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MEROMORPHIC OBSERVER–BASED POLE ASSIGNMENT IN TIME DELAY SYSTEMS

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The paper deals with a novel method of control system design which applies meromorphic transfer functions as models for retarded linear time delay systems. After introducing an auxiliary state model a finite-spectrum observer is designed to close a stabilizing state feedback. The observer finite spectrum is the key to implement a state feedback stabilization scheme and to apply the affine parametrization in controller design. On the basis of the so-called RQ-meromorphic functions an algebraic solution to the problem of time-delay system stabilization and control is presented that practically provides a finite spectrum assignment of the control loop.

Keywords: retarded time-delay system, meromorphic transfer function, reduced-order observer, state feedback, affine parametrization of stabilizing controllers

AMS Subject Classification: 93C05, 93B55, 93D15

1. INTRODUCTION

The trend to open the algebraic design from the control systems described by rational transfer functions to the linear systems with delays has attracted a large research attention. The time-delay systems can be described by rational functions in two variables s and $e^{\vartheta s}$ [5], provided the delays are assumed as commensurate. However, the delays in real plants are real valued and then ratios of *quasi-polynomials* or meromorphic functions are to be used as algebraic models [10]. In principle, the algebraic approaches to controller design, e. g. the pole placement approach or affine controller parametrization, may be adopted for this kind of infinite order systems [17], however, with serious constraints. Most of these constraints are more or less connected with the transcendental nature of the models, i. e. with the infinite spectrum of these models. As to the role of models, it is a specific feature of plants with significant delays that an application of a kind of model or observer becomes essential in stabilizing controller design [9]. A functional scheme of state observers for time delay systems has been introduced by Trinh [13]. The state feedback control has been proposed by Kim and Park [4] and the meaning of a feedback pre-compensation in linear time-delay systems has been shown by Picard et al. [12]. As examples of model based methods the finite spectrum assignment approach to controller design [16], or a meromorphic extension of internal model control design [18, 20], can be mentioned. As regards the state feedback and the pole placement these design approaches are severely limited by the fact that an infinite spectrum of poles is amended by means of a relatively low number of control parameters, these limitations have been qualified by Michiels and Roose [7]. The so-called continuous pole placement in time delay systems has been presented by Michiels et al. [6], Michiels and Vyhlídal [8]. The final result of a pole shifting is significantly limited by the well-known effect that the desirable shift of the rightmost poles to the left may be wasted with a tendency of the rest of the spectrum to a counter-movement to the right. An efficient rootfinder is necessary to apply in checking the spectrum changes, see Breda et al. [1] or Vyhlídal and Zítek [14, 15]. In addition, the parametrization based controller design is subject to significant limitations due to much larger variety of quasi-polynomials and due to the causality requirements in the time delay systems [17, 19].

The structure of the paper is as follows. In Section 2 a meromorpfic model of a retarded time-delay system is introduced, its decomposition to a state model is proposed in Section 3 and with the use of this auxiliary model a reduced-order observer is designed in Section 4. The observer-based state feedback then serves to stabilizing the system for which a final controller is designed by means of affine parametrization procedure in Section 5. An application example and concluding remarks are added in Sections 6 and 7.

2. MEROMORPHIC MODELS OF TIME–DELAY SYSTEMS

In extending the class of admissible functions from rational to meromorphic, the natural requirements of causality and feasibility of both the plant and the controller have to be respected in the ultimate control system implementation. To satisfy these conditions in rational algebraic design, the plant and controller models are constrained to proper rational functions. An equivalent restriction is to be introduced for *meromorphic functions* as well. In order to avoid impulsive modes in system's responses the so-called *internal stability* condition is adopted. To apply the algebraic approach to the feedback design of time-delay systems, it is necessary to define an admissible class of these systems, particularly as to the delays. The time-delay systems are supposed containing lumped delays only and with the so-called retarded structure [2]. This class of systems is defined below.

Definition 1. (*RQ meromorphic function*) A ratio $G(s)$ of quasi-polynomials $G(s) = B(s)/A(s)$ is said to be a retarded quasi-polynomial (RQ) meromorphic function if

- $A(s)$ is a retarded quasi-polynomial of the generic form
$$A(s) = s^n + \sum_{i=0}^{n-1}\sum_{j=1}^{h} a_{ij} s^i \exp(-\vartheta_{ij} s) \qquad (1)$$
where the highest power s^n represents a *delayless* term of the model and ϑ_{ij} are non-negative delays,

- $B(s)$ can be factorized as $B(s) = \bar{B}(s)\exp(-s\tau)$, where $\bar{B}(s)$ is a retarded and stable quasi-polynomial

$$\bar{B}(s) = b_m s^m + \sum_{i=0}^{m-1}\sum_{j=1}^{h} b_{ij} s^i \exp(-\tau_{ij} s) \tag{2}$$

  where τ and τ_{ij} are non-negative delays,

- the fraction is strictly proper, i. e., it holds for the highest s-power s^m in $B(s)$, that $m \le (n-1)$.

3. CONVERTING THE PLANT MODEL TO A STATE FORM

Consider an RQ meromorphic function $G(s)$ introduced in Definition 1 as the transfer function of a single-input-single-output (SISO) plant. With the aim to close a memoryless feedback of the form $u(t) = -\boldsymbol{K}\boldsymbol{x}(t)$, $\boldsymbol{x} \in \mathbb{R}^n$ we need to convert $G(s)$ into a state model of the generic Laplace transfer form

$$s\boldsymbol{x}(s) = \boldsymbol{A}(s)\boldsymbol{x}(s) + \boldsymbol{B}(s)u(s) \tag{3}$$

$$y(s) = \boldsymbol{C}\boldsymbol{x}(s), \tag{4}$$

where $\boldsymbol{x}$, u and y are state variable vector, single input and single output, respectively [4]. Zero initial conditions are considered in (3) for both the output and state variables. The following proposition points out that it is always possible to select such a state variable vector $\boldsymbol{x}(s)$ that all $G(s)$ parameters are located in the first column of $\boldsymbol{A}(s)$.

