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ON QUASI–HOMOGENEOUS COPULAS

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Quasi-homogeneity of copulas is introduced and studied. Quasi-homogeneous copulas are characterized by the convexity and strict monotonicity of their diagonal sections. As a by-product, a new construction method for copulas when only their diagonal section is known is given.

Keywords: copula, diagonal section, quasi-homogeneity

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1. INTRODUCTION

Homogeneity of order k of real functions reflects their regularity with respect to the inputs with the same ratio in the form

$$F(\lambda x_1, \dots, \lambda x_n) = \lambda^k F(x_1, \dots, x_n). \tag{1}$$

In several classes of special functions, such as *triangular norms* or *copulas*, the homogeneity is a rather restrictive property. A generalized homogeneity should reflect the multiplicative constant λ as well as the original value $F(x_1, \dots, x_n)$, and thus it should be expressed on the form

$$F(\lambda x_1, \dots, \lambda x_n) = G(\lambda, F(x_1, \dots, x_n)) \tag{2}$$

where G is a binary function. In [9], the concept of quasi-homogeneity was introduced by considering $G(a, b) = \varphi^{-1}(f(a)\varphi(b))$, with φ an injective function and f an arbitrary function. Hence a function F is called quasi-homogeneous if

$$F(\lambda x_1, \dots, \lambda x_n) = \varphi^{-1}(f(\lambda)\varphi(F(x_1, \dots, x_n))). \tag{3}$$

The aim of this paper is to discuss the class of *quasi-homogeneous* copulas. In the next section, we recall some preliminary notions and results on homogeneity of t-norms and copulas, and on quasi-homogeneity of t-norms. In Section 3, we represent quasi-homogeneous copulas by means of their diagonal sections, while in Section 4 we characterize all diagonal sections of quasi-homogeneous copulas. As a consequence, a new construction method for copulas when only their diagonal section is known, is obtained. Finally several concluding remarks are included.

2. PRELIMINARIES

We will suppose the reader to be familiar with some basic concepts and results on copulas, that can be found in [15]. Recall that a binary function $C : [0,1]^2 \to [0,1]$ is said to be a *copula* if it satisfies the following properties:

C1) $C(x,0) = C(0,x) = 0$ for all $x \in [0,1]$,

C2) $C(x,1) = C(1,x) = x$ for all $x \in [0,1]$,

C3) for all x, x', y, y' in $[0,1]$ with $x \leq x'$ and $y \leq y'$,

$$C(x',y') - C(x,y') - C(x',y) + C(x,y) \geq 0.$$

The weakest copula is the Łukasiewicz copula whereas the strongest one is the minimum. They are respectively given by

$$W(x,y) = \max\{0, x+y-1\} \qquad \text{and} \qquad M(x,y) = \min\{x,y\}$$

for all $x, y \in [0,1]$.

Similarly, basic notions on *t-norms* are also assumed and they can be found in [14]. Recall that a binary function $T : [0,1]^2 \to [0,1]$ is said to be a t-norm if it is associative, commutative, non-decreasing in each variable and has neutral element 1, that is $T(x,1) = T(1,x) = x$ for all $x \in [0,1]$. Thus, we only recall here some definitions and results that will be used in the paper.

Definition 1. A function $F : [0,1]^2 \to [0,1]$ is said to be *homogeneous* of degree $k > 0$ if it satisfies

$$F(\lambda x, \lambda y) = \lambda^k F(x,y) \qquad \text{for all} \quad x, y, \lambda \in [0,1]. \tag{4}$$

The homogeneity condition has been characterized for t-norms as well as for copulas and the results are as follows.

Theorem 1. (Alsina et al. [2], Theorem 3.4.1) A t-norm T is homogeneous if and only if either $k = 1$ and T is the minimum t-norm, or $k = 2$ and T is the product t-norm.

Theorem 2. (Nelsen [15], Theorem 3.4.2) A copula C is homogeneous if and only if $1 \leq k \leq 2$ and C is the member C_θ of the Cuadras–Augé family with $\theta = 2 - k$.

Recall that the Cuadras–Augé family is the parametric family of copulas given by

$$C_\theta(x,y) = (\min\{x,y\})^\theta (xy)^{1-\theta} \qquad \text{for all} \quad x, y \in [0,1]$$

with $\theta \in [0,1]$.

