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FAULT TOLERANT CONTROL FOR UNCERTAIN TIME–DELAY SYSTEMS BASED ON SLIDING MODE CONTROL

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Fault tolerant control for uncertain systems with time varying state-delay is studied in this paper. Based on sliding mode controller design, a fault tolerant control method is proposed. By means of the feasibility of some linear matrix inequalities (LMIs), delay dependent sufficient condition is derived for the existence of a linear sliding surface which guarantees quadratic stability of the reduced-order equivalent system restricted to the sliding surface. A reaching motion controller, which can be seen as a fault tolerant controller, can retain the stability of the closed loop system in the present of uncertainties, disturbances and actuator fault is designed. A numerical simulation shows the effectiveness of the approach.

Keywords: fault tolerant control, sliding mode control, time varying state-delay, uncertainty, linear matrix inequality (LMI)

AMS Subject Classification: 34D20, 93D09, 15A39, 37L45

1. INTRODUCTION

To increase overall system reliability and safety in safety-critical systems, Fault-Tolerant Control (FTC) of the systems become extremely important. Therefore, FTC has been an important topic of research. Generally speaking, FTC systems can be categorized into two main classes: passive and active. Passive FTC systems are designed with the consideration of a set of presumed failure modes without the need to detect their presence. No alternation or adaptation is made to the control law so these controllers may be thought of as a specific class of robust controllers. The resulting control system performance tends to be conservative. It also has the limitation to deal with unanticipated faults. But the design of the controller is easy. In contrast, active FTC systems react to the occurrence of system faults on-line in an attempt to maintain the overall system stability and performance. They can deal with unanticipated faults with Fault Diagnosis (FD) and Controller Reconfiguration (CR); and therefore can achieve better performance than the passive methods. But the active FTC depends on the fast and accurate information provided by FD. In other words, the active FTC system would be useless without an effective FD scheme. What's more, the overall system becomes more complicated and costly.

This paper is concerned with the use of sliding mode idea to design FTC system for uncertain time-delay systems. Recent work has explored how sliding mode ideas can be used in FTC system design. The methods can be also divided into two categories: a) First, a sliding mode observer is used for fault estimation and then based on which a CR is adopted; b) Design a sliding mode controller directly to retain the stability and some performances of the closed-loop system. That's because a sliding mode controller which can drive the system onto the sliding surface in the present of fault input can be looked on as a fault tolerant controller. The former ones have been widely used, see [2, 7, 15, 16, 18, 19] and the references therein for some new results of this field. The latter ones have been studied, too. But unfortunately, few works of them concerned with time-delay systems. As we all known, time delays are commonly encountered in practical applications and might lead to poor system performance or even instability. Therefore, over the past decades, the analysis and synthesis of time-delay systems have been one of the most active research areas in system sciences. The recent developments on sliding mode control involving time-delay systems can be found in [3, 4, 5, 8, 11, 12, 17]. [11] considers sliding mode control for nonlinear state-delay systems based on neural network. It's an approximate calculation method. And the theoretical analysis of the influence of the approximation error to the control performance should be presented. The other methods deal with state-delay systems and input-delay systems. Most of them require the states of the time-delay system are available, and in these SMC schemes, the control laws usually utilize full-state feedback. But in practice, this is often not possible. To overcome this, [12] adopt state observer to obtain the unknown states, and then synthesize a sliding mode control law based on state estimates. To the best of our knowledge, the results of them are delay independent and thus are conservative.

In this paper, the problem of designing fault tolerant control based on sliding mode control for a class of uncertain systems with time-varying state-delay has been considered. The time-delay is not a constant but defined with some restrictions. So the result is less conservative than the results mentioned before and is more close to real life. Both parametric uncertainties and disturbance are considered. In terms of LMI, delay dependent sufficient condition is derived for the existence of a linear sliding surface which guarantees quadratic stability of the reduced-order equivalent system restricted to the sliding surface. And a reaching motion controller is proposed. The sliding motion and the reaching motion are robust against the mismatched uncertainties matched disturbance and actuator fault.

The paper is organized as follows. Section 2 gives the problem formulation. The main results are given in Section 3 which is composed of 2 parts. Subsection 3.1 shows how to design the sliding surface and 3.2 gives the design method of the reaching motion control law. The effectiveness of the approach proposed will be demonstrated via an example in Section 4, which will be followed by some concluding remarks in Section 5.

2. PROBLEM FORMULATION

Consider the time-delay system with the form

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= (\boldsymbol{A}+\boldsymbol{\Delta A})(t) + (\boldsymbol{A}_{\mathrm{d}}+\boldsymbol{\Delta A}_{\mathrm{d}}(t))\boldsymbol{x}(t-\tau(t)) + \boldsymbol{Bu}(t) + \boldsymbol{Hw}(t)\\ \boldsymbol{x}(t) &= \phi(t), \quad t\in[-\bar{\tau},0)\end{aligned} \tag{1}$$

where $\boldsymbol{x}(t)\in\mathbb{R}^n$, $\boldsymbol{u}(t)\in\mathbb{R}^m$ are the state vector and control input respectively. $\boldsymbol{w}(t)\in\mathbb{R}^l$ is the disturbance. $\boldsymbol{A}$, $\boldsymbol{A}_{\mathrm{d}}$, $\boldsymbol{B}$ and $\boldsymbol{H}$ are real constant matrices with appropriate dimensions and $\mathrm{rank}(\boldsymbol{B})=m$; $\tau(t)$ is the variable time-delay function which satisfies $0\le\tau(t)\le\bar{\tau}$, $|\dot{\tau}(t)|\le\mu$, $\forall\, t\ge 0$. $\boldsymbol{\Delta A}(t)$, $\boldsymbol{\Delta A}_{\mathrm{d}}(t)$ are real-unknown but norm-bounded matrix functions representing time-varying uncertainties. The admissible uncertainties are defined as:

$$[\boldsymbol{\Delta A}\ \ \boldsymbol{\Delta A}_{\mathrm{d}}] = \boldsymbol{DF}(t)\,[\boldsymbol{E}_{\mathrm{a}}\ \ \boldsymbol{E}_{\mathrm{d}}] \tag{2}$$

where $\boldsymbol{D}$, $\boldsymbol{E}_{\mathrm{a}}$, $\boldsymbol{E}_{\mathrm{d}}$ are real-known constant matrices and $\boldsymbol{F}(t)$ is a real-unknown time-varying matrix with Lebesgue measurable elements satisfying $\boldsymbol{F}^{\mathrm{T}}(t)\boldsymbol{F}(t)\le\boldsymbol{I}$.

Assumption 1. $\boldsymbol{Hw}(t)$ satisfies the matching condition $\boldsymbol{Hw}(t)\in\mathrm{range}\,\boldsymbol{B}$, thus there exists $\boldsymbol{d}(t)\in\mathbb{R}^m$ such that $\boldsymbol{Hw}(t)=\boldsymbol{Bd}(t)$. And each component of $\boldsymbol{d}(t)$ is bounded by the known $\bar{d}_i(t)$, i.e. $d_i(t)\le\bar{d}_i(t)$.

Assumption 2. The actuator faults can be denoted by $\boldsymbol{B}_f=\boldsymbol{BL}$, where

$$\boldsymbol{L}=\begin{bmatrix}1-\ell_1 & 0 & \cdots & 0\\ 0 & 1-\ell_2 & \ddots & 0\\ \vdots & \ddots & \ddots & \vdots\\ 0 & \cdots & 0 & 1-\ell_m\end{bmatrix}$$

and ℓ_i, $i=1,\dots,m$ denote the control effectiveness factors. $\ell_i=0$, $i=1,\dots,m$ denote the healthy ith actuator and $\ell_i=1$ corresponds to total failure of the ith actuator.

By the assumption $\mathrm{rank}(\boldsymbol{B})=m$, one can get the singular value decomposition (SVD) of $\boldsymbol{B}$:

$\boldsymbol{B}=[\boldsymbol{U}_1\ \ \boldsymbol{U}_2]\begin{bmatrix}\sum\\ \boldsymbol{0}_{(n-m)\times m}\end{bmatrix}\boldsymbol{V}^{\mathrm{T}}$, where $\sum\in\mathbb{R}^{m\times m}$ is a diagonal positive-definite matrix and $\boldsymbol{V}$, $[\boldsymbol{U}_1\ \ \boldsymbol{U}_2]$ are unitary matrices.

Define $\boldsymbol{T}=\begin{bmatrix}\boldsymbol{U}_1^{\mathrm{T}}\\ \boldsymbol{U}_2^{\mathrm{T}}\end{bmatrix}$ then we can have $\boldsymbol{TB}=\begin{bmatrix}\boldsymbol{B}_1\\ \boldsymbol{0}_{(n-m)\times m}\end{bmatrix}$. By the state transformation $\boldsymbol{z}=\boldsymbol{Tx}$, (1) has the form (with Assumptions 1 and 2):

$$\begin{aligned}\dot{\boldsymbol{z}}(t) &= (\bar{\boldsymbol{A}}+\boldsymbol{\Delta}\bar{\boldsymbol{A}}(t))\boldsymbol{z}(t)+(\bar{\boldsymbol{A}}_{\mathrm{d}}+\boldsymbol{\Delta}\bar{\boldsymbol{A}}_{\mathrm{d}}(t))\boldsymbol{z}(t-\tau(t))+\begin{bmatrix}\boldsymbol{B}_1\\ \boldsymbol{0}_{(n-m)\times m}\end{bmatrix}\boldsymbol{L}\,(\boldsymbol{u}(t)+\boldsymbol{d}(t))\\ \boldsymbol{z}(t) &= \bar{\phi}(t), \quad t\in[-\bar{\tau},0)\end{aligned} \tag{3}$$

where $\bar{\boldsymbol{A}} = \boldsymbol{TAT}^{-1}$, $\bar{\boldsymbol{A}}_{\mathrm{d}} = \boldsymbol{TA}_{\mathrm{d}}\boldsymbol{T}^{-1}$, $\boldsymbol{\Delta}\bar{\boldsymbol{A}} = \boldsymbol{T\Delta AT}^{-1}$, $\boldsymbol{\Delta}\bar{\boldsymbol{A}}_{\mathrm{d}} = \boldsymbol{T\Delta A}_{\mathrm{d}}\boldsymbol{T}^{-1}$, and $\bar{\phi}(t) = \boldsymbol{T}\phi(t)$.