Proposition 1. If an RQ meromorphic function $G(s) = B(s)/A(s)$ is given by quasi-polynomials $A(s)$ and $B(s)$ as in (1) and (2) respectively, it can be identified with the nth order state model (3) where $\boldsymbol{x}$ is the state variable vector and where the system matrices are of the form

$$\boldsymbol{A}(s) = \begin{bmatrix} -\sum_{j=1}^{h} a_{n-1,j}\exp(-\vartheta_{n-1,j}s), & 1, & \cdots & 0 \\ ------ & - & - & - \\ -\sum_{j=1}^{h} a_{1,j}\exp(-\vartheta_{1,j}s), & 0, & \cdots & 1 \\ -\sum_{j=1}^{h} a_{0,j}\exp(-\vartheta_{0,j}s), & 0, & \cdots & 0 \end{bmatrix}, \tag{5}$$

$$\boldsymbol{B}(s) = \bar{\boldsymbol{B}}(s)\exp(-s\tau) = \begin{bmatrix} b_m \\ --- \\ \sum_{j=1}^{h} b_{1,j}\exp(-\tau_{1,j}s) \\ \sum_{j=1}^{h} b_{0,j}\exp(-\tau_{0,j}s) \end{bmatrix}\exp(-\tau s), \tag{6}$$

$\boldsymbol{C} = [1, 0, \dots, 0]$, and where it holds that $x_1 \equiv y$ and all the delay elements in $\boldsymbol{A}(s)$ are placed in the first column only. The state variables are introduced in the manner given in the proof.

P r o o f. The transfer function $G(s)$ represents a linear time delay differential equation which has the Laplace transform $A(s)y(s) = B(s)u(s)$ when the case of zero valued initial conditions is considered. In order to get the coefficients and delays of $A(s)$ fixed to the first column of $\boldsymbol{A}(s)$ the method of *nested integrations* is applied to this equation – beginning with x_n variable in the first integration

$$sx_n(s) = -\sum_{j=1}^{h} a_{0,j} \exp(-\vartheta_{0,j})x_1(s) + \sum_{j=1}^{h} b_{0,j} \exp(-(\tau_{0,j} + \tau)s)u(s), \quad (7)$$

where the elements in the last row of $\boldsymbol{A}(s)$ and $\boldsymbol{B}(s)$ are obtained from. In the same manner the next integration is performed and the second state variable is introduced

$$sx_{n-1}(s) = x_n(s) - \sum_{j=1}^{h} a_{1,j} \exp(-\vartheta_{1,j})x_1(s) + \sum_{j=1}^{h} b_{1,j} \exp(-(\tau_{1,j} + \tau)s)u(s). \quad (8)$$

Similarly further integrations are performed and the rest of n state variables is introduced while all the summation elements with delay operators are placed in the first column of $\boldsymbol{A}(s)$ only. Finishing with $x_1 \equiv y$ the set of equations (3) with the matrices (5) and (6) is obtained. The state variable x_1 always results as identical with the output y. □

Matrix $\boldsymbol{A}(s)$ is in the observable Frobenius form and therefore system (3) is spectrally observable [11] and the first column elements determine its characteristic equation. It is also apparent that if either $\sum_{j=1}^{h} b_{0j} \neq 0$ or $\sum_{j=1}^{h} a_{0,j} \neq 0$ then the obtained model given by the matrices (5), (6) is completely spectrally controllable since due to the unit-element positions in the controllability condition

$$\operatorname{rank}\,[s\boldsymbol{I} - \boldsymbol{A}(s), \boldsymbol{B}(s)] = n \quad (9)$$

just these non-zero elements can ensure the full rank [11]. As regards the intent to close the state feedback, only the first state variable is available as the plant output, $x_1 \equiv y$, and for closing the state feedback $u(t) = -\boldsymbol{K}\boldsymbol{x}(t)$ the other state variables can only be estimated by an observer. □

4. REDUCED–ORDER OBSERVER FOR STATE FEEDBACK DESIGN

Only the states $x_2, \ldots, x_n$ are not available for closing the state feedback $u = -\boldsymbol{K}\boldsymbol{x}$. To avoid the idle estimation of $x_1 \equiv y$ let the vector of unavailable states $\boldsymbol{x}_E = [x_2, \ldots, x_n]^T$ be separated from y by the partition $\boldsymbol{x} = \left[y, \boldsymbol{x}_E^T\right]^T$ and in the sequel, the appropriate partitions of the system matrices are considered

$$\boldsymbol{A}(s) = \begin{bmatrix} A_{yy}(s) & \boldsymbol{A}_{yE} \\ \boldsymbol{A}_{Ey}(s) & \boldsymbol{A}_{EE} \end{bmatrix}, \quad (10)$$

$$\boldsymbol{B}(s) = \begin{bmatrix} B_{yy}(s) \\ \boldsymbol{B}_{E}(s) \end{bmatrix}. \quad (11)$$

The following reduced-order observer can then be applied to provide an estimate $\widehat{\boldsymbol{x}}_E$ of $\boldsymbol{x}_E$ [3].

Proposition 2. If for a system given as $G(s)$ the state model (3) with system matrices (5) and (6) is introduced the following model given by the L-transform equations (in the case of zero initial conditions) can serve as $(n-1)$-order observer providing the estimate $\widehat{\boldsymbol{x}}_E$ of $\boldsymbol{x}_E$

$$s\boldsymbol{v}(s) = [\boldsymbol{A}_{EE} - \boldsymbol{H}\boldsymbol{A}_{yE}]\,\widehat{\boldsymbol{x}}_E(s) + [\boldsymbol{v}_E(s) - \boldsymbol{H}\boldsymbol{B}_y(s)]\,u(s) + [\boldsymbol{A}_{Ey}(s) - \boldsymbol{H}\boldsymbol{A}_{yy}(s)]\,y(s) \quad (12)$$

where $\widehat{\boldsymbol{x}}_E = \boldsymbol{v} + \boldsymbol{H}y$ and $\boldsymbol{H} = [h_1, h_2, \ldots, h_{n-1}]^T$.

