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WEIGHTED SCALARIZATION RELATED TO L_p-METRIC AND PARETO OPTIMALITY

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Relations between (proper) Pareto optimality of solutions of multicriteria optimization problems and solutions of the minimization problems obtained by replacing the multiple criteria with L_p-norm related functions (depending on the criteria, goals, and scaling factors) are investigated.

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1. INTRODUCTION

The problem of finding "simultaneous maxima" of a family of real functions f_i ($1 \le i \le m$) over a set X is well known as multicriteria optimization problem. While each function f_i may have a maximum (say, f_i^*) on X, there may not exist a point in X at which all the functions attain their maxima. Nevertheless, under certain mild conditions there will exist points $\boldsymbol{x} \in X$ such that moving away from $\boldsymbol{x}$ in any "direction" within X will always decrease the value of some f_i (or keep all values the same). Points with this property are known as Pareto optimal (or, undominated) solutions.

The rather unprecise term in quotation marks therefore allows for various interpretations and numerous approaches to its "meaning" have been developed in the literature; we refer to [1] and [2] for surveys. A particularly important approach consists in replacing the family of the m functions f_i by a single function G depending on all the f_i as well as on additional parameters. Some of the extra parameters may be "scaling factors" accompanying the functions f_i, others may include "goals", that is, certain target values g_i which may be related to the individual maxima f_i^*.

In [7] and later in [5, 8] the authors proposed to replace the "simultaneous maximization" of the above family of functions by minimization of the L_p-norm related function

$$G = \left(\sum_{i=1}^{m} (w_i |g_i - f_i(\boldsymbol{x})|)^p \right)^{1/p}$$

over all $\boldsymbol{x} \in X$ for some p such that $1 \leq p \leq \infty$, some chosen target values g_i $(1 \leq i \leq m)$, and some scaling factors $w_i \geq 0$ $(1 \leq i \leq m)$. The paper [7] also gives some evidence about universality (in a certain sense) of the choice of the replacement function G in the given form. Due to the presence of scaling factors such functions are sometimes called scalarization replacements.

Since G depends on the choice of target values and scaling factors, different choice in general results in different point of minima of G in X (all of which can be shown to be Pareto optimal). Therefore one may ask about the relationship between the undominated solutions of the original multicriteria optimization problem and points of minima of G. It is also interesting to ask about the ways a pre-assignment of goals and scaling factors affects solutions of the original problem.

We will present results related to both questions, confining ourselves to the most commonly studied case when, for $1 \leq i \leq m$, the maximum value f_i^* of each f_i on X exists and the goal value g_i is taken to be f_i^*. The necessary background to multicriteria optimization problems can be found in Section 2 together with a discussion of Pareto optimality and scalarization. In Section 3 we show that for any p such that $1 \leq p \leq \infty$ and for any given Pareto optimal point of the multicriteria optimization problem one can "manipulate" the scaling factors in such a way that the chosen point will be the point of minimum of the replacement function G. It is interesting to point out that one of our main tools in the proofs is Hölder's inequality, well known in functional analysis. A similar behaviour is presented in Section 4 for *proper* Pareto optimality if $1 \leq p < \infty$, with a discussion about the exceptional case $p = \infty$. This type of considerations and the corresponding results have apparently not been discussed in the available literature in the generality indicated above, except for cases when the functions are linear an X is a polyhedral region, see e. g. [3]. The last section contains a handful of remarks.

2. PARETO OPTIMALITY AND SCALARIZATION

Let X be a non-empty subset of $\mathbb{R}^n$ and, for $m \geq 2$ and all i with $1 \leq i \leq m$, let $f_i : X \to \mathbb{R}$ be functions. The *multicriteria optimization problem* is the task to find a "simultaneous maximum" over all the functions, that is, finding

$$\text{“max”} \ f_i(\boldsymbol{x}) \quad \text{subject to } \boldsymbol{x} \in X \ . \tag{1}$$

The set X is usually called the *feasible set*, the functions f_i are the *objectives*, or *criteria*, and the set $Y = \{(f_1(\boldsymbol{x}), \dots, f_m(\boldsymbol{x}));\ \boldsymbol{x} \in X\}$ is the *target set*. At this point we do not want to make any assumptions neither on the objectives nor on the feasible set or the target set. We note that terminology and notation varies in the literature; our basic reference monograph will be [1].