Some generalizations of the homogeneity condition have been studied, specially in the framework of t-norms (see [2]). One of these generalizations is introduced by substituting λ^k by any arbitrary function $f : [0,1] \to [0,1]$, but this leads to no new solutions for t-norms (see [2], Corollary 3.4.2). The widest generalization of homogeneity introduces the so-called quasi-homogeneity in the following terms.

Definition 2. A function $F : [0,1]^2 \to [0,1]$ is said to be *quasi-homogeneous* if there exists a continuous, strictly monotonic function $\varphi : [0,1] \to \mathbb{R}$ and a function $f : [0,1] \to [0,1]$ such that

$$F(\lambda x, \lambda y) = \varphi^{-1}(f(\lambda)\varphi(F(x,y))) \qquad \text{for all} \quad x, y, \lambda \in [0,1]. \tag{5}$$

In this case it will be said that F is (φ, f)-quasi-homogeneous.

Quasi-homogeneous t-norms have been also characterized allowing for new solutions. The result is due to Ebanks in 1998 (see [9]), see also [2] for the current version.

Theorem 3. (Alsina et al. [2], Theorem 3.4.3) A t-norm T is quasi-homogeneous if and only if it is a member of the family T_α with $0 \leq \alpha \leq +\infty$, where

$$T_\alpha(x,y) = \begin{cases} (x^{-\alpha} + y^{-\alpha} - 1)^{-1/\alpha} & \text{if } \min\{x,y\} > 0 \\ 0 & \text{otherwise} \end{cases}$$

for α such that $0 < \alpha < +\infty$, and $T_0 = T_{\mathbf{P}}$ is the product t-norm and $T_{+\infty} = T_{\mathbf{M}}$ is the minimum t-norm.

Here, $f_\alpha(\lambda) = \lambda^c$ with arbitrary $c > 0$ for all $\alpha \in [0, +\infty]$, and the φ_α are given by

$$\varphi_\alpha(x) = k(1 + x^\alpha)^{-c/\alpha}, \qquad \text{for} \quad 0 < \alpha < +\infty$$

and

$$\varphi_0(x) = kx^{c/2} \qquad \text{and} \qquad \varphi_{+\infty}(x) = kx^c.$$

3. QUASI–HOMOGENEOUS COPULAS

In this section we want to characterize quasi-homogeneous copulas, that is, those copulas that satisfy Definition 2. Firstly, let us deal with the easier generalization of homogeneity that consists in substituting λ^k by an arbitrary function f.

Proposition 1. Let $f : [0,1] \to [0,1]$ be an arbitrary function and let C be a copula such that

$$C(\lambda x, \lambda y) = f(\lambda) C(x,y) \qquad \text{for all} \quad x, y, \lambda \in [0,1].$$

Then $f(\lambda) = \lambda^k$ with $1 \leq k \leq 2$ and C is a member of the Cuadras–Augé family.

Proof. Taking $x = y = 1$ we have $C(\lambda, \lambda) = f(\lambda)$ for all $\lambda \in [0,1]$, that is, f has to be the diagonal section of C and, in particular, f must be continuous with $f(0) = 0$ and $f(1) = 1$. On the other hand,

$$f(\lambda x) = C(\lambda x, \lambda x) = f(\lambda) f(x)$$

for all $\lambda, x \in [0,1]$. Consequently, f must be of the form $f(\lambda) = \lambda^k$ for some $k > 0$ (see for instance [1]). That is C must be homogeneous of degree k and hence the result. □

Thus, as for the case of t-norms, no new solutions appear for copulas with this generalization. On the contrary, for the quasi-homogeneity condition, we will see that there are a lot of copulas satisfying Equation (5). First let us characterize the structure of such copulas.

Theorem 4. A copula C is quasi-homogeneous if and only if its diagonal section is strictly increasing and C is given by

$$C(x,y) = \begin{cases} \delta\left((x\vee y)\delta^{-1}\left(\frac{x\wedge y}{x\vee y}\right)\right) & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0). \end{cases} \tag{6}$$

In this case, C is (φ, f)-quasi-homogeneous with $f(\lambda) = \lambda^c$ and $\varphi(x) = \left(\delta^{-1}(x)\right)^c$ for arbitrary $c > 0$.