P r o o f. From the partition (8) it follows that the state model (3) can be split into two equations

$$\begin{aligned} sy(s) &= A_{yy}(s)y(s) + \boldsymbol{A}_{yE}\boldsymbol{x}_E(s) + B_y(s)u(s) && (13)\\ s\boldsymbol{x}_E(s) &= \boldsymbol{A}_{Ey}(s)y(s) + \boldsymbol{A}_{EE}\boldsymbol{x}_E(s) + \boldsymbol{B}_E(s)u(s). && (14) \end{aligned}$$

Since the only state vector to be reconstructed is $\boldsymbol{x}_E$ the observer is to be based merely on equation (14) while (13) is taken as an additional relationship between u and y as measured quantities and the estimate $\widehat{\boldsymbol{x}}_E$. This first equation is not satisfied in observer operation and just the difference between its left and right side is used as the observer error and the so-called innovation term is established by multiplying it by $(n-1,1)$ feedback gain matrix $\boldsymbol{H}$ as follows

$$\begin{aligned} s\widehat{\boldsymbol{x}}_E(s) = {} & \boldsymbol{A}_{Ey}(s)y(s) + \boldsymbol{A}_{EE}\widehat{\boldsymbol{x}}_E(s) + \boldsymbol{B}_E(s)u(s) \\ & + \boldsymbol{H}\left[sy(s) - A_{yy}(s)y(s) - \boldsymbol{A}_{yE}\widehat{\boldsymbol{x}}_E(s) - B_y(s)u(s)\right]. \end{aligned} \quad (15)$$

Both u and y are considered as observer inputs, however, the undesired property of this form – the derivative of y in (15) – can be overcome by introducing the observer state vector $\boldsymbol{v}$ by the substitution $\boldsymbol{v} = \widehat{\boldsymbol{x}}_E - \boldsymbol{H}y$. The observer differential equation is then obtained in the form

$$\begin{aligned} s\boldsymbol{v}(s) = {} & [\boldsymbol{A}_{EE} - \boldsymbol{H}\boldsymbol{A}_{yE}]\,\widehat{\boldsymbol{x}}_E(s) + [\boldsymbol{B}_E(s) - \boldsymbol{H}B_y(s)]\,u(s) \\ & + [\boldsymbol{A}_{Ey}(s) - \boldsymbol{H}A_{yy}(s)]\,y(s), \end{aligned} \quad (16)$$

where $\widehat{\boldsymbol{x}}_E = \boldsymbol{v} + \boldsymbol{H}y$, and the form (12) is obtained. □

The main merit of observer (12) is not only its lower order but primarily the algebraic nature of the characteristic equation of (12) and its independence of the parameters specifying the observed plant.

Lemma 2. (*Observer characteristic equation*) If the reduced-order observer is designed according to the scheme given in (12) its characteristic equation does not contain any delay factor, i. e. it is algebraic, with $n-1$ roots only. Furthermore, the observer matrix $[\boldsymbol{A}_{EE} - \boldsymbol{H}\boldsymbol{A}_{yE}]$ is in observable canonical form and, moreover, since only 1 and 0 are the elements of $\boldsymbol{A}_{EE}$ and $\boldsymbol{A}_{yE}$, the observer characteristic equation

$$\det\left[s\boldsymbol{I} - \boldsymbol{A}_{EE} + \boldsymbol{H}\boldsymbol{A}_{yE}\right] = 0 \quad (17)$$

is *algebraic* and does not depend on the $G(s)$ parameters at all. The observer gain coefficients $h_i,\ i = 1,\dots,n-1$ become the coefficients of the observer characteristic equation.

Proof. Consider the observer error $\Delta \boldsymbol{x}_E = \widehat{\boldsymbol{x}}_E - \boldsymbol{x}_E$ and investigate its dynamics. After subtracting (14) from (15) we obtain

$$s\Delta\boldsymbol{x}_E(s) = \boldsymbol{A}_{EE}\Delta\boldsymbol{x}_E(s) - \boldsymbol{H}\left[\boldsymbol{A}_{yE}\widehat{\boldsymbol{x}}_E + B_y(s)u(s) + A_{yy}(s)y(s) - sy(s)\right]. \quad (18)$$

Using the equation (13) to remove the derivative of y from (18) the following homogenous state equation is obtained for the observer error dynamics

$$s\Delta\boldsymbol{x}_E(s) = \left[\boldsymbol{A}_{EE} - \boldsymbol{H}\boldsymbol{A}_{yE}\right]\Delta\boldsymbol{x}_E(s). \quad (19)$$