A number of concepts related to the problem (1) have been introduced and studied in the literature (see [1, 2] and references therein), most of which depend on the way the symbol "max" is interpreted. However, an important example of a concept independent on the meaning of maximization in (1) is Pareto optimality, which will be central to our study. A point $\boldsymbol{x}^* \in X$ is called *Pareto optimal* for the multicriteria optimization problem (1) if there is no $\boldsymbol{x} \in X$ such that

$$f_i(\boldsymbol{x}) \geq f_i(\boldsymbol{x}^*) \quad \text{for } 1 \leq i \leq m, \quad \text{with a strict inequality for some } i\ . \tag{2}$$

Pareto optimal points are also called *undominated solutions* of the problem (1).

Many variations of Pareto optimality have been studied, such as weak, strong, and proper (in the sense of Geoffrion, of Benson, or Borwein, or Kuhn–Tucker), and also variations with respect to any partial order on the set Y; for a detailed overview of related facts and results we again refer to [1]. In this paper we shall consider solely the variation known as proper Pareto optimality in the sense of Geoffrion, first introduced in [4]. A point $\boldsymbol{x}^* \in X$ is *properly Pareto optimal* for the problem (1) in the sense of Geoffrion if there exists a constant $d > 0$ such that for all i, $1 \leq i \leq m$, the system of inequalities

$$f_i(\boldsymbol{x}) > f_i(\boldsymbol{x}^*)\ , \quad \text{and} \quad f_i(\boldsymbol{x}) + df_j(\boldsymbol{x}) > f_i(\boldsymbol{x}^*) + df_j(\boldsymbol{x}^*) \quad \text{for all } j \neq i \tag{3}$$

has no solution in X. For brevity, we will just refer to proper Pareto optimality and omit the "in the sense of Geoffrion" appendix in what follows. An equivalent way to state the definition is to say that $\boldsymbol{x}^* \in X$ is properly Pareto optimal for the problem (1) if there exists a $d > 0$ such that for all $\boldsymbol{x} \in X \setminus \{\boldsymbol{x}^*\}$ and for all i, $1 \leq i \leq m$, there exists a j with $1 \leq j \leq m$ such that $f_i(\boldsymbol{x}) + df_j(\boldsymbol{x}) \leq f_i(\boldsymbol{x}^*) + df_j(\boldsymbol{x}^*)$.

Of course, proper Pareto optimality implies Pareto optimality. To see this, suppose that $\boldsymbol{x}^*$ is properly Pareto optimal for (1). If $f_i(\boldsymbol{x}) > f_i(\boldsymbol{x}^*)$ for some i and some $\boldsymbol{x} \in X$, then by (3) there exists a $j \neq i$ such that $f_j(\boldsymbol{x}) \leq f_j(\boldsymbol{x}^*) - d^{-1}(f_i(\boldsymbol{x}) - f_i(\boldsymbol{x}^*))$. This not only shows that $\boldsymbol{x}^*$ is Pareto optimal, but it also means that proper Pareto optimality implies a "gap" of a prescribed minimum size between $f_j(\boldsymbol{x})$ and $f_j(\boldsymbol{x}^*)$ provided that $f_i(\boldsymbol{x}) > f_i(\boldsymbol{x}^*)$.

Returning to specification of the meaning of the symbol "max" in the formulation of the problem (1), there appear to be two basic approaches [1, 2]: (i) introduction of a total order on the target set Y, and (ii) replacement of the collection of the criteria f_i, $1 \leq i \leq m$, by a single function (a *compromise criterion*) that in some way reflects the individual objectives. In this contribution we will focus on replacements stemming from weighted-scalarization type functions used in goal programming. In the most general form, in goal programming one first chooses a collection of "goal values" g_i for $1 \leq i \leq m$ and then replaces the "maximization" in (1) by maximization or minimization (depending on the setting) of a single criterion $G(f_1(\boldsymbol{x}), \ldots, f_m(\boldsymbol{x}))$ subject to $\boldsymbol{x} \in X$, where G depends also on the goal values (and possibly on additional parameters).