Proof. Let us consider a copula C that verifies equation (5) for a continuous strictly monotonic function $\varphi : [0,1] \to \mathbb{R}$ and an arbitrary function $f : [0,1] \to [0,1]$. We can write

$$\varphi(C(x,x)) = f(x)\varphi(C(1,1)) = f(x)\varphi(1)$$

for all $x \in [0,1]$. It is clear that if C satisfies equation (5) for functions φ, f it also satisfies it for functions $k\varphi, f$ with $k \neq 0$, and consequently we can suppose $\varphi(1) = 1$. Thus we have $f(x) = \varphi(C(x,x)) = \varphi(\delta(x))$ where δ is the diagonal section of C, and

$$f(xy) = \varphi(C(xy,xy)) = f(x)\varphi(C(y,y)) = f(x)f(y).$$

That is, f satisfies the multiplicative Cauchy equation and since φ is strictly monotonic this implies that also f is monotonic, and, hence that $f(\lambda) = \lambda^c$ for all $\lambda \in [0,1]$ with $c > 0$ (see again [1]). Thus C satisfies

$$\varphi(C(\lambda x, \lambda y)) = \lambda^c \varphi(C(x,y)) \quad \text{for all} \quad x,y,\lambda \in [0,1] \tag{7}$$

with $c > 0$. Now, taking $x = y = 1$ we obtain $\varphi(\delta(\lambda)) = \lambda^c$ which implies that δ must be strictly increasing and that $\varphi(x) = \left(\delta^{-1}(x)\right)^c$ with $c > 0$.

Finally, equation (7) can be written now as

$$\left(\delta^{-1}(C(\lambda x, \lambda y))\right)^c = \lambda^c \left(\delta^{-1}(C(x,y))\right)^c$$

or equivalently

$$\delta^{-1}(C(\lambda x, \lambda y)) = \lambda\delta^{-1}(C(x,y))$$

for all $x, y, \lambda \in [0,1]$. If we consider the function $F : [0,1]^2 \to [0,1]$ defined by $F(x,y) = \delta^{-1}(C(x,y))$ we obtain that F is homogeneous of degree 1. Moreover, whereas $\max\{x,y\} > 0$ we can write

- if $x \leq y$
$$F(x,y) = F\left(y\frac{x}{y}, y\right) = yF\left(\frac{x}{y}, 1\right) = y\delta^{-1}\left(C\left(\frac{x}{y}, 1\right)\right) = y\delta^{-1}\left(\frac{x}{y}\right)$$
- if $y \leq x$
$$F(x,y) = F\left(x, x\frac{y}{x}\right) = xF\left(1, \frac{y}{x}\right) = x\delta^{-1}\left(C\left(1, \frac{y}{x}\right)\right) = x\delta^{-1}\left(\frac{y}{x}\right).$$

That is, F is given by

$$F(x,y)=\begin{cases}(x\vee y)\delta^{-1}\left(\frac{x\wedge y}{x\vee y}\right) & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

where $\vee$ stands for maximum and $\wedge$ for minimum. Thus C must be given by equation (6).

Reciprocally, if C is a copula given by equation (6) with diagonal section δ strictly increasing then clearly C is quasi-homogeneous with functions $f(\lambda)=\lambda^c$ and $\varphi(x)=(\delta^{-1}(x))^c$ with $c>0$. □

Remark 1. i) Due to special properties of copulas, it is possible to relax the requirements of continuity and strict monotonicity of the function φ in Definition 2 into the requirements that $Rang\,(\varphi)$ contains at least three elements.

ii) Observe that a function $S:[0,1]^2\to[0,1]$ is called a semi-copula whenever it is non-decreasing in both coordinates and $S(1,x)=S(x,1)=x$ for all $x\in[0,1]$ (see [8]). A semi-copula $Q:[0,1]^2\to[0,1]$ is called a quasi-copula (see [8, 12, 13] or [15]) if it is 1-Lipschitz, i. e.,

$$|Q(x_1,y_1)-Q(x_2,y_2)|\leq|x_1-x_2|+|y_1-y_2| \qquad \text{for all} \quad x_1,x_2,y_1,y_2\in[0,1].$$

Note that Proposition 1 as well as Theorem 4 can be applied to continuous semi-copulas and quasi-copulas without any modification.

iii) Note also that all quasi-homogeneous copulas are symmetric in view of the representation (6).