Since the matrices $\boldsymbol{A}_{yE}, \boldsymbol{A}_{EE}$ of the partition (10),(11) are of the following simple form

$$\boldsymbol{A}_{yE} = \begin{bmatrix} 1, & 0, & \cdots & 0 \end{bmatrix}, \quad (20)$$

$$\boldsymbol{A}_{EE} = \begin{bmatrix} 0, & 1, & \cdots & 0 \\ - & - & - & - \\ 0, & 0, & \cdots & 1 \\ 0, & 0, & \cdots & 0 \end{bmatrix}, \quad (21)$$

it is obvious that the observer matrix $[\boldsymbol{A}_{EE} - \boldsymbol{H}\boldsymbol{A}_{yE}]$ does not contain any delay terms or plant parameters. Since the matrices $\boldsymbol{A}_{EE}, \boldsymbol{A}_{yE}$ have only 1 and 0 elements, the observer matrix is completely independent of $G(s)$ parameters. In addition, due to the simple form of matrix product $\boldsymbol{H}\boldsymbol{A}_{yE}$ (only one nonzero column) the following polynomial form of observer characteristic equation results

$$\det\left[s\boldsymbol{I} - \boldsymbol{A}_{EE} + \boldsymbol{H}\boldsymbol{A}_{yE}\right] = s^{n-1} + h_1 s^{n-2} + \cdots + h_{n-2}s + h_{n-1} = H(s) = 0 \quad (22)$$

independent of $G(s)$ parameters. □

The observer characteristic matrix is in Frobenius form and hence, independently of the form of $G(s)$, the observer (12) is always obtained in the observable canonical form. Due to the form (22) independent of $G(s)$, the zeros of $H(s)$ as the observer poles can be easily placed at the prescribed positions $s = \sigma_i,\ i = 1, 2, \dots, n-1$ by means of satisfying the identity

$$s^{n-1} + h_1 s^{n-2} + \cdots + h_{n-2}s + h_{n-1} \equiv (s-\sigma_1)(s-\sigma_2)\cdots(s-\sigma_{n-1}) \quad (23)$$

for any s, i.e. by equating the coefficients of the powers $s^i,\ i = 1, 2, \dots n-1$, on both sides.

It is worth noting that due to the true value of y the state observer designed in (12) provides the so-called innovation process $\eta(t) = y(t) - \boldsymbol{C}\widehat{\boldsymbol{x}}(t)$ identically equal to zero, $\eta(t) \equiv 0$, and therefore the observer error does not affect any state feedback closed from the error provided the observer (12) is used [3].

The observer (12) has been designed to implement the state feedback

$$u = -k_1 y - k_2\widehat{x}_2 - k_3\widehat{x}_3 \dots - k_n\widehat{x}_n = -\boldsymbol{K}\begin{bmatrix} y \\ \widehat{\boldsymbol{x}}_E \end{bmatrix} \quad (24)$$

where again $\widehat{\boldsymbol{x}}_E$ denotes the vector of observer state estimates. After closing the feedback (24) a system of order $2n-1$ is generated where the spectrum of the observer remains separated from the rest of the system even though the reduced-order observer is applied.

Theorem 1. (*Separation property*) Consider a time delay plant $G(s)$ and its decomposition to the state model (3), (5), (6). If the reduced-order state observer of $\widehat{\boldsymbol{x}}_E$ according to (12) is provided and the feedback (24) is closed, the characteristic equation of the complete closed-loop system is given as a product of the following two determinants

$$\begin{aligned} M_{CL}(s) &= \det\left[s\boldsymbol{I}-\boldsymbol{A}(s)+\boldsymbol{B}\boldsymbol{K}\right]\det\left[s\boldsymbol{I}-\boldsymbol{A}_{EE}+\boldsymbol{H}\boldsymbol{A}_{yE}\right] \\ &= H(s)\det\left[s\boldsymbol{I}-\boldsymbol{A}(s)+\boldsymbol{B}\boldsymbol{K}\right]=0 \end{aligned} \tag{25}$$

so that the well-known separation property holds in the proposed reduced-order observer feedback structure. The eigenvalue spectrum of the observer (12) and the spectrum of the state feedback system itself are independent of each other.

Proof. The separability of $H(s)$ directly follows from equation (19). Having closed the state feedback (24) around system (3) and using $\Delta\boldsymbol{x}_E$ to describe the observer operation, the following state model of order $2n-1$ is obtained

$$s\begin{bmatrix}\boldsymbol{x}\\ \Delta\boldsymbol{x}_E\end{bmatrix}=\begin{bmatrix}\boldsymbol{A}(s)-\boldsymbol{B}(s)\boldsymbol{K} & -\boldsymbol{B}(s)\boldsymbol{K}_E\\ 0 & \boldsymbol{A}_{EE}-\boldsymbol{H}\boldsymbol{A}_{yE}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}\\ \Delta\boldsymbol{x}_E\end{bmatrix} \tag{26}$$

where $\boldsymbol{K}_E=[k_2,k_3,\ldots,k_n]$. Since the determinant of the block triangular characteristic matrix is given only by the product of diagonal blocks the property (25) is proved. □

5. AFFINE PARAMETERIZATION BASED CONTROLLER DESIGN

With regard to the separability of the polynomial $H(s)$ and the characteristic quasi-polynomial $\det\left[s\boldsymbol{I}-\boldsymbol{A}(s)+\boldsymbol{B}(s)\boldsymbol{K}\right]=M(s)$, the poles of $M(s)$ can be placed separately from those of $H(s)$ as if the state feedback were closed directly from the true state variables. But it is necessary to be aware of the limited potential in stabilizing the time-delay system by the state feedback (24). Basically these limitations arise from the infinite spectrum of time-delay systems contrasting with the number n of feedback parameters. The stabilizability of the system by the state feedback depends not only on the number of the right-half-plane (RHP) system poles but also on the values of delays [6]. Just the retarded character of $M(s)$ renders the set of possible RHP system poles finite. Nevertheless, if the number of RHP poles is less than n and if the delay values are in a limited range it is possible to find such a $\boldsymbol{K}$ that stabilizes $M(s)$. An approach to finding a stabilizing $\boldsymbol{K}$ will be given below.