A criterion of the above type, claimed to be sufficiently universal, was proposed in [7] and, in an improved setting, in [5, 8] where the authors suggested to use a function G defined by

$$\min \left(\sum_{i=1}^{m} (w_i |g_i - f_i(\boldsymbol{x})|)^p \right)^{1/p} \qquad \text{subject to } \ \boldsymbol{x} \in X \tag{4}$$

where $1 \leq p \leq \infty$ is a chosen real number, the values g_i $(1 \leq i \leq m)$ are the chosen goals, and $w_i \geq 0$ $(1 \leq i \leq m)$ are predefined real numbers. In other words, the function to be minimized is just the well known L_p-norm distance function.

If $p < \infty$, one may of course disregard the pth root in (4), since the points of minima of the function (4) are the same as the points of minima of the corresponding function without the pth root. Nevertheless, we prefer to keep the pth root in (4) because it makes a difference when $p = \infty$. Indeed, by the usual convention in calculus that $p = \infty$ means the limit when $p \to \infty$, after taking the limit the sum in (4) reduces to $\sup_{1\le i\le m} w_i|g_i - f_i(\boldsymbol{x})|$ and the replacement problem reduces to determining

$$\min\ \left\{\sup_{1\le i\le m} w_i|g_i - f_i(\boldsymbol{x})|\right\} \quad \text{subject to } \boldsymbol{x} \in X\ . \tag{5}$$

At the other extreme, if $p = 1$ then the replacement problem is equivalent to determining the maximum of the function $w_1 f_1(\boldsymbol{x}) + w_2 f_2(\boldsymbol{x}) + \ldots + w_m f_m(\boldsymbol{x})$ subject to $\boldsymbol{x} \in X$, where $w_i \ge 0$ and not all the w_i are equal to zero, giving the well-known *weighted sum scalarization.* If necessary or if desirable, one can norm the weighs so that they sum to 1, but we will not do so in our exposition.

For completeness we note that the distance function (4) is a special case of a more general approach of replacing the multicriteria problem (1) with minimization of a single function of the form

$$\sum_{i=1}^{m} w_i \varphi_i(f_i(\boldsymbol{x})) \tag{6}$$

with $w_i \ge 0$, where, for $1 \le i \le m$, the domain of φ_i includes the range of f_i.

As stated in the Introduction, our point of interest is the study of effects of pre-setting goals and weights on the solutions of multicriteria optimization problems, with particular emphasis on Pareto optimality. The following are the two basic questions in this area: 1. What is the relationship between Pareto optimal solutions of (1) and solutions of the L_p-norm based goal programming replacement (4)? Secondly, what are the effects of pre-setting goals and weights in (4) on the relationship between Pareto optimal solutions of (1) and solutions of the goal programming replacement? In the next two sections we will address both questions, separately considering Pareto optimality (Section 3) and proper Pareto optimality (Section 4).

3. PARETO OPTIMALITY AND THE CHOICE OF WEIGHTS

We begin with a fairly general result concerning Pareto optimality of solutions of (1) and weighted scalarization of compositions of the objectives f_i in (1) with certain suitable functions. In most cases we will be assuming that the feasible set X is convex. Then, a real function g on X is *concave up* if $g(\lambda\boldsymbol{x} + \lambda'\boldsymbol{x}') \le \lambda g(\boldsymbol{x}) + \lambda' g(\boldsymbol{x}')$ for all $\boldsymbol{x}, \boldsymbol{x}' \in X$ and all non-negative real λ, λ' such that $\lambda + \lambda' = 1$. If the opposite inequality holds for all the parameters, g is called *concave down.*

Proposition 1. Assume that, in the problem (1), the feasible set is convex. Further, for $1 \le i \le m$ let φ_i be decreasing functions defined on the range of f_i such that the compositions $\varphi_i \circ f_i$ are all concave up on X. Then, for any Pareto optimal solution $\boldsymbol{x}^*$ of (1) there exist $w_i \ge 0$, $1 \le i \le m$, such that $\boldsymbol{x}^*$ minimizes the function $w_1\varphi_1(f_1(\boldsymbol{x})) + \ldots + w_m\varphi_m(f_m(\boldsymbol{x}))$ on X.

