})_{(\beta)}(x,y) = \frac{F_{(\alpha)}(x,\beta y)}{\beta} = \frac{C(x,\alpha\,\beta y)}{\alpha\,\beta} = F_{(\alpha\,\beta)}(x,y)$.

3. G–ORDINAL SUMS AND CONDITIONING

Recently, a gluing construction method for copulas was introduced in [13].

Proposition 2. Let $n \in \mathbb{N}$, $0 = a_0 < a_1 < \ldots a_n = 1$, and $C_1, \ldots, C_n$ be copulas. Then the function $C : [0,1]^2 \to [0,1]$ given by

$$C(x,y) = a_{i-1}y + (a_i - a_{i-1})C_i\left(\frac{x-a_{i-1}}{a_i - a_{i-1}}, y\right) \text{ if } x \in [a_{i-1}, a_i]$$

is a copula.

Note that $C(a_i, y) = a_i\, y = \Pi(a_i,\, y)$ for $i = 0, 1, \ldots, n$. Similarly to the ordinal sum of copulas, gluing methods can be introduced also for an infinite number of intervals (and copulas), the result being both a construction method and a representation.

Theorem 1. A function $C : [0,1]^2 \to [0,1]$ is a copula if and only if there is a disjoint system $(]a_j,\, b_j[)_{j\in J}$ of nonempty open subintervals of $[0,\, 1]$ (thus J is at most countable) and a system $(C_j)_{j\in J}$ of copulas such that for each $C_j, j \in J$, and for each $x \in]0,\, 1[$ there is $y_{x,j} \in]0,1[$ such that $C_j(x,\, y_{x,j}) \neq x y_{x,j}$ (copulas with no trivial product vertical section), so that

$$C(x,y) = \begin{cases} a_j y + (b_j - a_j)C_j\left(\frac{x-a_j}{b_j-a_j}, y\right) & \text{if } x \in]a_j, b_j[, \\ xy & \text{otherwise.} \end{cases} \quad (2)$$

P r o o f. The sufficiency is trivial, following the same ideas as in [13]. To see the necessity, denote $S = \{x \in [0,1] \mid C(x,y) = xy \text{ for all } y \in [0,1]\}$. Evidently, S is a closed subset of $[0,1]$ containing as trivial members 0 and 1. Then the complement $[0,1] \setminus S$ is an open (possibly empty) subset of $[0,1]$ and hence there is a disjoint system $(]a_j,\, b_j[)_{j\in J}$ such that $S = \bigcup_{j\in J}]a_j, b_j[$.

For $j \in J$, define $C_j : [0,1]^2 \to [0,1]$ by

$$C_j(x,y) = \frac{C(a_j + (b_j - a_j)x, y) - a_j y}{b_j - a_j}.$$

Then C_j is a copula and the representation (2) of C is immediate. Moreover, $S_j = \{x \in [0,1] \mid C_j(x,y) = xy \text{ for all } y \in [0,1]\} = \{0,1\}$ is trivial for all $j \in J$. □

Note that the construction (2) can be applied to any system $(C_j)_{j\in J}$ of copulas, still yielding a copula C. This construction will be called *g-ordinal sum* (gluing ordinal sum), with notation $C = g - (\langle a_j, b_j, C_j \rangle \mid j \in J)$. Formally, J can be also empty and then $C = \Pi$ is the product copula. Observe that g-ordinal sums belongs to patchwork techniques for copulas studied recently by [7]. Moreover, the idea of g-ordinal sums (based on the product copula Π) is similar to the idea of ordinal sums based on M, or W-ordinal sums based on W, see [11]. In all cases, the first step is based on the set of all elements x of $[0,1]$ for which the vertical sections of copula C coincide with the vertical section of the background copula. Note that Durante, Saminger-Platz and Sarkoci [7] have used the construction (2) as a rectangular patchwork with the notation $\langle (a_j, b_j, C_j) \rangle^{\Pi}_{j\in J}$. To stress the representation part (2) and its relationships with ordinal sums based on M and W, we prefer to call (2) a g-ordinal sum. Based on Theorem 1, it is not difficult to show the next results.

Corollary 1. A nontrivial g-ordinal sum copula C with all summands equal to C_j, $C_j = C$ is necessarily the product copula, $C = \Pi$. Moreover, a g-ordinal sum copula C is PQD (positive quadrant dependent, see [12]) if and only if all summands C_j are PQD.