From the previous theorem, it is clear that for finally characterizing quasi-homogeneous copulas we only need to find those admissible diagonals of copulas, that are strictly increasing and for which equation (6) effectively gives a copula. This will be done in next section, but in the more general case where the function δ needs not to be strict.

4. DIAGONAL SECTIONS OF QUASI–HOMOGENEOUS COPULAS

Given a copula C it is well known that its diagonal section is a function $\delta:[0,1]\to[0,1]$ that satisfies:

d1) $\delta(x)\leq x$ for all $x\in[0,1]$ with $\delta(0)=0$ and $\delta(1)=1$,

d2) δ is non-decreasing,

d3) δ is 2-Lipschitz, i. e. $|\delta(x)-\delta(y)|\leq 2|x-y|$ for all $x,y\in[0,1]$.

Let us denote by $\boldsymbol{D}$ the set of all functions $\delta:[0,1]\to[0,1]$ that can be the diagonal section of a copula, that is, satisfying conditions from d1) to d3). In general, there are a lot of copulas with the same diagonal section $\delta\in\boldsymbol{D}$. In this sense many authors have studied, fixing a function $\delta\in\boldsymbol{D}$, how to construct a copula

C with diagonal section δ. This has been done in different manners and contexts (see [7, 10, 11] obtaining respectively Bertino copulas, diagonal copulas, MT-copulas and so on (see [5]).

Our interest now is to study in what cases a copula C can be obtained from its diagonal through equation (6). In fact, note that such equation can be generalized for diagonals $\delta \in \boldsymbol{D}$ in general, not necessarily strictly increasing, by using the pseudo-inverse function $\delta^{(-1)}$. Specifically, given a non-decreasing function $\delta : [0,1] \to [0,1]$ its pseudo-inverse $\delta^{(-1)} : [0,1] \to [0,1]$ is given by (see [14])

$$\delta^{(-1)}(x) = \sup\{t \in [0,1] \mid \delta(t) \leq x\} \qquad \text{for all} \quad x \in [0,1]. \tag{8}$$

Now we can study when a copula C can be constructed from its diagonal through the expression

$$C_{(\delta)}(x,y) = \delta\left((x \vee y)\delta^{(-1)}\left(\frac{x \wedge y}{x \vee y}\right)\right) \qquad \text{for all} \quad (x,y) \neq (0,0) \tag{9}$$

and $C(0,0) = 0$. This generalization is important because it will allow us to obtain many more representative examples. For instance the following one.

Example 1. The weakest copula $W(x,y) = \max\{x+y-1, 0\}$ has diagonal section δ_W given by $\delta_W(x) = \max\{2x-1, 0\}$ and it can be constructed from δ through equation (9). Note that however W is not quasi-homogeneous since its diagonal is not strictly increasing.

Theorem 5. Let $\delta \in \boldsymbol{D}$. If δ is convex then the binary operation $C_{(\delta)}$ given by equation (9) is a (commutative) copula with diagonal section δ.

Proof. Evidently, $C_{(\delta)}(x,1) = C_{(\delta)}(1,x) = x, C_{(\delta)}(x,0) = C_{(\delta)}(0,x) = 0$ and $C_{(\delta)}(x,y) = C_{(\delta)}(y,x)$ for all $x,y \in [0,1]$. Thus, the only thing to show $C_{(\delta)}$ is a copula is its 2-increasingness. Denote

$$a = \sup\{x \in [0,1] \mid \delta(x) = 0\},$$

since δ is convex it must be strictly increasing on $[a,1]$. We denote by d^{-1} the inverse of $\delta : [a,1] \to [0,1]$, then $d^{-1} : [0,1] \to [a,1]$ is given by

$$d^{-1}(x) = \delta^{(-1)}(x) = \sup\{z \in [0,1] \mid \delta(z) \leq x\},$$

where $\delta^{(-1)}$ is the pseudo-inverse of δ (see (8)) and $C_{(\delta)}(x,y)$ can be written as

$$C_{(\delta)}(x,y) = \delta\left((x \vee y)d^{-1}\left(\frac{x \wedge y}{x \vee y}\right)\right)$$

for all $(x,y) \neq (0,0)$. It is easy to see that for $y \leq x$, it is

$$C_{(\delta)}(x,y) = 0 \qquad \Longleftrightarrow \qquad y \leq x\delta\left(\frac{a}{x}\right) \quad \text{or} \quad x \leq a.$$