Furthermore, similarly to [17] one can get

$$\begin{aligned}
\dot{\boldsymbol{z}}_1(t) &= (\bar{\boldsymbol{A}}_{11} + \boldsymbol{\Delta}\bar{\boldsymbol{A}}_{11}(t))\boldsymbol{z}_1(t) + (\bar{\boldsymbol{A}}_{\mathrm{d}11} + \boldsymbol{\Delta}\bar{\boldsymbol{A}}_{\mathrm{d}11}(t))\boldsymbol{z}_1(t-\tau(t)) \\
&\quad + (\bar{\boldsymbol{A}}_{12} + \boldsymbol{\Delta}\bar{\boldsymbol{A}}_{12}(t))\boldsymbol{z}_2(t) + (\bar{\boldsymbol{A}}_{\mathrm{d}12} + \boldsymbol{\Delta}\bar{\boldsymbol{A}}_{\mathrm{d}12}(t))\boldsymbol{z}_2(t-\tau(t)) \\
&\quad + \boldsymbol{B}_1\boldsymbol{L}(\boldsymbol{u}(t) + \boldsymbol{d}(t)) \\
\dot{\boldsymbol{z}}_2(t) &= (\bar{\boldsymbol{A}}_{21} + \boldsymbol{\Delta}\bar{\boldsymbol{A}}_{21}(t))\boldsymbol{z}_1(t) + (\bar{\boldsymbol{A}}_{\mathrm{d}21} + \boldsymbol{\Delta}\bar{\boldsymbol{A}}_{\mathrm{d}21}(t))\boldsymbol{z}_1(t-\tau(t)) \\
&\quad + (\bar{\boldsymbol{A}}_{22} + \boldsymbol{\Delta}\bar{\boldsymbol{A}}_{22}(t))\boldsymbol{z}_2(t) + (\bar{\boldsymbol{A}}_{\mathrm{d}22} + \boldsymbol{\Delta}\bar{\boldsymbol{A}}_{\mathrm{d}22}(t))\boldsymbol{z}_2(t-\tau(t)) \\
\boldsymbol{z}_1(t) &= \bar{\phi}_1(t), \quad t \in [-\bar{\tau}, 0), \qquad \boldsymbol{z}_2(t) = \bar{\phi}_2(t), \quad t \in [-\bar{\tau}, 0)
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
&\boldsymbol{z}_1(t) \in \mathbb{R}^m, \quad \boldsymbol{z}_2(t) \in \mathbb{R}^{n-m}, \quad \boldsymbol{B}_1 = \sum \boldsymbol{V}^{\mathrm{T}}, \quad \bar{\boldsymbol{A}}_{\mathbf{11}} = \boldsymbol{U}_1^{\mathrm{T}}\boldsymbol{A}\boldsymbol{U}_1, \\
&\bar{\boldsymbol{A}}_{12} = \boldsymbol{U}_1^{\mathrm{T}}\boldsymbol{A}\boldsymbol{U}_2, \quad \bar{\boldsymbol{A}}_{21} = \boldsymbol{U}_2^{\mathrm{T}}\boldsymbol{A}\boldsymbol{U}_1, \quad \bar{\boldsymbol{A}}_{22} = \boldsymbol{U}_2^{\mathrm{T}}\boldsymbol{A}\boldsymbol{U}_2
\end{aligned}$$

and the other parameter matrices can be got in the same way.

Then one can see that the second equation of (4) represents the sliding motion dynamics of (3), and the sliding mode surface can be chosen as follows:

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{C} \end{bmatrix} \boldsymbol{z} = \boldsymbol{z}_1 + \boldsymbol{C}\boldsymbol{z}_2 = \mathbf{0} \tag{5}$$

where $\boldsymbol{C} \in \mathbb{R}^{n-m}$. Substituting (5) into (4) shows the sliding motion:

$$\begin{aligned}
\dot{\boldsymbol{z}}_2(t) &= (\bar{\boldsymbol{A}}_{22} + \boldsymbol{\Delta}\bar{\boldsymbol{A}}_{22}(t) - \bar{\boldsymbol{A}}_{21}\boldsymbol{C} - \boldsymbol{\Delta}\bar{\boldsymbol{A}}_{21}(t)\boldsymbol{C})\boldsymbol{z}_2(t) \\
&\quad + (\bar{\boldsymbol{A}}_{\mathrm{d}22} + \boldsymbol{\Delta}\bar{\boldsymbol{A}}_{\mathrm{d}22}(t) - \bar{\boldsymbol{A}}_{\mathrm{d}21}\boldsymbol{C} - \boldsymbol{\Delta}\bar{\boldsymbol{A}}_{\mathrm{d}21}(t)\boldsymbol{C})\boldsymbol{z}_2(t-\tau(t)) \\
\boldsymbol{z}_2(t) &= \bar{\phi}_2(t), \quad t \in [-\bar{\tau}, 0).
\end{aligned} \tag{6}$$

The objective of this paper is to design constant gain $\boldsymbol{C}$ and control law $\boldsymbol{u}(t)$ such that:

1. the sliding motion (6) is quadratically stable;
2. system (4) is asymptotically stable with the reaching control law $\boldsymbol{u}(t)$ in the present of actuator fault and disturbance $\boldsymbol{d}(t)$.

3. MAIN RESULTS

3.1. Sliding surface design

Let us consider the nominal time-delay system that can be obtained from (6) by setting $\boldsymbol{\Delta A} \equiv \mathbf{0}$, $\boldsymbol{\Delta A}_{\mathrm{d}}(\mathrm{t}) \equiv \mathbf{0}$.

$$\begin{aligned}
\dot{\boldsymbol{z}}_2(t) &= (\bar{\boldsymbol{A}}_{22} - \bar{\boldsymbol{A}}_{21}\boldsymbol{C})\boldsymbol{z}_2(t) + (\bar{\boldsymbol{A}}_{\mathrm{d}22} - \bar{\boldsymbol{A}}_{\mathrm{d}21}\boldsymbol{C})\boldsymbol{z}_2(t-\tau(t)) \\
\boldsymbol{z}_2(t) &= \bar{\phi}_2(t), \quad t \in [-\bar{\tau}, 0).
\end{aligned} \tag{7}$$

Using descriptor type model transformation, one can represent (7) in descriptor form as:

$$\begin{aligned}\dot{\boldsymbol{z}}_2(t) &= \boldsymbol{y}(t)\\ \boldsymbol{y}(t) &= (\bar{\boldsymbol{A}}_{22} - \bar{\boldsymbol{A}}_{21}\boldsymbol{C})\boldsymbol{z}_2(t) + (\bar{\boldsymbol{A}}_{\mathrm{d}22} - \bar{\boldsymbol{A}}_{\mathrm{d}21}\boldsymbol{C})\boldsymbol{z}_2(t-\tau(t))\end{aligned} \tag{8}$$

where $\boldsymbol{y}(t)$ is the 'fast varying' state variable.

Theorem 1. Consider the linear time-delay sliding motion (7). Then given the scalars $\bar{\tau} > 0$ and μ, this sliding motion is quadratically stable for any time-delay satisfying $0 \le \tau(t) \le \bar{\tau}$, $|\dot{\tau}(t)| \le \mu$, $\forall t \ge 0$ if there exist matrix $\boldsymbol{C}$, symmetric positive definite (s.p.d.) matrices $\boldsymbol{P}_1$, $\boldsymbol{P}_{22}$, $\boldsymbol{Q}_{11}$, $\boldsymbol{Q}_{22}$, $\boldsymbol{R}$, $\boldsymbol{S}$, $\boldsymbol{T}$ and arbitrary matrices $\boldsymbol{P}_{12}$, $\boldsymbol{Q}_{12}$, $\boldsymbol{P}_i\,(i = 2,\dots,11)$ with appropriate dimensions satisfying[1]

$$\begin{bmatrix} \boldsymbol{P}_1 & \boldsymbol{P}_{12} \\ * & \boldsymbol{P}_{22}\end{bmatrix} \ge \boldsymbol{0}, \qquad \begin{bmatrix} \boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} \\ * & \boldsymbol{Q}_{22}\end{bmatrix} \ge \boldsymbol{0}$$

and

$$\boldsymbol{\Pi} = \begin{bmatrix} \boldsymbol{\Pi}_{11} & \boldsymbol{\Pi}_{12} & \boldsymbol{\Pi}_{13} & \boldsymbol{\Pi}_{14} & \boldsymbol{\Pi}_{15} & \boldsymbol{0} \\ * & \boldsymbol{\Pi}_{22} & \boldsymbol{\Pi}_{23} & \boldsymbol{\Pi}_{24} & \boldsymbol{\Pi}_{25} & \boldsymbol{0} \\ * & * & \boldsymbol{\Pi}_{33} & \boldsymbol{\Pi}_{34} & \boldsymbol{\Pi}_{35} & \boldsymbol{0} \\ * & * & * & \boldsymbol{\Pi}_{44} & \boldsymbol{\Pi}_{45} & \boldsymbol{\Pi}_{46} \\ * & * & * & * & \boldsymbol{\Pi}_{55} & \boldsymbol{0} \\ * & * & * & * & * & \boldsymbol{\Pi}_{66} \end{bmatrix} < \boldsymbol{0} \tag{9}$$