Corollary 2. A g-ordinal sum copula C is absolutely continuous if and only if all its summands C_j are absolutely continuous. The same holds for singular copulas, but additionally we should require that the intervals $([a_j, b_j])_{j\in J}$ form a covering of the unit interval [0,1].

Based on ([13], Theorem 3.2) one can show the next result.

Proposition 3. Let $C = g - (\langle a_j, b_j, C_j \rangle \mid j \in J)$, then Spearmann rho

$$\rho_C = \sum_{j\in J} (b_j - a_j)^2 \rho_{C_j}.$$

Similarly, for Kendall tau we have

$$\tau_C = \sum_{j\in J} (b_j - a_j)^2 \tau_{C_j}.$$

g-ordinal sums are well compatible with the conditioning of copulas.

Theorem 2. Let $C = g - (\langle a_j, b_j, C_j\rangle \mid j \in J)$ be a g-ordinal sum copula and $\alpha \in]0,1[$. Then $C_{(\alpha)} = g - (\langle a_j, b_j, C_{j(\alpha)}\rangle \mid j \in J)$.

Proof. The proof is based on the commuting of pseudo-inverse operation and increasing affine transform. It holds $C_{(\alpha)}(a_j, y) = a_j y$ for all $j \in J,\ y \in [0,1]$, and similarly $C_{(\alpha)}(b_j, y) = b_j y$. Due to $C(a_j, y) = a_j y$ we have that, for a fixed $\alpha \in]0,1[,\ \varphi^{(-1)}(a_i) = \sup\{t \in [0,1] \mid C(t,\alpha) \le \alpha a_i\} = c \le a_i$ and $C(c,\alpha) = \alpha a_i$. However, then the volume $V_C([c, a_i] \times [0,\alpha]) = 0$, i.e., $C(c,y) = C(a_i, y) = a_i y$ for all $y \in [0,1]$. Similarly, $C_{(\alpha)}(b_j, y) = b_j y,\ y \in [0,1],\ j \in J$. Consequently, $C_{(\alpha)}$ can be represented as a g-ordinal sum, $C_{(\alpha)} = g - (\langle a_j, b_j, D_j\rangle \mid j \in J)$ for some copulas $D_j,\ j \in J$.

For $x \in [0,1]\setminus\bigcup_{j\in J}]a_j, b_j[,\ C(x,y) = C_{(\alpha)}(x,y) = g - (\langle a_j, b_j, C_{j(\alpha)}\rangle \mid j \in J)(x,y) = xy$. If $x \in]a_k, b_k[$ for some $k \in J$, and $C(x,\alpha) = C(a_k,\alpha) = a_k\alpha$, evidently $C\left(\frac{x-a_k}{b_k-a_k}, \alpha\right) = 0$ and $C_{(\alpha)}(x,y) = g - (\langle a_j, b_j, C_{j(\alpha)}\rangle \mid j \in J)(x,y) = a_k y$. Finally, if $x \in]a_k, b_k[$ for some $k \in J$ and $C(x,\alpha) > \alpha x_k$, i.e., $C\left(\frac{x-a_k}{b_k-a_k}, \alpha\right) > 0$, then $\varphi^{(-1)}(x) = \sup\left\{t \in [0,1] \mid \frac{C(t,\alpha)}{\alpha} < x\right\} \in [a_k, b_k]$ and thus, for all $y \in [0,1]$

$$
\begin{aligned}
C_{(\alpha)}(x,y) &= \frac{C(\varphi^{(-1)}(x), \alpha y)}{\alpha} = a_k y + \frac{(b_k - a_k) C_k\left(\frac{\varphi^{(-1)}(x) - a_k}{b_k - a_k}, \alpha y\right)}{\alpha} \\
&= a_k y + \frac{(b_k - a_k) C_k\left(\varphi_k^{(-1)}\left(\frac{x-a_k}{b_k-a_k}\right), \alpha y\right)}{\alpha} \\
&= a_k y + (b_k - a_k) C_{k(\alpha)}\left(\frac{x - a_k}{b_k - a_k}, y\right) \\
&= g - (\langle a_j, b_j, C_{j(\alpha)}\rangle \mid j \in J)(x,y),
\end{aligned}
$$

where $\varphi_k^{(-1)}(u) = \sup\left\{t \in [0,1] \mid \frac{C(t,\alpha)}{\alpha} < u\right\}$.