Moreover, for $y < x$, $C_{(\delta)}$ is non-decreasing in y, and this fact together with the symmetry of $C_{(\delta)}$ reduces the cases for 2-increasingness to be checked for two cases:

i) 2-increasingness on squares whose diagonal is on the main diagonal of $[0,1]^2$,

ii) 2-increasingness on rectangles contained in the region where $C_{(\delta)}$ is positive for $y < x$.

In the first case it should be shown that for $0 \le u < v \le 1$ it holds

$$\delta(u) + \delta(v) - 2\delta\left(vd^{-1}\left(\frac{u}{v}\right)\right) \ge 0. \tag{10}$$

Since $\delta \ge \delta_W$ we have

$$\delta\left(\frac{u+v}{2v}\right) \ge 2\frac{u+v}{2v} - 1 = \frac{u}{v}, \qquad \text{that is,} \qquad \frac{u+v}{2} \ge vd^{-1}\left(\frac{u}{v}\right),$$

and thus

$$v - vd^{-1}\left(\frac{u}{v}\right) \ge vd^{-1}\left(\frac{u}{v}\right) - u.$$

The convexity of δ then ensures

$$\delta(v) - \delta\left(vd^{-1}\left(\frac{u}{v}\right)\right) \ge \delta\left(vd^{-1}\left(\frac{u}{v}\right)\right) - \delta(u)$$

which is equivalent to (10).

To prove case ii), one should show that for $[u,u'] \times [v,v']$ in positive area of $C_{(\delta)}$ and such that $v' \le u$, it holds

$$\delta\left(ud^{-1}\left(\frac{v}{u}\right)\right) + \delta\left(u'd^{-1}\left(\frac{v'}{u'}\right)\right) - \delta\left(ud^{-1}\left(\frac{v'}{u}\right)\right) - \delta\left(u'd^{-1}\left(\frac{v}{u'}\right)\right) \ge 0 \tag{11}$$

Due to concavity of d^{-1}, the function $h(x) = d^{-1}(x)/x$ is decreasing and it holds

$$\frac{d^{-1}\left(\frac{v'}{u'}\right) - d^{-1}\left(\frac{v}{u'}\right)}{\frac{v'-v}{u'}} \ge \frac{d^{-1}\left(\frac{v'}{u}\right) - d^{-1}\left(\frac{v}{u}\right)}{\frac{v'-v}{u}},$$

that is,

$$u'\left(d^{-1}\left(\frac{v'}{u'}\right) - d^{-1}\left(\frac{v}{u'}\right)\right) \ge u\left(d^{-1}\left(\frac{v'}{u}\right) - d^{-1}\left(\frac{v}{u}\right)\right).$$

Now, due to convexity of δ and decreasingness of h, the last inequality reads

$$\delta\left(u'd^{-1}\left(\frac{v'}{u'}\right)\right) - \delta\left(u'd^{-1}\left(\frac{v}{u'}\right)\right) \ge \delta\left(ud^{-1}\left(\frac{v'}{u}\right)\right) - \delta\left(ud^{-1}\left(\frac{v}{u}\right)\right)$$

which is equivalent to (11). □

Example 2. Fix $\alpha \in [0, 1/2]$ and let $\delta_\alpha : [0,1] \to [0,1]$ be the convex function in $\boldsymbol{D}$ given by

$$\delta_\alpha(x) = \begin{cases} 0 & \text{if } x \le \alpha \\ \frac{x-\alpha}{1-\alpha} & \text{otherwise.} \end{cases}$$

By applying the previous theorem to these functions δ_α, we obtain a family of parametric copulas $C_{(\delta_\alpha)}$ given by

$$C_{(\delta_\alpha)}(x,y) = \max\left\{0, \frac{\alpha(x\vee y)+(1-\alpha)(x\wedge y)-\alpha}{1-\alpha}\right\},$$

with boundary members $C_{(\delta_0)} = M$ and $C_{(\delta_{1/2})} = W$. In Figure 1 we can see the parametric family of diagonals (δ_α) with $\alpha \in [0,1/2]$, and the corresponding copulas $C_{(\delta_\alpha)}$.