where

$$\begin{aligned}
\boldsymbol{\Pi}_{11} &= \boldsymbol{P}_2^{\mathrm{T}}(\bar{\boldsymbol{A}}_{22} - \bar{\boldsymbol{A}}_{21}\boldsymbol{C}) + (\bar{\boldsymbol{A}}_{22}^{\mathrm{T}} - \boldsymbol{C}^{\mathrm{T}}\bar{\boldsymbol{A}}_{21}^{\mathrm{T}})\boldsymbol{P}_2 + \boldsymbol{P}_7 + \boldsymbol{P}_7^{\mathrm{T}} + \mu\boldsymbol{P}_{12}\boldsymbol{S}^{-1}\boldsymbol{P}_{12}^{\mathrm{T}} + \boldsymbol{R} + \bar{\tau}^2\boldsymbol{Q}_{11},\\
\boldsymbol{\Pi}_{12} &= \boldsymbol{P}_1 - \boldsymbol{P}_2^{\mathrm{T}} + \left(\bar{\boldsymbol{A}}_{22}^{\mathrm{T}} - \boldsymbol{C}^{\mathrm{T}}\bar{\boldsymbol{A}}_{21}^{\mathrm{T}}\right)\boldsymbol{P}_3 + \boldsymbol{P}_8 + \bar{\tau}^2\boldsymbol{Q}_{12},\\
\boldsymbol{\Pi}_{13} &= \boldsymbol{P}_2^{\mathrm{T}}(\bar{\boldsymbol{A}}_{\mathrm{d}22} - \bar{\boldsymbol{A}}_{21}\boldsymbol{C}) + \left(\bar{\boldsymbol{A}}_{22}^{\mathrm{T}} - \boldsymbol{C}^{\mathrm{T}}\bar{\boldsymbol{A}}_{21}^{\mathrm{T}}\right)\boldsymbol{P}_4 - \boldsymbol{P}_7^{\mathrm{T}} + \boldsymbol{P}_9,\\
\boldsymbol{\Pi}_{14} &= \left(\bar{\boldsymbol{A}}_{22}^{\mathrm{T}} - \boldsymbol{C}^{\mathrm{T}}\bar{\boldsymbol{A}}_{21}^{\mathrm{T}}\right)\boldsymbol{P}_5 + \boldsymbol{P}_{10}, \quad \boldsymbol{\Pi}_{15} = \left(\bar{\boldsymbol{A}}_{22}^{\mathrm{T}} - \boldsymbol{C}^{\mathrm{T}}\bar{\boldsymbol{A}}_{21}^{\mathrm{T}}\right)\boldsymbol{P}_6 + \boldsymbol{P}_{11} + \boldsymbol{P}_{12} - \boldsymbol{P}_7^{\mathrm{T}},\\
\boldsymbol{\Pi}_{22} &= \bar{\tau}^2\boldsymbol{Q}_{22} - \boldsymbol{P}_3 - \boldsymbol{P}_3^{\mathrm{T}}, \qquad \boldsymbol{\Pi}_{23} = \boldsymbol{P}_3^{\mathrm{T}}(\bar{\boldsymbol{A}}_{\mathrm{d}22} - \bar{\boldsymbol{A}}_{\mathrm{d}21}\boldsymbol{C}) - \boldsymbol{P}_8^{\mathrm{T}} - \boldsymbol{P}_4,\\
\boldsymbol{\Pi}_{24} &= \boldsymbol{P}_{12} - \boldsymbol{P}_5, \qquad \boldsymbol{\Pi}_{25} = -\boldsymbol{P}_6 - \boldsymbol{P}_8^{\mathrm{T}},\\
\boldsymbol{\Pi}_{33} &= \boldsymbol{P}_4^{\mathrm{T}}(\bar{\boldsymbol{A}}_{\mathrm{d}22} - \bar{\boldsymbol{A}}_{\mathrm{d}21}\boldsymbol{C}) + \left(\bar{\boldsymbol{A}}_{\mathrm{d}22}^{\mathrm{T}} - \boldsymbol{C}^{\mathrm{T}}\bar{\boldsymbol{A}}_{\mathrm{d}21}^{\mathrm{T}}\right)\boldsymbol{P}_4 - \boldsymbol{P}_9 - \boldsymbol{P}_9^{\mathrm{T}} + \mu(\boldsymbol{S} + \boldsymbol{T}) + (\mu - 1)\boldsymbol{R},\\
\boldsymbol{\Pi}_{34} &= \left(\bar{\boldsymbol{A}}_{\mathrm{d}22}^{\mathrm{T}} - \boldsymbol{C}^{\mathrm{T}}\bar{\boldsymbol{A}}_{\mathrm{d}21}^{\mathrm{T}}\right)\boldsymbol{P}_5 - \boldsymbol{P}_{10}, \quad \boldsymbol{\Pi}_{35} = \left(\bar{\boldsymbol{A}}_{\mathrm{d}22}^{\mathrm{T}} - \boldsymbol{C}^{\mathrm{T}}\bar{\boldsymbol{A}}_{\mathrm{d}21}^{\mathrm{T}}\right)\boldsymbol{P}_6 - \boldsymbol{P}_{11} - \boldsymbol{P}_9^{\mathrm{T}},\\
\boldsymbol{\Pi}_{44} &= -\boldsymbol{Q}_{11}, \qquad \boldsymbol{\Pi}_{45} = \mathbf{P}_{22} - \boldsymbol{P}_{10}^{\mathrm{T}} - \boldsymbol{Q}_{12}, \qquad \boldsymbol{\Pi}_{46} = \mu\boldsymbol{P}_{22},\\
\boldsymbol{\Pi}_{55} &= -\boldsymbol{P}_{11} - \boldsymbol{P}_{11}^{\mathrm{T}} - \boldsymbol{Q}_{22}, \qquad \boldsymbol{\Pi}_{66} = -\mu\boldsymbol{T}.
\end{aligned}$$

And the sliding surface can be chosen as $\boldsymbol{S} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{C}\end{bmatrix}\boldsymbol{z} = \boldsymbol{z}_1 + \boldsymbol{C}\boldsymbol{z}_2 = \boldsymbol{0}$.

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[1] $*$ denotes the symmetricity of the matrix.

P r o o f. Choosing a candidate Lyapunov–Krasovskii functional as in [13]:

$$V(\boldsymbol{z}_2(t),t)=\boldsymbol{Z}^{\mathrm{T}}(t)\boldsymbol{E}\boldsymbol{P}\boldsymbol{Z}(t)+\begin{bmatrix}\boldsymbol{z}_2(t)\\ \int_{t-\tau(t)}^{t}\boldsymbol{z}_2(s)\,\mathrm{d}s\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\mathbf{0} & \boldsymbol{P}_{12}\\ * & \boldsymbol{P}_{22}\end{bmatrix}\begin{bmatrix}\boldsymbol{z}_2(t)\\ \int_{t-\tau(t)}^{t}\boldsymbol{z}_2(s)\,\mathrm{d}s\end{bmatrix}$$
$$+\bar{\tau}\int_{-\bar{\tau}}^{0}\int_{t+\theta}^{t}\begin{bmatrix}\boldsymbol{z}_2(s)\\ \boldsymbol{y}(s)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{Q}_{11} & \boldsymbol{Q}_{12}\\ * & \boldsymbol{Q}_{22}\end{bmatrix}\begin{bmatrix}\boldsymbol{z}_2(s)\\ \boldsymbol{y}(s)\end{bmatrix}\mathrm{d}s\,\mathrm{d}\theta$$
$$+\int_{t-\tau(t)}^{t}\boldsymbol{z}_2^{\mathrm{T}}(s)\boldsymbol{R}\boldsymbol{z}_2(s)\,\mathrm{d}s \tag{10}$$

where

$$\boldsymbol{Z}(t)=\begin{bmatrix}\boldsymbol{z}_2^{\mathrm{T}}(t) & \boldsymbol{y}(t) & \boldsymbol{z}_2^{\mathrm{T}}(t-\tau(t)) & \left(\int_{t-\tau(t)}^{t}\boldsymbol{z}(s)\,\mathrm{d}s\right)^{\mathrm{T}} & \left(\int_{t-\tau(t)}^{t}\mathbf{y}(s)\,\mathrm{d}s\right)^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}},$$

$$\boldsymbol{E}=\begin{bmatrix}\boldsymbol{I} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix},\qquad \boldsymbol{P}=\begin{bmatrix}\boldsymbol{P}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \boldsymbol{P}_2 & \boldsymbol{P}_3 & \boldsymbol{P}_4 & \boldsymbol{P}_5 & \boldsymbol{P}_6\\ \boldsymbol{P}_7 & \boldsymbol{P}_8 & \boldsymbol{P}_9 & \boldsymbol{P}_{10} & \boldsymbol{P}_{11}\end{bmatrix}.$$