P r o o f. For $1 \le i \le m$ and all $\boldsymbol{x} \in X$, let $h_i(\boldsymbol{x}) = \varphi_i(f_i(\boldsymbol{x})) - \varphi_i(f_i(\boldsymbol{x}^*))$. It is easy to see that the system of inequalities $h_i(\boldsymbol{x}) < 0$, $1 \le i \le m$, has no solution in X. Indeed, assume the contrary and let $\boldsymbol{x}$ be such a solution. Then, $\varphi_i(f_i(\boldsymbol{x})) < \varphi_i(f_i(\boldsymbol{x}^*))$. As φ_i is decreasing for $1 \le i \le m$, it follows that $f_i(\boldsymbol{x}) > f_i(\boldsymbol{x}^*)$ for each i, contrary to Pareto optimality of $\boldsymbol{x}^*$. By an appropriate variation of Theorem 2.24 of [1] to functions that are concave up (with the corresponding inequalities reversed), there exist real numbers $w_i \ge 0$ such that $w_1 h_1(\boldsymbol{x}) + \ldots + w_m h_m(\boldsymbol{x}) \ge 0$ for all $\boldsymbol{x} \in X$. Consequently, $w_1\varphi_1(f_1(\boldsymbol{x})) + \ldots + w_m\varphi_m(f_m(\boldsymbol{x})) \ge w_1\varphi_1(f_1(\boldsymbol{x}^*)) + \ldots + w_m\varphi_m(f_m(\boldsymbol{x}^*))$ for all $\boldsymbol{x} \in X$. This means that the function $w_1\varphi_1(f_1(\boldsymbol{x})) + \ldots + w_m\varphi_m(f_m(\boldsymbol{x}))$ is minimized by $\boldsymbol{x}^*$ on X. □

From now on, assume that for each i, $1 \le i \le m$, the maximum value of f_i on X exists and is equal to f_i^*. Then, a particularly important application of Proposition 1 arises by taking $\varphi_i(y) = (f_i^* - y)^p$ for any fixed $p \ge 1$ and all i, $1 \le i \le m$. The function (6) then turns into $\sum_{i=1}^m w_i(f_i^* - f_i(\boldsymbol{x}))^p$, which (up to taking the pth root) gives the distance function (4) with goals equal to the individual maxima of the criteria. The functions φ_i are obviously decreasing. Moreover, if the criteria f_i are concave down on X, then each composition $\varphi_i \circ f_i$ is obviously concave up on X. We therefore have the following immediate corollary:

Proposition 2. Let $\boldsymbol{x}^*$ be a Pareto optimal solution of the multicriteria optimization problem (1) with a convex feasible set X, in which all the criteria are concave down on X. Assume that for each i, $1 \le i \le m$, the maximum of f_i on X exists and is equal to f_i^*. Then for any fixed p such that $1 \le p < \infty$ there exist non-negative weights w_i, $1 \le i \le m$, such that the function

$$\sum_{i=1}^{m} w_i(f_i^* - f_i(\boldsymbol{x}))^p\ , \quad \boldsymbol{x} \in X$$

is minimized at the point $\boldsymbol{x}^*$.

In the light of this result it is natural to ask if a similar result can be proved without the fairly restrictive assumptions on the objectives. The answer is straightforward if $p = 1$:

Proposition 3. Let the target set Y of (1) be convex. Let $\boldsymbol{x}^*$ be a Pareto optimal solution of (1). Further, suppose that for each i, $1 \le i \le m$, the maximum of f_i on X exists and is equal to f_i^*.Then there exist non-negative real numbers w_i, $1 \le i \le m$, such that the function

$$\sum_{i=1}^{m} w_i(f_i^* - f_i(\boldsymbol{x}))\ , \quad \boldsymbol{x} \in X$$

has minimum at $\boldsymbol{x}^*$.