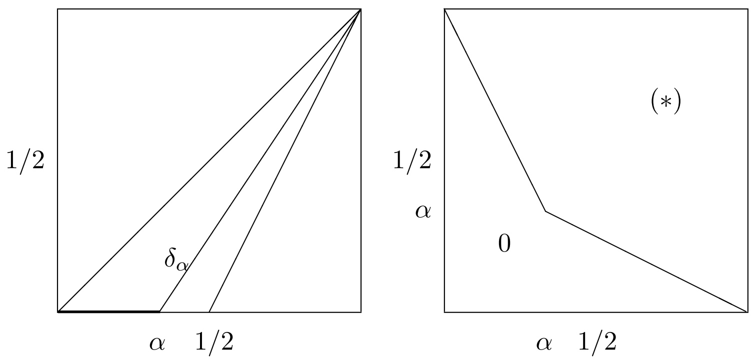

Fig 1. Parametric family of diagonals (δ_α) (left) and copulas $C_{(\delta_\alpha)}$ (right) of Example 2, where $(*)$ stands for $\frac{\alpha(x\vee y)+(1-\alpha)(x\wedge y)-\alpha}{1-\alpha}$.

Theorem 6. Let $\delta \in \boldsymbol{D}$ be strictly increasing. Then the binary operation $C_{(\delta)}$ given by equation (6) is a (commutative) copula with diagonal section δ if and only if δ is convex.

Proof. From the previous theorem we only need to prove that when $C_{(\delta)}$ is a copula then δ must be convex. But if $C_{(\delta)}$ is a copula (quasi-copula is enough) it is 1-Lipschitz, i.e.,

$$C_{(\delta)}(x,y_2) - C_{(\delta)}(x,y_1) \le y_2 - y_1 \qquad \text{for all} \quad x, y_1, y_2 \text{ with } y_1 \le y_2.$$

For $z \in]0,1[$ and $\varepsilon \in]0, 1-z[$ put

$$x = \frac{z}{z+\varepsilon}, \quad y_1 = \frac{z\delta(z)}{z+\varepsilon}, \quad y_2 = \frac{z\delta(z+\varepsilon)}{z+\varepsilon}.$$

Then, since δ is strictly increasing we have $\delta^{(-1)} = \delta^{-1}$, and thus

$$C_{(\delta)}(x,y_2) - C_{(\delta)}(x,y_1) = \delta(z) - \delta\left(\frac{z^2}{z+\varepsilon}\right) \le y_2 - y_1 = \frac{\delta(z+\varepsilon)-\delta(z)}{\frac{z+\varepsilon}{z}}.$$

Note that $\frac{z^2}{z+\varepsilon} = z - \varepsilon \frac{z}{z+\varepsilon}$ and thus the equation above can be written as

$$\frac{\delta(z) - \delta\left(z - \varepsilon\frac{z}{z+\varepsilon}\right)}{\varepsilon\frac{z}{z+\varepsilon}} \le \frac{\delta(z+\varepsilon) - \delta(z)}{\varepsilon}. \tag{12}$$

Finally, since δ is 2-Lipschitz, it has continuous derivative on a union of open subintervals of $[0,1]$ of the form $\cup_{k\in K}]a_k, b_k[$ with $\sum_{k\in K}(b_k - a_k) = 1$. This fact together with equation (12) implies the convexity of δ. □

Corollary 1. Let $C : [0,1]^2 \to [0,1]$ be a binary operation with continuous diagonal $\delta(x) = C(x,x)$. Then C is a quasi-homogeneous copula if and only if δ is a strictly increasing convex function and C is given by equation (6).