Suppose that a feedback gain vector $\boldsymbol{K}$ *has been found* rendering the quasi-polynomial $M(s)$ *stable*. This state feedback option stabilizes the system independently of the $H(s)$ design but due to the mentioned infinite system spectrum

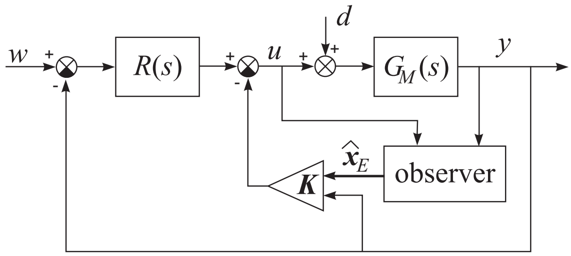

Fig. 1. Overall scheme of the control system, y, w – controlled variable and the set-point, u, d – control and disturbance variables.

the stabilization does not yet provide the system with desirable dynamic properties. First of all, in spite of placing n poles, the majority of the infinitely many system poles is placed in a way which is hardly predictable. Furthermore, in spite of the $H(s)$ separability, to some extent, the system response is obviously influenced by the observer dynamics. Besides, the above stabilization feedback was not considered as a control scheme so that the reference input and control error have not been introduced so far. On the other hand, with regard to the achieved system stability it is possible to apply the standard approach of affine parametrization with the aim to add a control loop providing the system with prescribed dynamics. The complete scheme is in Figure 1 and the whole system can be viewed as a single loop control where the state feedback scheme developed above is considered as a pre-stabilized plant to be controlled.

After closing the state feedback (24) by means of the observer (12) with a stabilizing set of the gain parameters $\boldsymbol{K} = [k_1, k_2, \ldots, k_n]$ a stable transfer function $G_F(s)$ between the control input u and the output y is obtained. Due to the use of the observer (12) this $G_F(s)$ is relatively complicated, its order is $2n-1$. Nevertheless if the observer and the state feedback design are performed so that the observer eigenvalues are placed substantially farther to the left from those placed by the state feedback design, their influence on the final system response becomes fairly weak or even negligible. On the assumption that this is the case it is possible to consider the state feedback model (given only by the characteristic quasi-polynomial $M(s) = \det[s\boldsymbol{I} - \boldsymbol{A}(s) + \boldsymbol{B}(s)\boldsymbol{K}]$, i. e. neglecting the observer dynamics) as an approximation of the real state feedback performance. Having this in mind and aiming at simplifying the controller design, let us introduce an approximate transfer function $G_M(s)$, where $G_M(s) \cong G_F(s)$.

The transfer function $G_M(s)$ of the state feedback scheme neglecting the observer dynamics is given by the quasi-polynomial ratio

$$G_M(s) = \frac{\boldsymbol{C}\operatorname{adj}[s\boldsymbol{I} - \boldsymbol{A}(s) + \boldsymbol{B}(s)\boldsymbol{K}]\,\boldsymbol{B}(s)}{\det[s\boldsymbol{I} - \boldsymbol{A}(s) + \boldsymbol{B}(s)\boldsymbol{K}]} = \frac{\boldsymbol{C}\operatorname{adj}[s\boldsymbol{I} - \boldsymbol{A}(s)]\,\boldsymbol{B}(s)}{\det[s\boldsymbol{I} - \boldsymbol{A}(s) + \boldsymbol{B}(s)\boldsymbol{K}]} = \frac{N(s)}{M(\boldsymbol{K}, s)} \tag{27}$$

where the quasi-polynomial $N(s)$ is independent of $\boldsymbol{K}$ due to the well-known matrix equality $\operatorname{adj}[\boldsymbol{M} + \boldsymbol{B}\boldsymbol{K}]\,\boldsymbol{B} = \operatorname{adj}[\boldsymbol{M}]\,\boldsymbol{B}$ which holds for any (n, n) non-singular matrix $\boldsymbol{M}$ and a pair of arbitrary $(n, 1))$ and $(1, n)$ matrices $\boldsymbol{B}$ and $\boldsymbol{K}$, respectively [3].

Using this approximation of the actual state feedback performance, the following affine parametrization approach can be applied in controller design.

Theorem 2. Consider the stabilized plant as an augmented meromorphic function

$$G_M(s) = \frac{\widetilde{B}(s)}{\widetilde{A}(s)}, \quad \widetilde{A}(s) = \frac{M(\boldsymbol{K}, s)}{F(s)}, \quad \widetilde{B}(s) = \frac{N(s)}{F(s)}, \tag{28}$$

where $F(s)$ is selected as stable polynomial of order n, such that:

- the functions $\widetilde{A}(s), \widetilde{B}(s)$ are coprime RQ-meromorphic functions,
- the quasi-polynomial $F(s) - 2N(s)$ is stable with $F(0) \geq 2N(0)$.

Then any RQ-meromorphic function

$$R(s) = \frac{2M(\boldsymbol{K}, s)}{F(s) - 2N(s)} \tag{29}$$

is a stabilizing anisochronic controller for $G_M(s)$, operating with time-shifted data.

P r o o f. The functions $\widetilde{A}(s)$, $\widetilde{B}(s)$ are coprime and stable RQ-meromorphic functions. By analogy with the rational algebraic approach the ring of RQ-meromorphic stable functions $\boldsymbol{R}_{MS}$ can be considered [19] and $\widetilde{A}(s), \widetilde{B}(s) \in \boldsymbol{R}_{MS}$. If RQ-meromorphic functions $X_0(s), Y_0(s) \in \boldsymbol{R}_{MS}$ satisfying the Bézout equation

$$\widetilde{A}(s)X_0(s) + \widetilde{B}(s)Y_0(s) = 1 \tag{30}$$

are found as an internally stabilizing controller $Y_0(s)/X_0(s)$ for the plant $G_M(s)$ then the set of all stabilizing controllers is given by the ratio

$$R(s) = \frac{Y_0(s) + \widetilde{A}(s)W(s)}{X_0(s) - \widetilde{B}(s)W(s)}, \tag{31}$$

where $W(s)$ is a parametrizing stable and RQ-meromorphic function, $W(s) \in \boldsymbol{R}_{MS}$. For the option $Y_0(s) = 1$ the denominator function satisfying (30) is

$$X_0(s) = \frac{1 - \widetilde{B}(s)}{\widetilde{A}(s)} = \frac{F(s) - N(s)}{M(\boldsymbol{K}, s)} \tag{32}$$

and for $W(s) = F(s)/M(\boldsymbol{K}, s)$ the affine parametrization (31) results in the controller (29). Since $F(s) - 2N(s)$ is selected to be stable and $F(s)$ is of the same degree as the quasi-polynomial $M(\boldsymbol{K}, s)$, and since $G_M(s)$ is strictly proper, $R(s)$ is not only stable and proper meromorphic function but due to the delay-free term at s^n in $F(s)$ it is retarded as well. □