P r o o f. Finding the minimum of the function above is equivalent to finding the maximum of $\sum_{i=1}^m w_i f_i(\boldsymbol{x})$ over $\boldsymbol{x} \in X$. The result now follows from Theorem 3.3

of [1], by which for any Pareto-optimal solution $\boldsymbol{x}^*$ of (1) with a convex target set Y there exist a non-negative linear combination $\sum_{i=1}^m w_i f_i(\boldsymbol{x})$ maximized at $\boldsymbol{x}^*$ over $\boldsymbol{x} \in X$. □

For the remaining values of p, that is, for $1 < p < \infty$, relaxation of the assumptions appearing in Proposition 2 becomes delicate. We offer here a relaxation obtained using Hölder's inequality which states that for any family of real numbers $a_i \geq 0$ and $b_i \geq 0$ $(1 \leq i \leq m)$ one has

$$\sum_{i=1}^m a_i b_i \leq \left(\sum_{i=1}^m a_i^p\right)^{1/p} \left(\sum_{i=1}^m b_i^q\right)^{1/q} , \tag{7}$$

where $p, q > 1$ are tied by $1/p + 1/q = 1$. Moreover, one has equality in (7) if and only if the vector (a_i^p) is a scalar multiple of the vector (b_i^q). In the case when entries of the second vector have sum equal to 1, the inequality reduces to

$$\sum_{i=1}^m a_i b_i \leq \left(\sum_{i=1}^m a_i^p\right)^{1/p} \quad \text{if} \quad \sum_{i=1}^m b_i^q = 1 . \tag{8}$$

For compatibility with Hölder's inequality we will keep the pth root in the expression for the L_p-distance function (4) in what follows.

Proposition 4. Let $\boldsymbol{x}^*$ be a Pareto optimal solution of the multicriteria optimization problem (1) with a convex target set Y. Assume that for each i, $1 \leq i \leq m$, the maximum of f_i on X exists and is equal to f_i^*. Then there exist non-negative real numbers v_i, $1 \leq i \leq m$, such that the L_p-distance function

$$\left(\sum_{i=1}^m (v_i(f_i^* - f_i(\boldsymbol{x}))^p\right)^{1/p} , \quad \boldsymbol{x} \in X$$

with $1 < p < \infty$ is minimized at $\boldsymbol{x}^*$.

Proof. For $1 \leq i \leq m$ let us define $a_i(\boldsymbol{x}) = f_i^* - f_i(\boldsymbol{x})$ for each $\boldsymbol{x} \in X$; note that $a_i(\boldsymbol{x})$ is always non-negative. If $a_i(\boldsymbol{x}^*) = 0$ for all i, $1 \leq i \leq m$, then there is nothing to prove. We may therefore assume that the set J of all those i for which $a_i(\boldsymbol{x}^*) > 0$ is non-empty. In this case we may replace the problem (1) with the reduced problem where only the constraints f_i such that $i \in J$ are considered for taking the "simultaneous maximum"; we will refer to this reduced problem as (1'). The reason we may do this is that any Pareto optimal solution of (1) is also a Pareto optimal solution of (1').

By one of our assumptions, the target set Y of (1) is convex. The target set Y' of the reduced problem (1') obtained by reducing all vectors in Y through deleting the coordinates with subscripts not in J is obviously convex as well. We may thus apply Proposition 3 to the reduced problem (1') to conclude that there exist non-negative numbers w_i, $i \in J$, such that the function $\sum_{i \in J} w_i a_i(\boldsymbol{x})$ subject to $\boldsymbol{x} \in X$

has minimum at $\boldsymbol{x}^*$. For $i \in J$ let $v_i = w_i^{1/p} a_i(\boldsymbol{x}^*)^{-1/q}$; note that v_i are well defined since we only consider the indices i that belong to J. Let $\alpha = \left(\sum_{i \in J}(v_i a_i(\boldsymbol{x}^*))^p\right)^{-1}$ where $1/p + 1/q = 1$. Setting $b_i = \alpha^{1/q}(v_i a_i(\boldsymbol{x}^*))^{p/q}$ for $i \in J$, one may verify that $\sum_{i \in J} b_i^q = 1$. The special form of Hölder's inequality (8) gives, for the same p and q,

$$\sum_{i \in J} v_i a_i(\boldsymbol{x}) \cdot b_i \le \left(\sum_{i \in J}(v_i a_i(\boldsymbol{x}))^p\right)^{1/p} . \tag{9}$$