Then the time derivative of (9) along the trajectory (8) is

$$\dot{V}(\boldsymbol{z}_2(t),t)=2\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{Z}^{\mathrm{T}}(t)\right)\boldsymbol{E}\boldsymbol{P}\boldsymbol{Z}(t)+2\begin{bmatrix}\boldsymbol{z}_2(t)\\ \int_{t-\tau(t)}^{t}\boldsymbol{z}_2(s)\,\mathrm{d}s\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\mathbf{0} & \boldsymbol{P}_{12}\\ * & \boldsymbol{P}_{22}\end{bmatrix}$$
$$\times\begin{bmatrix}\boldsymbol{y}(t)\\ \int_{t-\tau(t)}^{t}\mathbf{y}(s)\,\mathrm{d}s+\dot{\tau}(t)\mathbf{z}_2(t-\tau(t))\end{bmatrix}$$
$$+\bar{\tau}^2\begin{bmatrix}\boldsymbol{z}_2(t)\\ \boldsymbol{y}(t)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{Q}_{11} & \boldsymbol{Q}_{12}\\ * & \boldsymbol{Q}_{22}\end{bmatrix}\begin{bmatrix}\boldsymbol{z}_2(t)\\ \boldsymbol{y}(t)\end{bmatrix} \tag{11}$$
$$-\bar{\tau}\int_{t-\bar{\tau}}^{t}\begin{bmatrix}\boldsymbol{z}_2(s)\\ \boldsymbol{y}(s)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{Q}_{11} & \boldsymbol{Q}_{12}\\ * & \boldsymbol{Q}_{22}\end{bmatrix}\begin{bmatrix}\boldsymbol{z}_2(s)\\ \boldsymbol{y}(s)\end{bmatrix}\mathrm{d}s+\boldsymbol{z}_2^{\mathrm{T}}(t)\boldsymbol{R}\boldsymbol{z}_2(t)$$
$$-\left[1-\dot{\tau}(t)\right]\boldsymbol{z}_2^{\mathrm{T}}(t-\tau(t))\boldsymbol{R}\boldsymbol{z}_2(t-\tau(t)).$$

Using (8) and Newton–Leibniz formula $\int_{t-\tau(t)}^{t}\boldsymbol{y}(s)\,\mathrm{d}s=\boldsymbol{z}_2(t)-\boldsymbol{z}_2(t-\tau(t))$, one can see:

$$2\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{Z}^{\mathrm{T}}\right)\boldsymbol{E}\boldsymbol{P}\boldsymbol{Z}(t)$$
$$=2\begin{bmatrix}\dot{\boldsymbol{z}}_2(t) & \dot{\boldsymbol{y}}(t) & \dot{\boldsymbol{z}}_2(t-\tau(t)) & \boldsymbol{z}_2(t)-\boldsymbol{z}_2(t-\tau(t)) & \boldsymbol{y}(t)-\boldsymbol{y}(t-\tau(t))\end{bmatrix}\boldsymbol{E}\boldsymbol{P}\boldsymbol{Z}(t)$$
$$=2\begin{bmatrix}\dot{\boldsymbol{z}}_2(t) & \mathbf{0} & \mathbf{0}\end{bmatrix}\boldsymbol{P}\boldsymbol{Z}(t)=\begin{bmatrix}\boldsymbol{y}(t)\\ \mathbf{0}\\ \mathbf{0}\end{bmatrix}$$
$$=2\boldsymbol{Z}^{\mathrm{T}}(t)\boldsymbol{P}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{y}(t)\\ -\boldsymbol{y}(t)+(\bar{\boldsymbol{A}}_{22}-\bar{\boldsymbol{A}}_{21}\mathbf{C})\boldsymbol{z}_2(t)+(\bar{\boldsymbol{A}}_{\mathrm{d}22}-\bar{\boldsymbol{A}}_{\mathrm{d}21}\boldsymbol{C})\boldsymbol{z}_2(t-\tau)\\ \boldsymbol{z}_2(t)-\boldsymbol{z}_2(t-\tau(t))-\int_{t-\tau(t)}^{t}\boldsymbol{y}(s)\,\mathrm{d}s\end{bmatrix}$$
$$=\boldsymbol{Z}^{\mathrm{T}}(t)\left(\boldsymbol{\Omega}_0^{\mathrm{T}}+\boldsymbol{\Omega}_0\right)\boldsymbol{Z}(t) \tag{12}$$

where $\boldsymbol{\Omega}_0 = \boldsymbol{P}^{\mathrm{T}} \begin{bmatrix} \mathbf{0} & \boldsymbol{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \bar{\boldsymbol{A}}_{22} - \bar{\boldsymbol{A}}_{21}\boldsymbol{C} & -\boldsymbol{I} & \bar{\boldsymbol{A}}_{\mathrm{d}22} - \bar{\boldsymbol{A}}_{\mathrm{d}21}\boldsymbol{C} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{I} & \mathbf{0} & -\boldsymbol{I} & \mathbf{0} & -\boldsymbol{I} \end{bmatrix}$.

Moreover,

$$
\begin{aligned}
&2 \begin{bmatrix} \boldsymbol{z}_2(t) \\ \int_{t-\tau(t)}^{t} \boldsymbol{z}_2(s)\,\mathrm{d}s \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{0} & \boldsymbol{P}_{12} \\ * & \boldsymbol{P}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{y}(t) \\ \int_{t-\tau(t)}^{t} \boldsymbol{y}(s)\,\mathrm{d}s + \dot{\tau}(t)\boldsymbol{z}_2(t-\tau(t)) \end{bmatrix} \\
&= 2 \begin{bmatrix} \boldsymbol{z}_2(t) \\ \int_{t-\tau(t)}^{t} \boldsymbol{z}_2(s)\,\mathrm{d}s \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{P}_{12}\left(\int_{t-\tau(t)}^{t} \boldsymbol{y}(s)\,\mathrm{d}s + \dot{\tau}(t)\boldsymbol{z}_2(t-\tau(t))\right) \\ \boldsymbol{P}_{12}^{\mathrm{T}}\boldsymbol{y}(t) + \boldsymbol{P}_{22}\left(\int_{t-\tau(t)}^{t} \boldsymbol{y}(s)\,\mathrm{d}s + \dot{\tau}(t)\boldsymbol{z}_2(t-\tau(t))\right) \end{bmatrix} \\
&= 2\boldsymbol{z}_2^{\mathrm{T}}(t)\boldsymbol{P}_{12}\left(\int_{t-\tau(t)}^{t} \boldsymbol{y}(s)\,\mathrm{d}s + \dot{\tau}(t)\boldsymbol{z}_2(t-\tau(t))\right) \\
&\quad +2\left(\int_{t-\tau(t)}^{t} \boldsymbol{z}_2(s)\,\mathrm{d}s\right)^{\mathrm{T}} \left(\boldsymbol{P}_{12}^{\mathrm{T}}\boldsymbol{y}(t) + \boldsymbol{P}_{22}\left(\int_{t-\tau(t)}^{t} \boldsymbol{y}(s)\,\mathrm{d}s + \dot{\tau}(t)\boldsymbol{z}_2(t-\tau(t))\right)\right) \\
&\le \boldsymbol{Z}^{\mathrm{T}}(t)\boldsymbol{\Omega}_1\boldsymbol{Z}(t)
\end{aligned}
\tag{13}
$$

where $\boldsymbol{\Omega}_1 = \begin{bmatrix} \mu\boldsymbol{P}_{12}\boldsymbol{S}^{-1}\boldsymbol{P}_{12}^{\mathrm{T}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{P}_{12} \\ * & \mathbf{0} & \mathbf{0} & \boldsymbol{P}_{12} & \mathbf{0} \\ * & * & \mu(\boldsymbol{S}+\boldsymbol{T}) & \mathbf{0} & \mathbf{0} \\ * & * & * & \mu\boldsymbol{P}_{22}\boldsymbol{T}^{-1}\boldsymbol{P}_{22} & \boldsymbol{P}_{22} \\ * & * & * & * & \mathbf{0} \end{bmatrix}$, $\boldsymbol{S}$, $\boldsymbol{T}$ are some s.p.d. matrices and $*$ denotes the symmetricity of the matrix.

The remainder of (10) can be dealt with as:

$$
\begin{aligned}
&\bar{\tau}^2 \begin{bmatrix} \boldsymbol{z}_2(t) \\ \boldsymbol{y}(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} \\ * & \boldsymbol{Q}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{z}_2(t) \\ \boldsymbol{y}(t) \end{bmatrix} \\
&\qquad - \bar{\tau}\int_{t-\bar{\tau}}^{t} \begin{bmatrix} \boldsymbol{z}_2(s) \\ \boldsymbol{y}(s) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} \\ * & \boldsymbol{Q}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{z}_2(s) \\ \boldsymbol{y}(s) \end{bmatrix} \mathrm{d}s \\
&\qquad + \boldsymbol{z}_2^{\mathrm{T}}(t)\boldsymbol{R}\boldsymbol{z}_2(t) - [1-\dot{\tau}(t)]\,\boldsymbol{z}_2^{\mathrm{T}}(t-\tau(t))\boldsymbol{R}\boldsymbol{z}_2(t-\tau(t)) \\
&\le \bar{\tau}^2 \begin{bmatrix} \boldsymbol{z}_2(t) \\ \boldsymbol{y}(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} \\ * & \boldsymbol{Q}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{z}_2(t) \\ \boldsymbol{y}(t) \end{bmatrix} \\
&\qquad - \tau(t)\int_{t-\tau(t)}^{t} \begin{bmatrix} \boldsymbol{z}_2(s) \\ \boldsymbol{y}(s) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} \\ * & \boldsymbol{Q}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{z}_2(s) \\ \boldsymbol{y}(s) \end{bmatrix} \mathrm{d}s \qquad\qquad (14) \\
&\qquad + \boldsymbol{z}_2^{\mathrm{T}}(t)\boldsymbol{R}\boldsymbol{z}_2(t) - \boldsymbol{z}_2^{\mathrm{T}}(t-\tau(t))\,(1-\mu)\,\boldsymbol{R}\boldsymbol{z}_2(t-\tau(t)) \\
&\le \bar{\tau}^2 \begin{bmatrix} \boldsymbol{z}_2(t) \\ \boldsymbol{y}(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} \\ * & \boldsymbol{Q}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{z}_2(t) \\ \boldsymbol{y}(t) \end{bmatrix} \\
&\qquad - \begin{bmatrix} \int_{t-\tau(t)}^{t} \boldsymbol{z}_2(s)\,\mathrm{d}s \\ \int_{t-\tau(t)}^{t} \boldsymbol{y}(s)\,\mathrm{d}s \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} \\ * & \boldsymbol{Q}_{22} \end{bmatrix} \begin{bmatrix} \int_{t-\tau(t)}^{t} \boldsymbol{z}_2(s)\,\mathrm{d}s \\ \int_{t-\tau(t)}^{t} \boldsymbol{y}(s)\,\mathrm{d}s \end{bmatrix} \\
&+ \boldsymbol{z}_2^{\mathrm{T}}(t)\boldsymbol{R}\boldsymbol{z}_2(t) - \boldsymbol{z}_2^{\mathrm{T}}(t-\tau(t))\,(1-\mu)\,\boldsymbol{R}\boldsymbol{z}_2(t-\tau(t)) \;=\; \boldsymbol{Z}^{\mathrm{T}}(t)\boldsymbol{\Omega}_2\boldsymbol{Z}(t)
\end{aligned}
$$