Complementary sensitivity function. Having applied controller (29) to the stabilized plant $G_M(s)$ the complementary sensitivity function $T(s) = y(s)/w(s)$ is obtained in the form independent of $M(\boldsymbol{K}, s)$

$$T(s) = \frac{R(s)G_M(s)}{1 + R(s)G_M(s)} = \frac{2N(s)}{F(s)}, \tag{33}$$

where the characteristic equation of the feedback loop is given only by the choice of $F(s)$. Thus the pole location of this system is independent of $M(\boldsymbol{K}, s)$ and $N(s)$ and the system obtains the character of finite spectrum assignment.

Disturbance rejection. The conditioning polynomial $F(s)$ is bounded by the condition $F(0) \geq 2N(0)$. Apparently if $F(0) > 2N(0)$ the controller $R(s)$ is proportional and leaves a steady state error. To avoid the steady state control error, it is necessary to select $F(s)$ such that $F(0) = 2N(0)$, to provide the integrating character of the controller action. Using this option the selection of $F(s)$ is to be modified as follows:

$$F(s) - 2N(s) = sf(s), \tag{34}$$

where $f(s)$ is required to be stable. The steady state value of the complementary sensitivity function is then equal to one, $T(0) = 2N(0)/F(0) = 1$.

Remark. Let us recall that the controller formula (29) is based on the assumption that the observer dynamics actually applied in the state feedback (24) can be neglected and only the quasi-polynomials $M(\boldsymbol{K}, s)$ and $N(s)$ constitute the model $G_M(s)$. Therefore in applying (29) it is necessary to know the positions of the rightmost (dominant) zeros of the obtained *stable* $M(\boldsymbol{K}, s)$ in order to place the zeros of $H(s)$ sufficiently far to the left. The verification of the $M(\boldsymbol{K}, s)$ spectrum can be performed by means of the tool described in [14, 15]. A possible way of adjusting the state feedback parameters $\boldsymbol{K}$ is proposed below.

State feedback gain assessment. The design of controller (29) is conditioned by having found such a state feedback gain vector $\boldsymbol{K}$ that makes the quasi-polynomial $M(\boldsymbol{K}, s) = \det[s\boldsymbol{I} - \boldsymbol{A}(s) + \boldsymbol{B}(s)\boldsymbol{K}]$ stable. However, to find such a $\boldsymbol{K}$ may become a hard problem even if the number of RHP eigenvalues of $\boldsymbol{A}(s)$ is not higher than n and the length of delays is not beyond the scope of the state feedback (24) potentials. Although it is quite easy to place n zeros of $M(\boldsymbol{K}, s)$ at prescribed positions, the problem is to identify these positions with the *dominant eigenvalues* of the system. These n placed zeros may stabilize the feedback system only if they become the *rightmost* eigenvalues of this system.

Let the following n zeros be prescribed for $M(\boldsymbol{K}, s) : s = r_i$, $\operatorname{Re}(r_i) < 0$, $i = 1, \ldots, n$. Since the dimensions of $\boldsymbol{B}(s)$ and $\boldsymbol{K}$ are $(n, 1)$, $(1, n)$, respectively, the quasi-polynomial $M(\boldsymbol{K}, s)$ is linear with respect to k_i, $i = 1, \ldots, n$, and therefore it can be considered in the form

$$M(\boldsymbol{K}, s) = M_0(s) + \sum_{j=1}^{n} \frac{\partial M(\boldsymbol{K}, s)}{\partial k_i} k_i, \tag{35}$$

where $M_0(s) = A(s) = \det[s\boldsymbol{I} - \boldsymbol{A}(s)]$. Apparently, in order to identify $s = r_i,\ i = 1, \dots, n$, with the zeros of $M(s)$ the appropriate parameters $k_i,\ i = 1, \dots, n$ are obtained as the roots of the following equation set

$$M_0(r_j) + \sum_{j=1}^{n} \left[\frac{\partial M(\boldsymbol{K}, s)}{\partial k_i} \right]_{s=r_j} k_i = 0, \quad j = 1, 2, \dots, n. \tag{36}$$

With regard to the problem of infinite spectrum the set of gain coefficients $\boldsymbol{K}$ obtained as the solution to (36) carries out a specimen of $M(\boldsymbol{K}, s)$ containing the prescribed $s = r_i,\ i = 1, \dots, n$ in its spectrum. However, this $\boldsymbol{K}$ does not give any guarantee that the prescribed eigenvalues will have the dominant meaning in system dynamics. That is why any solution to (36) has to be checked whether some of the other eigenvalues do not lie to the right from the prescribed $s = r_i,\ i = 1, \dots, n$. If this is the case the attempt is discarded and a new option of root prescription is to be performed. Only such a solution where the prescribed eigenvalues fall among the rightmost ones can be accepted as valid.