The relations between the parameters α, v_i and b_i imply that the $|J|$-dimensional vector $(b_i^q;\ i \in J)$ is an α-multiple of the $|J|$-dimensional vector $((v_i a_i(\boldsymbol{x}^*))^p;\ i \in J)$. Consequently, there is equality in (9) if the case when $\boldsymbol{x} = \boldsymbol{x}^*$, that is,

$$\left(\sum_{i \in J}(v_i a_i(\boldsymbol{x}^*))^p\right)^{1/p} = \sum_{i \in J} v_i a_i(\boldsymbol{x}^*) \cdot b_i . \tag{10}$$

We now temporarily set $v_i = 0$ for all $i \notin J$, which says that in the distance function one only needs focusing on subscripts in J. We will subsequently show that the distance function in the statement of our Proposition attains its minimum at $\boldsymbol{x}^*$.

Relations between our symbols give $v_i b_i = \alpha^{1/q} w_i$ for $i \in J$. This means that for every $\boldsymbol{x} \in X$,

$$\sum_{i \in J} v_i a_i(\boldsymbol{x}) \cdot b_i = \alpha^{1/q} \sum_{i \in J} w_i a_i(\boldsymbol{x}) . \tag{11}$$

As $\boldsymbol{x}^*$ minimizes the function $\sum_{i \in J} w_i a_i(\boldsymbol{x})$ on $\boldsymbol{x} \in X$, by combining (10) with (11) we obtain

$$\left(\sum_{i \in J}(v_i a_i(\boldsymbol{x}^*))^p\right)^{1/p} = \sum_{i \in J} v_i a_i(\boldsymbol{x}^*) \cdot b_i \le \alpha^{1/q} \sum_{i \in J} w_i a_i(\boldsymbol{x}) = \sum_{i \in J} v_i a_i(\boldsymbol{x}) \cdot b_i . \tag{12}$$

Recalling that $\sum_{i \in J} b_i^q = 1$, we may use Hölder's inequality in the form (9) to bound the sum on the right-hand side of (12), which yields

$$\left(\sum_{i \in J}(v_i a_i(\boldsymbol{x}^*))^p\right)^{1/p} \le \sum_{i \in J} v_i a_i(\boldsymbol{x}) \cdot b_i \le \left(\sum_{i \in J}(v_i a_i(\boldsymbol{x}))^p\right)^{1/p} . \tag{13}$$

By the definition of the set J, the inequality formed by the leftmost and rightmost parts of (13) extends, for an arbitrary *positive* choice of the constants v_i for $i \notin J$, to

$$\left(\sum_{i=1}^{m}(v_i a_i(\boldsymbol{x}^*))^p\right)^{1/p} \le \left(\sum_{i=1}^{m}(v_i a_i(\boldsymbol{x}))^p\right)^{1/p} .$$

Thus, $\boldsymbol{x}^*$ minimizes the L_p-distance function from the statement of our Proposition over $\boldsymbol{x} \in X$, as claimed. □

For completeness it remains to consider the case $p = \infty$ in the L_p-distance function.

Proposition 5. Let $\boldsymbol{x}^*$ be a Pareto optimal solution of the multicriteria optimization problem (1). Further, suppose that for each i, $1 \le i \le m$, the maximum of f_i on X exists and is equal to f_i^*.Then there exist non-negative real numbers w_i, $1 \le i \le m$, such that the function

$$\sup_{1\le i\le m} w_i(f_i^* - f_i(\boldsymbol{x}))\ , \quad \boldsymbol{x} \in X$$

is minimized x^*.