Thus, the proof is complete. □

4. LEFT AND RIGHT CONDITIONING OF COPULAS

As already shown in Proposition 1, $(C_{(\alpha)})_{(\beta)} = (C_{(\beta)})_{(\alpha)} = C_{(\alpha\,\beta)}$. Similarly we can show $(C_{[\alpha]})_{[\beta]} = (C_{[\beta]})_{[\alpha]} = C_{[\alpha\,\beta]}$. Moreover, $C_{(\alpha)}(x,y) = C_{[\alpha]}(y,x)$ whenever the copula C is symmetric (then $C_{(\alpha)}$ need not be symmetric, in general).

However, the left and the right conditioning of copulas do not commute, in general. This can be checked easily in the Cuadras–Augé family, taking as C any of its proper (not boundary) member. Moreover, this example shows also that $(C_{[\alpha]})_{(\beta)}$ as well as $(C_{(\beta)})_{[\alpha]}$ differs from Charpentier–Juri [3] conditioning $C_{[\alpha,\,\beta]}$, also in cases when all conditional copulas are well-defined (the first two copulas are always defined, while so is the third one only for copulas with strictly increasing horizontal and vertical sections, excluding the boundary sections).

Example 2. Let $C = M^{\frac{1}{2}}\Pi^{\frac{1}{2}}$ be the Chadras–Angé copula with parameter $\frac{1}{2}$. Then $C_{(\alpha)} = g-(\langle 0, \sqrt{\alpha}, C\rangle)$, $C_{[\alpha]}(x,y) = C_{(\alpha)}(y,x)$, $(C_{(\alpha)})_{[\beta]} = C_{(\frac{\alpha}{\beta^2})}$ if $\alpha \le \beta^2$ and $(C_{(\alpha)})_{[\beta]} = C_{[\frac{\beta}{\sqrt{\alpha}}]}$ if $\alpha > \beta^2$. Similarly, $(C_{[\beta]})_{(\alpha)} = C_{[\frac{\beta}{\alpha^2}]}$ if $\beta \le \alpha^2$ and $(C_{[\beta]})_{(\alpha)} = C_{[\frac{\alpha}{\sqrt{\beta}}]}$ if $\beta > \alpha^2$. Thus $(C_{(\frac{1}{4})})_{[\frac{1}{2}]} = C$ while $(C_{[\frac{1}{2}]})_{(\frac{1}{4})} = C_{(\frac{\sqrt{2}}{4})}$. Moreover, applying Charpentier–Juri approach [3], when copula $C_{[\alpha,\beta]}$ is related to the distribution function $F(x,y) = \frac{C(\beta x, \alpha y)}{C(\beta,\alpha)}$, $C(\beta,\alpha) > 0$, we have in our case $C_{[\alpha,\beta]} = C_{(\frac{\alpha^2}{\beta^2})}$ if $\alpha \le \beta$ and $C_{[\alpha,\beta]} = C_{(\frac{\beta^2}{\alpha^2})}$ if $\alpha \ge \beta$. Hence $C_{[\frac{1}{4},\frac{1}{2}]} = C_{(\frac{1}{4})}$.

We introduce another example of a copula C for which the right and the left conditioning do not commute.

Example 3. Let $C : [0,1]^2 \to [0,1]$ be a singular copula with support on segments connecting points $\left(0,\frac{1}{2}\right)$ with $\left(\frac{1}{2},1\right)$, and $\left(\frac{1}{2},\frac{1}{2}\right)$ with $(1,0)$. Then $C_{(0.5)} = W$, $C_{[0.5]} = M$ (the upper Fréchet–Hoeffding bound), and thus $(C_{(0.5)})_{[0.5]} = W \ne M = (C_{[0.5]})_{(0.5)}$.

Nevertheless, for strict Archimedean copulas we have the next important result connecting the left and right types of univariate conditioning of copulas.