Actually, obtaining an acceptable placement of n eigenvalues requires a series of attempts with various options of prescribed r_j. To facilitate this approach, it is easier to take the case of a repeated root of $M(\boldsymbol{K}, s)$ for prescribing r_j. Using this option the $M(\boldsymbol{K}, s)$ derivatives $\partial^j M / \partial s^j = M^{(j)}(s), j = 1, \dots, n-1$ are to be applied and instead of (36) the following set of conditions is solved

$$M_0^{(j)}(-\beta) + \sum_{j=1}^{n} \left[\frac{\partial M^{(j)}(\boldsymbol{K}, s)}{\partial k_i} \right]_{s=-\beta} k_i = 0, \quad j = 1, 2, \dots, n \tag{37}$$

where $-\beta$ is the multiple root. An increasing series of various β is then to be tried to find maximum value of this parameter still placing the repeated root on the rightmost position in the appropriate spectrum of $M(\boldsymbol{K}, s)$.

Alternatively, the method of continuous pole placement can be used to design the state feedback [6].

6. APPLICATION EXAMPLE

The feedback delay is a typical phenomenon encountered also in control performed in man-manoeuvred activities. So let us demonstrate the presented method on a non-traditional example of this kind.

The roller skater sketched in Figure 2 is to keep himself as close as possible to the top of a swaying bow by means of controlling a servomotor driving the slope of the bow and in this way influencing the skater's movement. Let the skater's deviation from the desired position be expressed by the angle ψ (between the skater and the bow symmetry axis) while the slope angle of the bow is φ and the servomotor force is P. If the skater's acceleration is influenced only by the slope angle $\psi + \varphi$ and the effect of friction is negligible the movement is described by the transfer function

$$G(s) = \frac{\psi(s)}{\varphi(s)} = \frac{\lambda \exp(-s(\tau + \vartheta))}{s^4 - \mu s^2 \exp(-s\vartheta)} \tag{38}$$

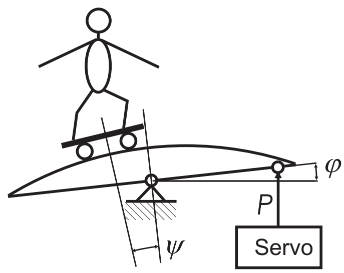

Fig. 2. The roller skater on the controlled swaying bow.

where μ, λ are constant coefficients and τ, ϑ are delays of the skater's and servomotor response. Let us consider the following model parameters $\mu = 1$, $\lambda = 0.2$, $\tau = 0.3$ s, $\vartheta = 0.1$s. The decomposition of the given model into the state space form (3) leads to the following set of equations

$$x_1 = \psi, \quad \dot{x}_1 = x_2, \quad \dot{x}_2 = x_3 + \mu x_1(t-\vartheta), \quad \dot{x}_3 = x_4, \quad \dot{x}_4 = \lambda P(t-\tau-\vartheta) \tag{39}$$

which are partitioned according to (10), (11) as follows

$$\boldsymbol{A}_{EE} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \tag{40}$$

$$\boldsymbol{A}_{Ey}(s) = \begin{bmatrix} \mu \exp(-s\vartheta) \\ 0 \\ 0 \end{bmatrix}, \tag{41}$$

$$\boldsymbol{A}_{yE}(s) = \begin{bmatrix} 1, & 0, & 0 \end{bmatrix}, \tag{42}$$

$$A_{yy} = 0, \qquad B_y = 0 \tag{43}$$

$$\boldsymbol{B}_E(s) = \begin{bmatrix} 0, & 0, & \lambda \exp(-s(\tau+\vartheta)) \end{bmatrix}^T. \tag{44}$$

After introducing the observer gain as vector $H = [h_1,\ h_2,\ h_3]^T$ the reduced order observer is then designed as in (12), and then its characteristic polynomial is as follows

$$H(s) = \det\left[s\boldsymbol{I} - \boldsymbol{A}_{EE} + \boldsymbol{H}\boldsymbol{A}_{yE}\right] = s^3 + h_1 s^2 + h_2 s + h_3. \tag{45}$$

It is simple to place the zeros of this polynomial, e.g. by prescribing a triple root $-\sigma$, so that $H(s) = (s+\sigma)^3$ and the following gain coefficients result $h_1 = 3\sigma, h_2 = 3\sigma^2,\ h_3 = \sigma^3$. For example the value $\sigma = 3s^{-1}$ renders the observer quick enough in removing the estimation error. The stabilizing state feedback is given by the four gains $k_i,\ i = 1,\dots,4$ and let four desirable poles $s = r_i,\ i = 1,\dots,4$ be prescribed. If the multiple root $s = -\beta$ for $i = 1,\dots,4$ is prescribed the appropriate gain coefficients are obtained by the help of solving $H(r_i) = 0,\ i = 1,\dots,4$. After prescribing $r_{1,2,3,4} = -\beta = -0.6$ the following state feedback gain coefficients are obtained from (37): $k_1 = 8.247,\ k_2 = 7.812,\ k_3 = 8.084,\ k_4 = 7.380$. In this case we