P r o o f. Let I be the set of all subscripts i, $1 \le i \le m$, such that $f_i^* = f_i(\boldsymbol{x}^*)$. If $I \ne \emptyset$, then we set $w_i > 0$ arbitrarily for all $i \in I$ and $w_i = 0$ for all $i \notin I$. With this choice of scaling factors the function in the statement of our Proposition attains at $\boldsymbol{x}^*$ the value 0, which is clearly its minimum value. If, on the other hand, $I = \emptyset$, then define the scaling factors w_i for $1 \le i \le m$ so that $(f_i^* - f_i(\boldsymbol{x}^*))w_i = 1$. We claim that with such scaling factors the function in the statement of our Proposition attains its minimum at $\boldsymbol{x}^*$. Indeed, for any $\boldsymbol{x} \in X$, either $f_i(\boldsymbol{x}) = f_i(\boldsymbol{x}^*)$ for all i, $1 \le i \le m$, or $f_i(\boldsymbol{x}) < f_i(\boldsymbol{x}^*)$ for some i, $1 \le i \le m$. In the first case there is nothing to prove. In the second case we have $1 = w_i(f_i^* - f_i(\boldsymbol{x}^*)) < w_i(f_i^* - f_i(\boldsymbol{x}))$, which shows that the function in our statement is minimized at $\boldsymbol{x}^*$. □

4. PROPER PARETO OPTIMALITY AND THE CHOICE OF WEIGHTS

The next natural step is to ask if the results from the previous section can be carried over to *proper* Pareto optimality. We do not see any obvious way to modify methods of proofs of Propositions 1 and 2 to arrive at results referring to proper Pareto optimality. Nevertheless, the method involving Hölder's inequality extends as follows.

Proposition 6. In the multicriteria optimization problem (1), assume that the feasible set X is convex and all the criteria f_i, $1 \le i \le m$, are concave down on X. Assume that for each i, $1 \le i \le m$, the maximum of f_i on X exists and is equal to f_i^*. If a point $\boldsymbol{x}^* \in X$ is properly Pareto optimal for (1), then for any p such that $1 < p < \infty$ there exist $v_i > 0$, $1 \le i \le m$, such that $\boldsymbol{x}^*$ minimizes the weighted scalarization L_p-distance function $\left(\sum_{i=1}^m v_i(f_i^* - f_i(\boldsymbol{x}))^p\right)^{1/p}$ on X.

P r o o f. We can almost exactly follow the lines of the proof of Proposition 4. Using the same notation, we first introduce the set J and argue that it only makes sense to consider the case when $J \ne \emptyset$. We then involve Theorem 2.23 of [1] to show that, for a given properly Pareto optimal point $\boldsymbol{x}^*$ restricted to the set of objectives indexed by J, there exist *positive* w_i, $i \in J$, such that $\boldsymbol{x}^*$ minimizes the function $\sum_{i\in J} w_i a_i(\boldsymbol{x})$ on X. Continuing in the proof of Proposition 4 step by step (using Hölder's inequality) we conclude that there exist *positive* constants v_i, $i \in J$, such that $\boldsymbol{x}^*$ minimizes the function $\left(\sum_{i\in J} v_i(f_i^* - f_i(\boldsymbol{x}))^p\right)^{1/p}$ on X. Finally, one observes that the weights v_i for $i \notin J$ may be taken to be arbitrary positive numbers. Checking of the individual steps is left to the reader. □

To complete our analysis it is natural to ask if Proposition 5 extends to proper Pareto optimality with all scaling factors positive, provided, of course, that for each i, $1 \leq i \leq m$, the maximum of f_i on X exists and is equal to f_i^*. The answer is in the negative in general. Before producing an example, let us observe that if $\boldsymbol{x}^*$ is a (not necessarily properly) Pareto optimal point for the problem (1) and if $f_i(\boldsymbol{x}^*) < f_i^*$ for all i, $1 \leq i \leq m$, then the proof of Proposition 5 automatically yields strictly positive scaling factors w_i for all i. However, if $f_i(x^*) = f_i^*$ for some i, an analogue of Proposition 5 with all scaling factors positive is impossible even in simplest cases, and the situation cannot be rescued by assuming proper Pareto optimality, as we now show.

Example. Let $X = \{(x,y);\ -1 \leq x \leq 1,\ 1 \leq y \leq 2\}$; let $f_1(x,y) = -(x+y)$ and $f_2(x,y) = x - y$. There exist two distinct properly Pareto optimal points (x^*, y^*) for the problem of determining "max" $\{f_1(x,y), f_2(x,y)\}$ over $(x,y) \in X$, such that for any positive scaling factors w_1 and w_2, the minimum of the function $\nu(x,y) = \sup\{w_1(f_1^* - f_1(x,y)), w_2(f_2^* - f_2(x,y))\}$ is never attained at (x^*, y^*).