where

$$\boldsymbol{\Omega}_2 = \begin{bmatrix} \boldsymbol{R} + \bar{\tau}^2 \boldsymbol{Q}_{11} & \bar{\tau}^2 \boldsymbol{Q}_{12} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & \bar{\tau}^2 \boldsymbol{Q}_{22} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & (\mu - 1)\boldsymbol{R} & \mathbf{0} & \mathbf{0} \\ * & * & * & -\boldsymbol{Q}_{11} & -\boldsymbol{Q}_{12} \\ * & * & * & * & -\boldsymbol{Q}_{22} \end{bmatrix}.$$

From (12) – (14), one can have:

$$\dot{\boldsymbol{V}}(\boldsymbol{z}_2(t), t) \le \boldsymbol{Z}^{\mathrm{T}}(t)\boldsymbol{\Omega}\boldsymbol{Z}(t) \quad \text{where} \quad \boldsymbol{\Omega} = \boldsymbol{\Omega}_0 + \boldsymbol{\Omega}_0^{\mathrm{T}} + \boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2.$$

According to Lyapunov–Krasovskii theorem, system (7) is asymptotically stable when $\dot{\boldsymbol{V}}(\boldsymbol{z}_2(t), t) < 0$. In order to guarantee $\dot{\boldsymbol{V}}(\boldsymbol{z}_2(t), t) < 0$, one needs $\boldsymbol{\Omega} < \mathbf{0}$. It also implies that there exists a sufficiently small $\xi > 0$ such that:

$$\boldsymbol{\Omega} + \begin{bmatrix} \xi\boldsymbol{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} < \mathbf{0}. \tag{15}$$

From the inequality above, one can get $\dot{\boldsymbol{V}}(\boldsymbol{z}_2(t), t) < \xi \left\|\boldsymbol{z}_2\right\|^2$. Thus the sliding motion (7) is quadratically stable (cf. [10]) with the matrices defined in Theorem 1. Note that in the term $\boldsymbol{\Omega}_{44} = \mu\boldsymbol{P}_{22}\boldsymbol{T}^{-1}\boldsymbol{P}_{22} - \boldsymbol{Q}_{11}$, $\boldsymbol{P}_{22}$ and $\boldsymbol{T}$ are both unknown variables and in a multiplicative form, so the inequality $\boldsymbol{\Omega} < 0$ is a bilinear one which maybe not dealt with easily. In fact, by applying Shur's complement (cf. [6]) to $\boldsymbol{\Omega} < 0$, the resulting inequality (9) is a linear matrix inequality (LMI) that can be solved effectively by using Matlab LMI toolbox. This completes the proof. □

The uncertain sliding motion (6) is considered in the next part of this subsection. The following theorem provides the sufficient conditions for the robust stability of sliding motion (6).

Theorem 2. Consider the uncertain time-delay sliding motion (6). Then given the scalars $\bar{\tau} > 0$ and μ, this sliding motion is quadratically stable for any time-delay satisfying $0 \le \tau(t) \le \bar{\tau}$, $|\dot{\tau}(t)| \le \mu$, $\forall t \ge 0$ if there exist matrix $\boldsymbol{C}$, positive scalars ε, γ, symmetric positive definite (s.p.d.) matrices $\boldsymbol{P}_1$, $\boldsymbol{P}_{22}$, $\boldsymbol{Q}_{11}$, $\boldsymbol{Q}_{22}$, $\boldsymbol{R}$, $\boldsymbol{S}$, $\boldsymbol{T}$ and arbitrary matrices $\boldsymbol{P}_{12}$, $\boldsymbol{Q}_{12}$, $\boldsymbol{P}_i$ $(i = 2, \ldots, 11)$ with appropriate dimensions satisfying

$$\begin{bmatrix} \boldsymbol{P}_1 & \boldsymbol{P}_{12} \\ * & \boldsymbol{P}_{22} \end{bmatrix} \ge \mathbf{0}, \begin{bmatrix} \boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} \\ * & \boldsymbol{Q}_{22} \end{bmatrix} \ge \mathbf{0}$$

and

$$\boldsymbol{\Pi}_u = \begin{bmatrix} \boldsymbol{\Pi}_{11} + \varepsilon \boldsymbol{E}_a \boldsymbol{U}_2 \boldsymbol{E}_a \boldsymbol{U}_2 & \boldsymbol{\Pi}_{12} & \boldsymbol{\Pi}_{13} + \varepsilon \boldsymbol{E}_a \boldsymbol{U}_2 \boldsymbol{E}_\mathrm{d} \boldsymbol{U}_2 & \boldsymbol{\Pi}_{14} & \boldsymbol{\Pi}_{15} & \boldsymbol{0} & \boldsymbol{P}_2^\mathrm{T} \boldsymbol{U}_2^\mathrm{T} \boldsymbol{D} & \boldsymbol{C}^\mathrm{T} \boldsymbol{U}_1^\mathrm{T} \boldsymbol{E}_a^\mathrm{T} & -\boldsymbol{P}_2^\mathrm{T} \boldsymbol{U}_2^\mathrm{T} \boldsymbol{D} \\ * & \boldsymbol{\Pi}_{22} & \boldsymbol{\Pi}_{23} & \boldsymbol{\Pi}_{24} & \boldsymbol{\Pi}_{25} & \boldsymbol{0} & \boldsymbol{P}_3^\mathrm{T} \boldsymbol{U}_2^\mathrm{T} \boldsymbol{D} & \boldsymbol{0} & -\boldsymbol{P}_3^\mathrm{T} \boldsymbol{U}_2^\mathrm{T} \boldsymbol{D} \\ * & * & \boldsymbol{\Pi}_{33} + \varepsilon \boldsymbol{E}_\mathrm{d} \boldsymbol{U}_2 \boldsymbol{E}_\mathrm{d} \boldsymbol{U}_2 & \boldsymbol{\Pi}_{34} & \boldsymbol{\Pi}_{35} & \boldsymbol{0} & \boldsymbol{P}_4^\mathrm{T} \boldsymbol{U}_2^\mathrm{T} \boldsymbol{D} & \boldsymbol{C}^\mathrm{T} \boldsymbol{U}_1^\mathrm{T} \boldsymbol{E}_\mathrm{d}^\mathrm{T} & -\boldsymbol{P}_4{}^\mathrm{T} \boldsymbol{U}_2^\mathrm{T} \boldsymbol{D} \\ * & * & * & \boldsymbol{\Pi}_{44} & \boldsymbol{\Pi}_{45} & \boldsymbol{\Pi}_{46} & \boldsymbol{P}_5^\mathrm{T} \boldsymbol{U}_2^\mathrm{T} \boldsymbol{D} & \boldsymbol{0} & -\boldsymbol{P}_5^\mathrm{T} \boldsymbol{U}_2^\mathrm{T} \boldsymbol{D} \\ * & * & * & * & \boldsymbol{\Pi}_{55} & \boldsymbol{0} & \boldsymbol{P}_6^\mathrm{T} \boldsymbol{U}_2^\mathrm{T} \boldsymbol{D} & \boldsymbol{0} & -\boldsymbol{P}_6^\mathrm{T} \boldsymbol{U}_2^\mathrm{T} \boldsymbol{D} \\ * & * & * & * & * & \boldsymbol{\Pi}_{66} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ * & * & * & * & * & * & -\varepsilon \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ * & * & * & * & * & * & * & -\gamma^{-1} \boldsymbol{I} & \boldsymbol{0} \\ * & * & * & * & * & * & * & * & -\gamma \boldsymbol{I} \end{bmatrix} < \boldsymbol{0} \tag{16}$$

where $\boldsymbol{\Pi}_{ij}$, i, $j = \{1, \ldots, 6\}$ are defined as in (9).

And the sliding surface is $\boldsymbol{S} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{C} \end{bmatrix} \boldsymbol{z} = \boldsymbol{z}_1 + \boldsymbol{C}\boldsymbol{z}_2 = \boldsymbol{0}$.