Example 3. Fix $k \in [0,1[$ and α such that $\max\{0, 2k-1\} \le \alpha \le k$. Let $\delta_{k,\alpha} : [0,1] \to [0,1]$ be the convex strictly increasing function in $\boldsymbol{D}$ given by

$$\delta_{k,\alpha}(x) = \begin{cases} \frac{\alpha x}{k} & \text{if } x \le k \\ \frac{\alpha-1}{k-1}x + \frac{k-\alpha}{k-1} & \text{otherwise.} \end{cases}$$

By applying Theorem 6 to these functions $\delta_{k,\alpha}$, we obtain a family of two-parametric quasi-homogeneous copulas $C_{(\delta_{k,\alpha})}$ given by $C_{(\delta_{k,\alpha})}(x,y) =$

$$\begin{cases} x \wedge y & \text{if } (x \vee y) \ge \alpha(x \wedge y) \\ x \wedge y + \frac{\alpha-k}{k-1}(x \vee y) + \frac{k-\alpha}{k-1} & \text{if } \frac{(k-1)(x\wedge y)+(\alpha-k)(x\vee y)}{\alpha-1} \le k \\ \frac{\alpha}{k(\alpha-1)}((k-1)(x\wedge y) + (\alpha-k)(x \vee y)) & \text{otherwise} \end{cases}$$

with boundary member $C_{(\delta_{0,0})} = M$ and whose limit when $k \to 1$ is given by the weakest copula W. This parametric family for the case $k = 1/2$ and $0 \le \alpha \le 1/2$ can be viewed in Figure 2.

Remark 2. In view of the fact that the class of all diagonals sections of copulas coincides with the class of all diagonal sections of quasi-copulas, it can be shown that there are no proper quasi-homogeneous quasi-copulas, i. e., each quasi-homogeneous quasi-copula is necessarily a copula. On the other hand, a continuous semi-copula S is quasi-homogeneous and given by (6) if and only if its diagonal section $\delta : [0,1] \to [0,1]$ given by $\delta(x) = S(x,x)$ is an automorphism of $[0,1]$ such that the function $h\colon]0,1] \to [0,1]$ given by $h(x) = \delta(x)/x$ is non-decreasing. Put, for example, $\delta(x) = x^c$ with $c > 0$. Then $h(x) = x^{c-1}$ is non-decreasing for $c \ge 1$, and the corresponding semi-copula S is given by

$$S(x,y) = (x \wedge y)(x \vee y)^{c-1} \qquad \text{for all} \quad x, y \in [0,1].$$

Recall that S is a copula (quasi-copula) only if δ is 2-Lipschitz, i. e., if $c \in [1,2]$ (and then it belongs to Cuadras–Augé family).

Similarly, $\delta(x) = \max\{x/3, 3x-2\}$ is an automorphism of $[0,1]$ such that $h(x) = \max\{1/3, 3-2/x\}$ is increasing. The corresponding semi-copula S is given on the triangle determined by points $(1,1/4), (1,1)$ and $(3/4,3/4)$ by $S(x,y) = y + 2x - 2$ and thus it is not a copula (quasi-copula).

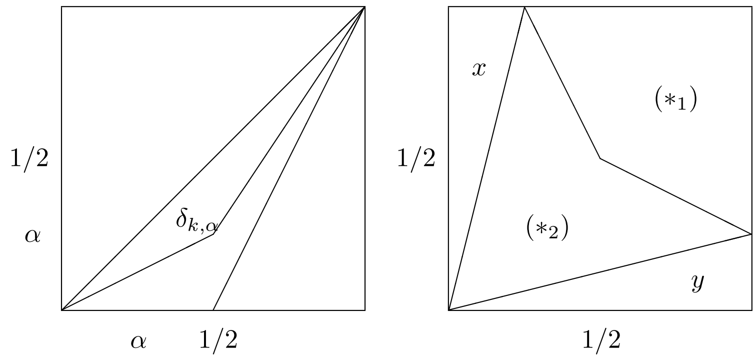

Fig. 2. Parametric family of diagonals $(\delta_{k,\alpha})$ (left) and copulas $C_{(\delta_{k,\alpha})}$ (right) of Example 3 with $k = 1/2$, where $(*_1)$ stands for $x \wedge y + (1 - 2\alpha)(x \vee y) + 2\alpha - 1$ and $(*_2)$ stands for $\frac{\alpha}{1-\alpha}((x \wedge y) + (1 - 2\alpha)(x \vee y))$.