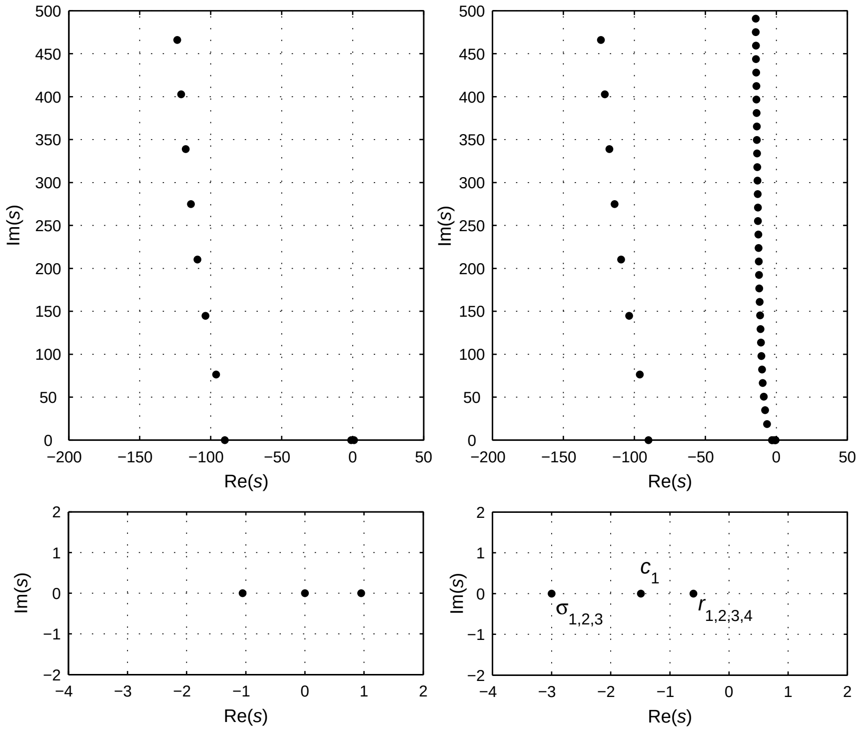

Fig. 3. Left – spectrum of the system (38), right – spectrum of the feedback system. Both with detailed view of the rightmost roots.

also see that $\beta = 0.2\sigma$, which means that the observer is able to provide effectively fast estimation in the state feedback.

The meromorphic transfer function $G_M(s) = N(s)/M(\boldsymbol{K}, s)$ of the system stabilized via state feedback is given by (27) and its denominator now results as follows

$$M(\boldsymbol{K}, s) = s^4 - m_0(s)s^2 + \left[k_4 s^3 + k_3 s^2 + (k_2 - k_4 m_0(s)) + k_1 - k_3 m_0(s)\right] N(s) \quad (46)$$

where $m_0(s) = \mu \exp(-s\vartheta)$ and $N(s) = \lambda \exp(-s(\tau + \vartheta))$. Since the relative order of $G_M(s)$ is $n - m = 4$, a fourth-order conditioning factor $F(s)$ is to be selected. One of the options satisfying the requirement that $2N(0) = F(0)$ can be selected as follows

$$F(s) = 2\lambda \left[1 + \frac{s}{\alpha}\right]^4. \quad (47)$$

Then the following controller function (29) results

$$R(s) = \frac{s^4 - m_0(s)s^2 + \left[k_4 s^3 + k_3 s^2 + (k_2 - k_4 m_0(s))s + k_1 - k_3 m_0(s)\right] N(s)}{\lambda(1 + s/\alpha)^4 - N(s)} \quad (48)$$

having at least one pole at $s = 0$, since $R^{-1}(0) = 0$. With this anisochronic controller the complementary sensitivity function of the complete control system is as simple

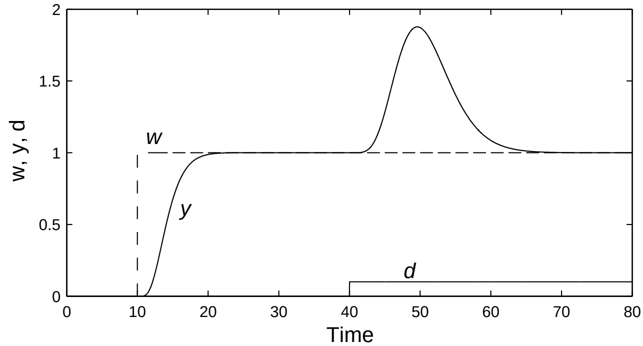

Fig. 4. Set-point responses of the whole feedback system.

as follows

$$T(s) = \frac{R(s)G_M(s)}{1 + R(s)G_M(s)} = \frac{\exp(-s(\tau + \vartheta))}{(1 + s/\alpha)^4} \tag{49}$$

provided that the plant model fits well the real plant.

Let us recall that when placing the poles of the feedback system at the new positions, we have to take into consideration the infinite spectra of both original and feedback systems. In Figure 3 left, we can see the spectrum of the original system (38). Beside four dominant roots (one unstable, one stable and double root at zero), there is an asymptotically exponential infinite chain of roots. However, the rightmost root of this chain is far to the left from the mentioned dominant roots and thus the modes corresponding to the chain have negligible effect on the system dynamics.

As can be seen in Figure 3 right, after closing the state feedback with the coefficients $k_i, i = 1, \ldots, 4$, the system is provided with the prescribed roots $r_{1,2,3,4} = -\beta$ and the system spectrum changes considerably. In virtue of the pole placement a new chain of roots appears in the original system spectrum, lying quite close to the imaginary axis of complex plane. However, since it holds for the rightmost root c_1 of this chain that $\mathrm{Re}\,(c_1) < \mathrm{Re}\,(r_{1,2,3,4})$, the prescribed roots are the rightmost dominant roots of the feedback system. Obviously the triple root $\sigma_{1,2,3}$ of the reduced order observer turns out to be only a finite part of the whole feedback system spectrum. The set-point response and disturbance rejection of the whole control system are shown in Figure 4 for $\alpha = 1$. The disturbance d is considered as an additional force acting on the skater, see Figure 1.

7. CONCLUDING REMARKS

The presented combination of the reduced-order observer and the pole placement synthesis with the affine parametrization approach to controller design proves effective to provide for a feasible time-delay plant stabilization and control. Although this combination may seem rather involved, it is necessary to take into consideration the infinite order of the problem. The proposed finite spectrum observer helps closing the stabilizing state feedback and considering its dynamics separate from the rest of the state feedback system spectrum. The final meromorphic parametrization results in a straightforward description of stabilizing controllers for the open-loop unstable linear time-delay plants. The controller performance formula (29) facilitates the design of *anisochronic controllers* that compensate for the delays in the plant despite the state feedback is designed only by means of proportional gains. The farther are the observer eigenvalues to the left from dominant eigenvalues of the feedback loop the better provides the final implementation of controller design the control loop practically with a *given finite spectrum*. The restriction to the RQ meromorphic functions limits the design to retarded time-delay systems only.

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