Proof. The proof follows from the proof of Theorem 1. Consider LMI (9) and replace $\bar{\boldsymbol{A}}_{22}$, $\bar{\boldsymbol{A}}_{21}$, $\bar{\boldsymbol{A}}_{\mathrm{d}22}$, $\bar{\boldsymbol{A}}_{\mathrm{d}21}$ with $\bar{\boldsymbol{A}}_{22} + \Delta\bar{\boldsymbol{A}}_{22}(t)$, $\bar{\boldsymbol{A}}_{21} + \Delta\bar{\boldsymbol{A}}_{21}(t)$, $\bar{\boldsymbol{A}}_{\mathrm{d}22} + \Delta\bar{\boldsymbol{A}}_{\mathrm{d}22}(t)$, $\bar{\boldsymbol{A}}_{\mathrm{d}21} + \Delta\bar{\boldsymbol{A}}_{\mathrm{d}21}(t)$, respectively. One can have:

$$\boldsymbol{\Pi}_n + \boldsymbol{\Pi}_{\mathrm{d}22} + \boldsymbol{\Pi}_{\mathrm{d}22}^\mathrm{T} + \boldsymbol{\Pi}_{\mathrm{d}21} + \boldsymbol{\Pi}_{\mathrm{d}21}^\mathrm{T} < \boldsymbol{0} \tag{17}$$

where $\boldsymbol{\Pi}_n = \boldsymbol{\Pi}$ in (9) and

$$\boldsymbol{\Pi}_{\mathrm{d}22} = \begin{bmatrix} \boldsymbol{P}_2^\mathrm{T}\Delta\bar{\boldsymbol{A}}_{22} & \boldsymbol{0} & \boldsymbol{P}_2^\mathrm{T}\Delta\bar{\boldsymbol{A}}_{\mathrm{d}22} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{P}_3^\mathrm{T}\Delta\bar{\boldsymbol{A}}_{22} & \boldsymbol{0} & \boldsymbol{P}_3^\mathrm{T}\Delta\bar{\boldsymbol{A}}_{\mathrm{d}22} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{P}_4^\mathrm{T}\Delta\bar{\boldsymbol{A}}_{22} & \boldsymbol{0} & \boldsymbol{P}_4^\mathrm{T}\Delta\bar{\boldsymbol{A}}_{\mathrm{d}22} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{P}_5^\mathrm{T}\Delta\bar{\boldsymbol{A}}_{22} & \boldsymbol{0} & \boldsymbol{P}_5^\mathrm{T}\Delta\bar{\boldsymbol{A}}_{\mathrm{d}22} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{P}_6^\mathrm{T}\Delta\bar{\boldsymbol{A}}_{22} & \boldsymbol{0} & \boldsymbol{P}_6^\mathrm{T}\Delta\bar{\boldsymbol{A}}_{\mathrm{d}22} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}$$

$$\boldsymbol{\Pi}_{\mathrm{d}21} = \begin{bmatrix} -\boldsymbol{P}_2^\mathrm{T}\Delta\bar{\boldsymbol{A}}_{21}\boldsymbol{C} & \boldsymbol{0} & -\boldsymbol{P}_2^\mathrm{T}\Delta\bar{\boldsymbol{A}}_{\mathrm{d}21}\boldsymbol{C} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ -\boldsymbol{P}_3^\mathrm{T}\Delta\bar{\boldsymbol{A}}_{21}\boldsymbol{C} & \boldsymbol{0} & -\boldsymbol{P}_3^\mathrm{T}\Delta\bar{\boldsymbol{A}}_{\mathrm{d}21}\boldsymbol{C} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ -\boldsymbol{P}_4^\mathrm{T}\Delta\bar{\boldsymbol{A}}_{21}\boldsymbol{C} & \boldsymbol{0} & -\boldsymbol{P}_4^\mathrm{T}\Delta\bar{\boldsymbol{A}}_{\mathrm{d}21}\boldsymbol{C} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ -\boldsymbol{P}_5^\mathrm{T}\Delta\bar{\boldsymbol{A}}_{21}\boldsymbol{C} & \boldsymbol{0} & -\boldsymbol{P}_5^\mathrm{T}\Delta\bar{\boldsymbol{A}}_{\mathrm{d}21}\boldsymbol{C} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ -\boldsymbol{P}_6^\mathrm{T}\Delta\bar{\boldsymbol{A}}_{21}\boldsymbol{C} & \boldsymbol{0} & -\boldsymbol{P}_6^\mathrm{T}\Delta\bar{\boldsymbol{A}}_{\mathrm{d}21}\boldsymbol{C} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}.$$

By substituting $\boldsymbol{U}_2^{\mathrm{T}}\boldsymbol{D}\boldsymbol{F}(t)\boldsymbol{E}_{\mathrm{a}}\boldsymbol{U}_2$ for $\Delta\bar{\boldsymbol{A}}_{22}$, one can decompose $\boldsymbol{\Pi}_{\mathrm{d}22}$ as $\boldsymbol{\Pi}_{\mathrm{d}22} = \boldsymbol{H}_2\boldsymbol{F}(t)\boldsymbol{E}_2$, where

$$\boldsymbol{H}_2 = \begin{bmatrix} \boldsymbol{P}_2^{\mathrm{T}}\boldsymbol{U}_2^{\mathrm{T}}\boldsymbol{D} \\ \boldsymbol{P}_3^{\mathrm{T}}\boldsymbol{U}_2^{\mathrm{T}}\boldsymbol{D} \\ \boldsymbol{P}_4^{\mathrm{T}}\boldsymbol{U}_2^{\mathrm{T}}\boldsymbol{D} \\ \boldsymbol{P}_5^{\mathrm{T}}\boldsymbol{U}_2^{\mathrm{T}}\boldsymbol{D} \\ \boldsymbol{P}_6^{\mathrm{T}}\boldsymbol{U}_2^{\mathrm{T}}\boldsymbol{D} \\ \mathbf{0} \end{bmatrix} \quad \text{and} \quad \boldsymbol{E}_2 = \begin{bmatrix} \boldsymbol{E}_{\mathrm{a}}\boldsymbol{U}_2 & \mathbf{0} & \boldsymbol{E}_{\mathrm{d}}\boldsymbol{U}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

For further analysis, the following lemma is needed.

Lemma 1. (Petersen [14]) Let $\boldsymbol{E}$, $\boldsymbol{F}$ and $\boldsymbol{H}$ be real matrices of appropriate dimensions, with $\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} \le \boldsymbol{I}$, then we have that for any scalar $\varepsilon > 0$,

$$\boldsymbol{E}\boldsymbol{F}(t)\boldsymbol{H} + \boldsymbol{H}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}(t)\boldsymbol{E}^{\mathrm{T}} \le \varepsilon\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E} + \varepsilon^{-1}\boldsymbol{H}\boldsymbol{H}^{\mathrm{T}}.$$

According to Lemma 1, with a given positive scalar ε, one can have

$$\boldsymbol{\Pi}_{\mathrm{d}22} + \boldsymbol{\Pi}_{\mathrm{d}22}^{\mathrm{T}} \le \varepsilon\boldsymbol{E}_2^{\mathrm{T}}\boldsymbol{E}_2 + \varepsilon^{-1}\boldsymbol{H}_2\boldsymbol{H}_2^{\mathrm{T}} \tag{18}$$

In the same way

$$\boldsymbol{\Pi}_{\mathrm{d}21} + \boldsymbol{\Pi}_{\mathrm{d}21}^{\mathrm{T}} \le \gamma\boldsymbol{E}_1^{\mathrm{T}}\boldsymbol{E}_1 + \gamma^{-1}\boldsymbol{H}_1\boldsymbol{H}_1^{\mathrm{T}} \tag{19}$$

where γ is a given positive scalar and

$$\boldsymbol{H}_1 = -\begin{bmatrix} \boldsymbol{P}_2^{\mathrm{T}}\boldsymbol{U}_2^{\mathrm{T}}\boldsymbol{D} \\ \boldsymbol{P}_3^{\mathrm{T}}\boldsymbol{U}_2^{\mathrm{T}}\boldsymbol{D} \\ \boldsymbol{P}_4^{\mathrm{T}}\boldsymbol{U}_2^{\mathrm{T}}\boldsymbol{D} \\ \boldsymbol{P}_5^{\mathrm{T}}\boldsymbol{U}_2^{\mathrm{T}}\boldsymbol{D} \\ \boldsymbol{P}_6^{\mathrm{T}}\boldsymbol{U}_2^{\mathrm{T}}\boldsymbol{D} \\ \mathbf{0} \end{bmatrix}, \quad \boldsymbol{E}_1 = \begin{bmatrix} \boldsymbol{E}_{\mathrm{a}}\boldsymbol{U}_1\boldsymbol{C} & \mathbf{0} & \boldsymbol{E}_{\mathrm{d}}\boldsymbol{U}_1\boldsymbol{C} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Substituting (18) – (19) into (16) and applying Schur's complement, the LMI given in (15) is obtained. Thus sliding motion (6) is robustly quadratically stable. This completes the proof.

It is worth to be pointed out that the results presented in Theorem 1 and Theorem 2 are both delay dependent. Comparing with the results given in [3, 4, 8, 17], our results are less conservative. For example, in [17], the Lyapunov–Krasovskii functional been adopted can be seen as a special case of the one introduced in this paper.

3.2. Fault tolerant controller design

As mentioned in the introduction, a sliding mode controller which can drive the system onto the sliding surface in the present of actuator fault together with the matched disturbances can be looked on as a fault tolerant controller. So based on Subsection 3.1, we should consider how to realize the design of the reaching motion controller.