5. CONCLUDING REMARKS

We have completely solved the problem of representing quasi-homogeneous copulas by means of their diagonal sections. Moreover, a new method of constructing copulas from convex diagonal sections was introduced. Recall that there are several methods of constructing a copula when a diagonal section $\delta : [0, 1] \to [0, 1]$ (non-decreasing, 2-Lipschitz, bounded from above by the identity function and $\delta(1) = 1$) is given. The weakest copula $C_{[\delta]} : [0, 1]^2 \to [0, 1]$ such that $C_{[\delta]}(x, x) = \delta(x)$ is the so-called Bertino copula given by

$$C_{[\delta]}(x, y) = (x \wedge y) - \min\{t - \delta(t) \mid t \in [x \wedge y, x \vee y]\}$$

see [3, 11], or [13]. On the other hand, the diagonal copula $C_\delta : [0, 1]^2 \to [0, 1]$ introduced in [10] and given by

$$C_\delta(x, y) = \min\left\{x, y, \frac{\delta(x) + \delta(y)}{2}\right\}$$

is the strongest symmetric copula satisfying $C_\delta(x, x) = \delta(x)$ (but not necessarily the strongest copula with diagonal section δ).

Other methods known from the literature, see, e. g., [4, 5] or [7], are restricted to special classes of diagonal sections. Observe that in the case of our construction method (restricted to convex diagonal sections), the only diagonal copula C_δ coinciding with $C_{(\delta)}$ is the strongest copula M ($\delta = id$ is the only diagonal section related to the unique copula $C = M$). On the other hand, the only Bertino copulas which can be obtained by our construction are related to a parametric class $(\delta_a)_{a \in [0,1/2]}$ of diagonal sections given by

$$\delta_a(x) = \max\left(0, \frac{x - a}{1 - a}\right)$$

and the corresponding copulas $C_{[\delta_a]} = C_{(\delta_a)} : [0,1]^2 \to [0,1]$ are given by

$$C_{(\delta_a)}(x,y) = \max\left\{0, (x \wedge y) + \frac{a}{1-a}((x \vee y) - 1)\right\}$$

with boundary members $C_{(\delta_0)} = M$ and $C_{(\delta_{1/2})} = W$.

For any copula $C : [0,1]^2 \to [0,1]$ and a diagonal section $\delta \in \boldsymbol{D}$, the function $C^{(\delta)} : [0,1]^2 \to [0,1]$ given by

$$C^{(\delta)}(x,y) = C\left(\delta(x \vee y), \frac{x \wedge y}{x \vee y}\right) \qquad \text{for all} \quad x,y \in [0,1] \tag{13}$$

(with the convention $0/0 = 1$) is a function fulfilling the boundary properties of copulas, and $C^{(\delta)}(x,x) = \delta(x)$. Note that the same is satisfied if we alternatively take the function

$$C^{(\delta)'}(x,y) = C\left(\frac{x \wedge y}{x \vee y}, \delta(x \vee y)\right) \qquad \text{for all} \quad x,y \in [0,1].$$

It is an interesting open problem for which C and δ also $C^{(\delta)}$ (or $C^{(\delta)'}$) is a copula. For the product copula Π, (13) can be written as

$$\Pi^{(\delta)}(x,y) = \delta(x \vee y)\,\frac{x \wedge y}{x \vee y} \qquad \text{for all} \quad x,y \in [0,1] \tag{14}$$

and the complete characterization of all copulas having the form $\Pi^{(\delta)}$ can be found in [6], where these copulas are called semilinear. Our representation of quasi-homogeneous copulas also contributes to the above mentioned open problem. Indeed, let $\delta \in \boldsymbol{D}$ be a convex strictly increasing diagonal section. Then δ^{-1} is concave and it is a multiplicative generator of a copula $D_\delta : [0,1]^2 \to [0,1]$, $D_\delta(x,y) = \delta(\delta^{-1}(x)\delta^{-1}(y))$. However, then

$$D_\delta^{(\delta)}(x,y) = D_\delta\left(\delta(x \vee y), \frac{x \wedge y}{x \vee y}\right) = \delta\left((x \vee y)\delta^{-1}\left(\frac{x \wedge y}{x \vee y}\right)\right) = C^{(\delta)}(x,y)$$

is a quasi-homogeneous copula (see Theorem 6).

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