Theorem 3. Let $C : [0,1]^2 \to [0,1]$ be a strict Archimedean copula, i. e., there is a decreasing convex bijection $f : [0,1] \to [0,\infty]$ such that

$$C(x,y) = f^{-1}(f(x) + f(y)). \tag{3}$$

Then for any $\alpha, \beta \in]0,1[$, it holds:

(i) $C_{(\alpha)} = C_{[\alpha]}$

(ii) $(C_{[\alpha]})_{(\beta)} = (C_{(\beta)})_{[\alpha]} = C_{(\alpha\,\beta)}$.

P r o o f.

(i) Recall that $C_{(\alpha)}(u,v) = \frac{C\left(\sigma^{(-1)}(u),\alpha v\right)}{\alpha}$,
where $\sigma^{(-1)}(u) = \sup\left\{t \in [0,1] \mid \frac{C(t,\alpha)}{\alpha} < u\right\}$. Due to (3), $\frac{C(t,\alpha)}{\alpha} = \frac{f^{-1}(f(t)+f(\alpha))}{\alpha}$ and thus $\sigma^{-1}(u) = f^{-1}(f(\alpha u) - f(\alpha))$. Consequently,

$$C_{(\alpha)}(u,v) = \frac{f^{-1}\left(f(\alpha u) + f(\alpha v) - f(\alpha)\right)}{\alpha}. \tag{4}$$

Observe that the family (C_α) given in (4) is, in general, a new parametric family of copulas (up to special case of invariant copulas).

Similarly, $C_{[\alpha]}(u,v) = \frac{f^{-1}(f(\alpha u)+f(\alpha v)-f(\alpha))}{\alpha}$, i. e., $C_{(\alpha)} = C_{[\alpha]}$.

(ii) This is a corollary of (i) and Proposition 1. □

Example 4. For $\lambda \geq 1$, the function $f_\lambda : [0,1] \to [0,\infty]$ given by $f_\lambda(x) = \left(\frac{1-x}{x}\right)^\lambda$ is an additive generator of a copula C_λ. The family $\left((C_\lambda)_{(\alpha)}\right)_{\alpha \in [0,1]}$ is given by

$$(C_\lambda)_{(\alpha)}(u,v) = \frac{1}{\alpha + \sqrt{\left(\left(\frac{1-\alpha u}{u}\right)^\lambda + \left(\frac{1-\alpha v}{v}\right)^\lambda - (1-\alpha)^\lambda\right)}^{\frac{1}{\lambda}}}$$

for $\alpha > 0$, and the limit member

$$(C_\lambda)_{(0)} = \lim_{\alpha \to 0^+} (C_\lambda)_{(\alpha)}$$

is the Clayton copula with parameter λ,

$$(C_\lambda)_{(0)}(u,v) = (u^{-\lambda} + v^{-\lambda} - 1)^{-\frac{1}{\lambda}}.$$

Remark 2.

(i) As already shown in Example 2, for any symmetric copula C it holds $C_{[\alpha]}(x,y) = C_{(\alpha)}(y,x)$, compare also Theorem 3 (i). However, $(C_{[\alpha]})_{(\beta)} = (C_{(\beta)})_{[\alpha]}$ need not hold for symmetric copulas in general, see Example 2.

(ii) Observe that also for non-strict copula C generated by an additive generator f, it can be shown that

$$C_{[\alpha]}(u,v) = C_{(\alpha)}(u,v) = f^{(-1)}\left(\min(f(0),\ f(\alpha u) + f(\alpha v) - f(\alpha))\right),\ \alpha \in]0,1[.$$

(iii) For any Archimedean copula C generated by an additive generator f, the copulas $C_{[\alpha]} = C_{(\alpha)}$, $\alpha \in]0,1[$, are again Archimedean and they are generated by an additive generator $f_\alpha : [0,1] \to [0,\infty]$ given by

$$f_\alpha(x) = f(\alpha x) - f(\alpha).$$

Compare [8], Proposition 3.2 for extreme tail dependence copulas.