Theorem 3. Suppose LMI (15) is feasible and the sliding motion is given by (5). Then the trajectory of the closed-loop system with fault input can be driven onto the sliding surface in limited time and keep it there for all subsequent time with the control:

$$\begin{aligned} \boldsymbol{u} = &-(\boldsymbol{B}_1\boldsymbol{L})^{\dagger}\left[\boldsymbol{K}\boldsymbol{S} + \alpha\,\mathrm{sign}(\boldsymbol{S}) + \bar{\boldsymbol{C}}\bar{\boldsymbol{A}}\boldsymbol{z}(t) + \bar{\boldsymbol{C}}\bar{\boldsymbol{A}}_{\mathrm{d}}\boldsymbol{z}(t-\tau(t))\right. \\ &\left.+\,\mathrm{diag}\left(\mathrm{sign}(s_1)\ \mathrm{sign}(s_2)\ \cdots\ \mathrm{sign}(s_m)\right)(\boldsymbol{N}_1+\boldsymbol{N}_2+\boldsymbol{N}_3)\right] \end{aligned} \tag{20}$$

where

$$\begin{aligned} &\boldsymbol{S} = \begin{bmatrix} s_1 & s_2 & \cdots & s_m \end{bmatrix}^{\mathrm{T}}, \bar{\boldsymbol{C}} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{C} \end{bmatrix}, \\ &\mathrm{sign}(\boldsymbol{S}) = \begin{bmatrix} \mathrm{sign}(s_1) & \mathrm{sign}(s_2) & \cdots & \mathrm{sign}(s_m) \end{bmatrix}^{\mathrm{T}}, \boldsymbol{B}_1 = \begin{bmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \cdots & \boldsymbol{b}_m \end{bmatrix}^{\mathrm{T}}, \\ &\boldsymbol{K} = \mathrm{diag}(k_i), \alpha = \mathrm{diag}(\alpha_i), k_i \end{aligned}$$

and α_i are positive constants.
$\boldsymbol{N}_{1i} = \left|\bar{\boldsymbol{C}}_i\boldsymbol{T}\boldsymbol{D}\boldsymbol{E}_{\mathrm{a}}\boldsymbol{T}^{-1}\boldsymbol{z}(t)\right|$, $\boldsymbol{N}_{2i} = \left|\bar{\boldsymbol{C}}_i\boldsymbol{T}\boldsymbol{D}\boldsymbol{E}_{\mathrm{d}}\boldsymbol{T}^{-1}\boldsymbol{z}(t-\tau(t))\right|$, $\boldsymbol{N}_i = |\boldsymbol{b}_i|\,\bar{\boldsymbol{d}}$ are the ith row of $\boldsymbol{N}_1$, $\boldsymbol{N}_2$ and $\boldsymbol{N}_3$, respectively. $(\times)^{\dagger}$ denotes the Pseudo-inverse of the argument $\times$.

Proof. From the sliding surface $\boldsymbol{S} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{C} \end{bmatrix}\boldsymbol{z}(t)$, one can have

$$\begin{aligned} \dot{\boldsymbol{S}} &= \begin{bmatrix} \boldsymbol{I} & \boldsymbol{C} \end{bmatrix}\dot{\boldsymbol{z}}(t) \\ &= \bar{\boldsymbol{C}}(\bar{\boldsymbol{A}} + \Delta\bar{\boldsymbol{A}}(t))\boldsymbol{z}(t) + \bar{\boldsymbol{C}}(\bar{\boldsymbol{A}}_{\mathrm{d}} + \Delta\bar{\boldsymbol{A}}_{\mathrm{d}}(t))\boldsymbol{z}(t-\tau(t)) + \boldsymbol{B}_1\boldsymbol{L}(\boldsymbol{u}(t)+\boldsymbol{d}(t)). \end{aligned} \tag{21}$$

Substituting (19) for $\boldsymbol{u}(t)$ in (20)

$$\begin{aligned} \dot{\boldsymbol{S}} &= -\boldsymbol{K}\boldsymbol{S} - \alpha\,\mathrm{sign}(\boldsymbol{S}) \\ &\quad-(\boldsymbol{N}_1\mathrm{diag}\left(\mathrm{sign}(s_1)\ \mathrm{sign}(s_2)\ \cdots\ \mathrm{sign}(s_m)\right) - \bar{\boldsymbol{C}}\Delta\bar{\boldsymbol{A}}(t)\boldsymbol{z}(t)) \\ &\quad-(\boldsymbol{N}_2\mathrm{diag}\left(\mathrm{sign}(s_1)\ \mathrm{sign}(s_2)\ \cdots\ \mathrm{sign}(s_m)\right) - \bar{\boldsymbol{C}}\Delta\bar{\boldsymbol{A}}_{\mathrm{d}}(t)\boldsymbol{z}(t-\tau(t))) \\ &\quad-(\boldsymbol{N}_3\mathrm{diag}\left(\mathrm{sign}(s_1)\ \mathrm{sign}(s_2)\ \cdots\ \mathrm{sign}(s_m)\right) - \boldsymbol{B}_1\boldsymbol{L}\boldsymbol{d}(t)). \end{aligned} \tag{22}$$

Then each element of $\dot{\boldsymbol{S}}$ is

$$\begin{aligned} \dot{\boldsymbol{s}}_i = &-k_i s_i - \alpha_i\,\mathrm{sign}(s_i) - (\boldsymbol{N}_{1i}\,\mathrm{sign}(s_i) - \bar{\boldsymbol{C}}_i\Delta\bar{\boldsymbol{A}}(t)\boldsymbol{z}(t)) \\ &-(\boldsymbol{N}_{2i}\,\mathrm{sign}(s_i) - \bar{\boldsymbol{C}}_i\Delta\bar{\boldsymbol{A}}_{\mathrm{d}}(t)\boldsymbol{z}(t-\tau(t))) \\ &-(\boldsymbol{N}_{3i}\,\mathrm{sign}(s_i) - \boldsymbol{b}_i\boldsymbol{L}\boldsymbol{d}(t)). \end{aligned} \tag{23}$$

Note that

$$\bar{\boldsymbol{C}}_i\Delta\bar{\boldsymbol{A}}(t)\boldsymbol{z}(t) \le \left|\bar{\boldsymbol{C}}_i\boldsymbol{T}\boldsymbol{D}\boldsymbol{F}(t)\boldsymbol{E}_{\mathrm{a}}\boldsymbol{T}^{-1}\boldsymbol{z}(t)\right| \le \left|\bar{\boldsymbol{C}}_i\boldsymbol{T}\boldsymbol{D}\boldsymbol{E}_{\mathrm{a}}\boldsymbol{T}^{-1}\boldsymbol{z}(t)\right| = \boldsymbol{N}_{1i}$$

$$\bar{\boldsymbol{C}}\Delta\bar{\boldsymbol{A}}_{d}(t)\boldsymbol{z}(t-\tau(t)) \le \left|\bar{\boldsymbol{C}}_i\boldsymbol{T}\boldsymbol{D}\boldsymbol{F}(t)\boldsymbol{E}_{\mathrm{d}}\boldsymbol{T}^{-1}\boldsymbol{z}(t-\tau(t))\right| \le \left|\bar{\boldsymbol{C}}_i\boldsymbol{T}\boldsymbol{D}\boldsymbol{E}_{\mathrm{d}}\boldsymbol{T}^{-1}\boldsymbol{z}(t-\tau(t))\right| = \boldsymbol{N}_{2i}$$

$$\boldsymbol{b}_i\boldsymbol{L}\boldsymbol{d}(t) \le |\boldsymbol{b}_i|\,|\boldsymbol{L}|\,\bar{\boldsymbol{d}} \le |\boldsymbol{b}_i|\,\bar{\boldsymbol{d}} = \boldsymbol{N}_{3i}.$$

So one can deduce that $\begin{cases} \dot{s}_i < 0, & \text{if } s_i > 0 \\ \dot{s}_i > 0, & \text{if } s_i < 0 \end{cases}$, which implies that the trajectory of the closed-loop system with actuator fault can be driven onto the sliding surface in limited time and keep it there for all subsequent time with the control law (19). This completes the proof. □

Remark 1. In this paper, if $\boldsymbol{L}$ is fixed a prior for the worst situation can be known, then the fault tolerant controller is a passive one. On the other hand, if $\boldsymbol{L}$ is estimated in real time, then the fault tolerant controller can be designed adaptively according to the fault. From this point of view, it's an active fault tolerant controller. The fault situations $\boldsymbol{L}$ can be getting use the method proposed in [9] and [20]. The interested reader can refer to these articles for details.

Remark 2. It is worth to be pointed out that, in this paper, quadratically stable of the close-loop system is chosen as the performance index. For further research, we can choose a more complex performance index such as an LQ one which is often used in design a passive fault tolerant controller, for example, in [1]. The proofs of the theorems will be a bit complex but in a similar way with the proofs of the corresponding theorems in this paper.

4. NUMERICAL EXAMPLE

Consider an approximate linear model of the lateral dynamics of an aircraft. The state space representation is given as follows:

$$\boldsymbol{A} = \begin{bmatrix} -1.6689 & 0.0759 & -0.0100 \\ -1.6920 & -22.3750 & 0.1712 \\ 5.6971 & -0.0615 & 6.6168 \end{bmatrix}, \boldsymbol{A}_{\mathrm{d}} = \begin{bmatrix} 1 & -2 & 0 \\ 0.03 & 0.26 & 0 \\ 0.1 & -0.23 & -5.36 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 2 & 0 \\ 1 & 0 \\ 4 & 1 \end{bmatrix}.$$

The state $\boldsymbol{x} = \begin{bmatrix} \delta\beta & \delta p & \delta r \end{bmatrix}^{\mathrm{T}}$, where $\delta\beta$ is the incremental sideslip angle, δp and δr are incremental roll rate and yaw rate, respectively; $\boldsymbol{u} = \begin{bmatrix} \dot{p}_c & \dot{r}_c \end{bmatrix}^{\mathrm{T}}$ are the generalized roll acceleration and yaw acceleration commands respectively. Disturbance is assumed as sine wave, which satisfies $|\boldsymbol{d}(t)| \leq 1$. The time-delay function can be chosen as $|\sin(0.5t)|$, with $\bar{\tau} = 1$ and $\mu = 0.5$ denoting the upper bound on the time-varying delay and its derivative, respectively. Assume that the model uncertainties in this example are as follows:

$$\boldsymbol{D} = \begin{bmatrix} 0.1 \\ 0 \\ 0.32 \end{bmatrix}, \boldsymbol{E}_{\mathrm{a}} = \begin{bmatrix} 0.1 & 0.2 & 0.3 \end{bmatrix}, \quad \boldsymbol{E}_{\mathrm{d}} = \begin{bmatrix} 0.3 & 0 & 0.1 \end{bmatrix}.$$

The initial condition of the states is $\begin{bmatrix} 2, & 0, & -1 \end{bmatrix}$ with the actuator fault denoted as $\boldsymbol{L} = \begin{bmatrix} 0.2 & 0 \\ 0 & 1 \end{bmatrix}$ which means the first actuator is 80 % fail and the second actuator is healthy.