Details. It is easy to check that the set of all Pareto optimal points for this problem is $P = \{(x,1);\ -1 \leq x \leq 1\}$ and that every point in P is, in fact, *properly* Pareto optimal (the corresponding constant d can be taken to be equal to 1). Also, note that $f_1^* = f_2^* = 0$. We show that the extreme points of the interval P satisfy our claim. Indeed, let $(x^*, y^*) = (-1, 1)$. Then, $f_1(x^*, y^*) = 0 = f_1^*$ while $f_2(x^*, y^*) = -2$. Let w_1, w_2 be arbitrary but fixed positive real numbers. Then, $\nu(x^*, y^*) = 2w_2$. Take now a positive $\varepsilon < 2w_2/(w_1 + w_2)$ and consider the point $(x,y) \in P$ where $x = -1 + \varepsilon$ and, of course, $y = 1$. Then, $w_1 f_1(x,y) = -w_1\varepsilon$ while $w_2 f_2(x,y) = -w_2(2-\varepsilon)$. Since the inequality $\varepsilon < 2w_2/(w_1+w_2)$ is equivalent with $w_1\varepsilon < w_2(2-\varepsilon)$, we have $\nu(x,y) = w_2(2-\varepsilon)$. In particular, $\nu(x,y) < 2w_2 = \nu(x^*, y^*)$, which indeed shows that the minimum of $\nu(x,y)$ over $(x,y) \in X$ cannot be attained at our (properly) Pareto optimal point $(x^*, y^*) = (-1, 1)$. A similar calculation is valid for the other extreme point $(x^*, y^*) = (1, 1)$ of P.

This example also shows that the conclusion of Proposition 5 cannot be strengthened by assuming that all criteria are bounded on X, or by stronger assumptions on the set X.

For completeness, let us mention that a result similar to our Proposition 5 can be found in [6]. In fact, the discussion in [6] is a little more general since it allows arbitrary goals. On the other hand, the minimax function in [6] lacks absolute values, a very strong condition of the so-called "perfect equilibration" is assumed, and comprehensiveness of explanations suffers from tautological statements in proofs.

5. CONCLUDING REMARKS

As we have mentioned in Section 2, variations of Pareto optimality such as weak, strong, and proper (in the sense of various founders) have been studied in the theory of multicriteria optimization. In principle, one could try to establish results analogous to those in Sections 3 and 4 for these variations. We chose to restrict ourselves to the two most important notions, which are Pareto optimality and proper Pareto optimality (in the sense of Geoffrion). As a matter of fact, under certain conditions certain variations of (proper) Pareto optimality are equivalent; we refer to the monograph [1] for details.

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REFERENCES

[1] M. Ehrgott: Multicriteria Optimization. Springer–Verlag, Berlin 2000. Second edition Springer, Berlin 2005.
[2] J. Figueira, S. Greco, and M. Ehrgott: Multiple criteria decision analysis: State of the art surveys. Kluwer Academic Publishers, Boston – Dordrecht – London 2005.
[3] J. Franklin: Methods of Mathematical Economics. Linear and Nonlinear Programming. Fixed-Point Theorems. Springer, New York 1980.
[4] A. M. Geoffrion: Proper efficiency and the theory of vector maximization. J. Math. Anal. Appl. *22* (1968) 618–630.
[5] M. Maňas: Game Theory and its Applications (in Czech). SNTL, Prague 1991.
[6] W. Ogryczak: Comments on properties of the minmax solution in goal programming. European J. Oper. Res. *132* (2001), 17–21.
[7] C. Romero: Multi-objective and goal-programming approaches as a distance function model. J. Operat. Res. Soc. *36* (1985), 3, 249–251.
[8] M. Tamiz, D. Jones, and C. Romero: Goal programming for decision making: An overview of the current state-of-the-art. European J. Oper. Res. *111* (1998), 569–581.

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