Consequently, for an associative copula C, also $C_{(\alpha)}$ is associative for all $\alpha \in]0,1[$. To be more precise, if $C = (\langle a_j, b_j, C_j \rangle \mid j \in J)$ is an ordinal sum with Archimedean summands C_j generated by additive generators f_j, $j \in J$, then

$$C_{(\alpha)} = \left(\left\langle \frac{a_j}{\alpha}, \frac{b_j}{\alpha}, C_j \right\rangle \mid j \in J,\ b_j \leq \alpha)\right),$$

if $\alpha \in [0,1] \setminus \bigcup_{j \in J}]a_j, b_j[$, and if $\alpha \in]a_k, b_k[$ for some $k \in J$, then

$$C_{(\alpha)} = \left(\left\langle \frac{a_k}{\alpha}, 1, C_k^* \right\rangle, \left\langle \frac{a_j}{\alpha}, \frac{b_j}{\alpha}, C_j \right\rangle \mid j \in J,\ b_j < \alpha\right),$$

where $C_k^* = (C_k)_{\left(\frac{\alpha - a_k}{b_k - a_k}\right)}$.

5. CONCLUDING REMARKS

We have studied a new method for constructing copula families from any given 2-copula C. This method preserves some special classes of copulas. Indeed, if C is absolutely continuous (singular, associative), then, for each $\alpha \in]0,1[$, also $C_{(\alpha)}$ is absolutely continuous (singular, associative). As a by-product, we have introduced the concept of g-ordinal sums, which is closely related to conditioning. Indeed, for any g-ordinal sum $C = g - (\langle a_j, b_j, C_j \rangle \mid j \in J)$ and any $\alpha \in [0,1] \setminus \bigcup_{j \in J}]a_j, b_j[$ it holds

$$C_{[\alpha]} = g - \left(\left\langle \frac{a_j}{\alpha}, \frac{b_j}{\alpha}, C_j \right\rangle \,\middle|\, j \in J,\, b_j \leq \alpha \right).$$

Especially, if $\alpha_k = 0$ for some $k \in J$, then $C_{[\beta_k]} = C_k$.

In our next investigation, we aim to discuss copulas invariant with respect to conditioning, i.e., copulas such that $C_{(\alpha)} = C$ ($C_{[\alpha]} = C$) for all $\alpha \in]0,1[$.

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REFERENCES

[1] A. Ahmadi Javid: Copulas with truncation-invariance property. Comm. Statist. A – Theory Methods, to appear.

[2] T. Calvo, A. Kolesárová, M. Komorníková, and R. Mesiar: Aggregation Operators: Properties, Classes and Construction Methods. (Studies in Fuzziness and Soft Computing – Aggregation Operators, New Trend and Applications.) Physica-Verlag, Heidelberg 2002, pp. 3–106.

[3] A. Charpentier and A. Juri: Limiting dependence structures for tail events, with applications to credit derivatives. J. Appl. Probab. *43* (2006), 563–586.

[4] F. Durante, B. Foschi, and S. Spizzichino: Threshold copulas and positive dependence. Statist. Probab. Lett., to appear.

[5] F. Durante, R. Mesiar, P. L. Papini, and C. Sempi: 2-Increasing binary aggregation operators. Inform. Sci. *177* (2007), 111–129.

[6] F. Durante, S. Saminger-Platz, and P. Sarkoci: On representations of 2-increasing binary aggregation functions. Inform. Sci. *178* (2008), 4534–4541.

[7] F. Durante, S. Saminger-Platz, and P. Sarkoci: Rectangular patchwork for bivariate copulas and tail dependence. Comm. Statist. A – Theory Methods, to appear.

[8] A. Juri and M. V. Wüthrich: Copula convergence theorems for tail events. Insurance Math. Econom. *30* (2002), 405–420.

[9] A. Juri and M. V. Wüthrich: Tail dependence from a distributional point of view. Extreme *6* (2003), 213–246.

[10] E. P. Klement, R. Mesiar, and E. Pap: Quasi- and pseudo-inverses of monotone functions, and the construction of t-norms. Fuzzy Sets and Systems *104* (1999), 3–14.

[11] R. Mesiar and J. Szolgay: W-ordinal sums of copulas and quasi-copulas. In: Proc. MAGIA 2004 Conference, Kočovce 2004, pp. 78–83.

[12] R. B. Nelsen: An Introduction to Copulas. Second edition. Springer, Berlin 2006.

[13] K. F. Siburg and P. A. Stoimenov: Gluing copulas. Comm. Statist. A – Theory Methods *37* (2008), 3124–3134.

[14] M. Sklar: Fonctions de répartition à n dimensions et leurs marges. Publ. Inst. Statist. Univ. Paris *8* (1959), 229–231.

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