Here $\operatorname{rank}(\boldsymbol{B}) = 2$ and the SVD of $\boldsymbol{B}$ is:

$$\boldsymbol{B} = \Bigg[\underbrace{\begin{matrix} -0.4209 & -0.7892 \\ -0.2105 & -0.3946 \\ -0.8824 & 0.4706 \end{matrix}}_{\boldsymbol{U}_1} \quad \underbrace{\begin{matrix} -0.4472 \\ 0.8944 \\ 0 \end{matrix}}_{\boldsymbol{U}_2} \Bigg] \begin{bmatrix} 4.6659 & 0 \\ 0 & 0.4792 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -0.9820 & -0.1891 \\ -0.1891 & 0.9820 \end{bmatrix}.$$

The state transformation is $\boldsymbol{z} = \boldsymbol{T}\boldsymbol{x}$ with $\boldsymbol{T} = \begin{bmatrix} \boldsymbol{U}_1^{\mathrm{T}} \\ \boldsymbol{U}_2^{\mathrm{T}} \end{bmatrix}$.

So that $\boldsymbol{B}_1 = \begin{bmatrix} -4.5817 & -0.8824 \\ -0.0906 & 0.4706 \end{bmatrix}$ and as shown in Section 2, we can get $\bar{\boldsymbol{A}}_{22}$ and other parameter matrices in (6).

It follows from Theorem 2 in Section 3 that $\boldsymbol{C} = \begin{bmatrix} 2.0691 \\ 1.6239 \end{bmatrix}$ and the sliding mode surface is:

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{C} \end{bmatrix} \boldsymbol{z}(t) = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{C} \end{bmatrix} \boldsymbol{T}\boldsymbol{x}(t) = \begin{bmatrix} -1.3462 & 1.6402 & -0.8824 \\ -1.5154 & 1.0579 & 0.4706 \end{bmatrix} \boldsymbol{x}(t) = 0.$$

With

$$\begin{aligned} \boldsymbol{N}_1 &= \begin{bmatrix} 0.0417 & 0.0834 & 0.1251 \\ 0.0001 & 0.0002 & 0.0003 \end{bmatrix} \left\| \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{x}_3 \end{bmatrix} \right\|, \\ \boldsymbol{N}_2 &= \begin{bmatrix} 0.1251 & 0 & 0.0417 \\ 0.0003 & 0 & 0.0001 \end{bmatrix} \left\| \begin{bmatrix} \boldsymbol{x}_1(t-\tau(t)) \\ \boldsymbol{x}_2(t-\tau(t)) \\ \boldsymbol{x}_3(t-\tau(t)) \end{bmatrix} \right\| \\ \boldsymbol{N}_3 &= \begin{bmatrix} 4.5817 & 0.8824 \\ 0.0906 & 0.4706 \end{bmatrix} \cdot |\sin t|, \end{aligned}$$

the reaching control law can be get from Theorem 3:

$$\begin{aligned} \boldsymbol{u} = &- \begin{bmatrix} -0.9174 & -0.8824 \\ -0.0181 & 0.4706 \end{bmatrix} \Bigg[\boldsymbol{K}\boldsymbol{S} + \alpha \operatorname{sign}(\boldsymbol{S}) + \begin{bmatrix} -5.5553 & -36.7475 & -5.5441 \\ 3.4202 & -23.8136 & 3.3101 \end{bmatrix} \boldsymbol{x}(t) \\ &+ \begin{bmatrix} -1.3853 & 3.3219 & 4.7294 \\ -1.4366 & 3.1977 & -2.5224 \end{bmatrix} \boldsymbol{x}(t-\tau(t)) + \operatorname{diag}(\operatorname{sign}(s_1)\ \operatorname{sign}(s_2))(\boldsymbol{N}_1 + \boldsymbol{N}_2 + \boldsymbol{N}_3) \Bigg]. \end{aligned}$$

Figures 1 – 3 illustrate the simulation results. It can be seen that the present fault tolerant control scheme effectively eliminated the effects of parameter uncertainties and fault input, and guaranteed the asymptotic stability of the closed-loop system.

Remark 3. The reaching control law $\boldsymbol{u}$ is an exponential one and the parameters $\boldsymbol{K}$ and α can be tuned to reduce the chattering on the sliding surface. In order to reduce the chattering and guarantee a relatively far reaching speed simultaneously, one should increase $\boldsymbol{K}$ and decrease α at the same time. Here we select $\boldsymbol{K} = 18$ and $\alpha = 1$.

5. CONCLUSION

In this paper, the problem of fault tolerant control for uncertain systems with time varying state-delay has been considered. Based on sliding mode control, a fault tolerant control design scheme is proposed. Delay dependent sufficient condition is derived for the existence of the sliding surface which guarantees the system restricted to the sliding surface. And a reaching motion controller that can be considered as a fault tolerant controller is proposed. The sliding motion and the reaching motion

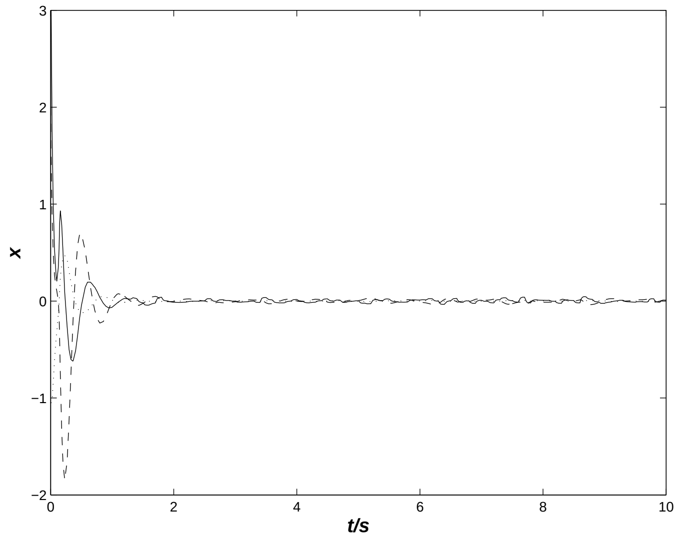

Fig. 1. State response.

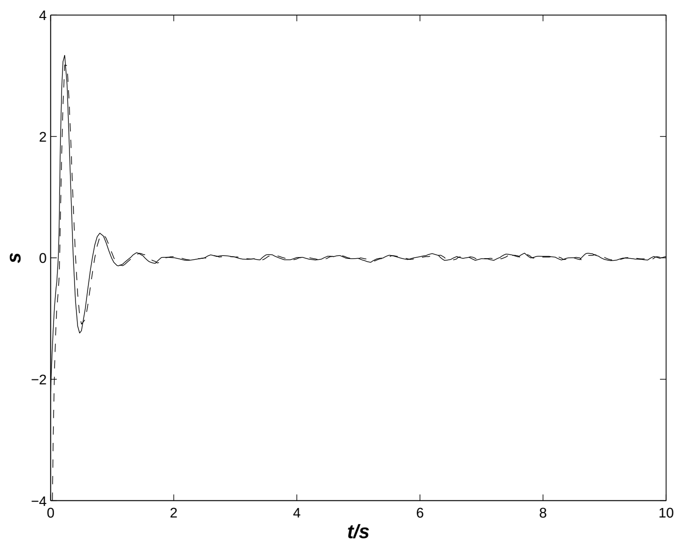

Fig. 2. Sliding surfaces.

are robust against the mismatched uncertainties, actuator fault and disturbance. The time-delay considered in this paper is the time-varying one, the results are delay dependent, which is less conservative than the delay-independent ones in the references. At last, a numerical example has been included to demonstrate the effectiveness of the presented method.

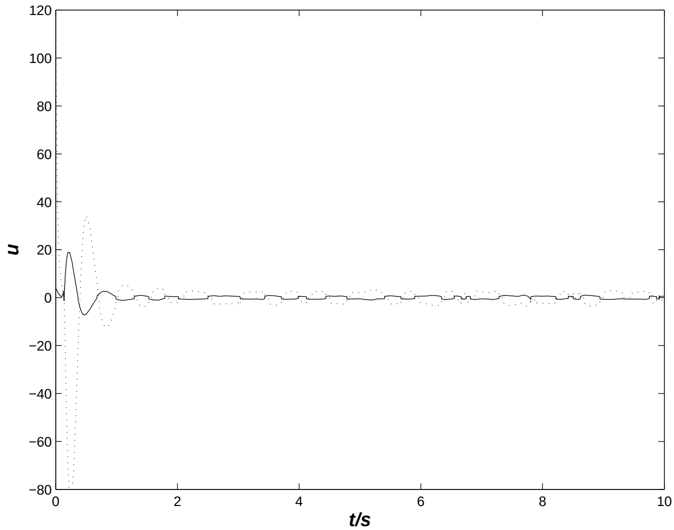

Fig. 3. Control input.

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