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# DISJOINTNESS OF FUZZY COALITIONS (Discussion)*

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The cooperative games with fuzzy coalitions in which some players act in a coalition only with a fraction of their total “power” (endeavor, investments, material, etc.) or in which they can distribute their “power” in more coalitions, are connected with some formal or interpretational problems. Some of these problems can be avoided if we interpret each fuzzy coalition as a fuzzy class of crisp coalitions, as shown by Mareš and Vlach in [9, 10, 11]. The relation between this model of fuzziness and the original one, where fuzzy coalition is a fuzzy set of players, is shown and the properties of the model are analyzed and briefly interpreted in this paper. The analysis is focused on very elementary properties of fuzzy coalitions and their payments like disjointness, superadditivity and also convexity. Three variants of their modelling are shown and their consistency is investigated. The derived results may be used for further development of the theory of fuzzy coalitions characterized by fuzzy sets of crisp coalitions. They show that the procedure developed in [11] appears to be the most adequate.

*Keywords:* fuzzy cooperative game, fuzzy coalition, vague cooperation, block of players

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## 1. INTRODUCTION

In this paper, we deal with cooperative (or coalitional) games with transferable utility, briefly TU-games, and with their fuzzification. Generally, they can be fuzzified in two principally different ways. It is possible to fuzzify their coalitions modelling the vague structure of cooperation, or the expected pay-offs if we wish to model the vagueness of expectations and motivations with which players enter the negotiation process (see, e. g., [7]). In the presented contribution, we are interested in the first problem, i. e., we deal with vaguely determined coalitions.

This model was formulated in seventies (see, e. g., [1, 2]) but in certain latent way, it can be found even in some papers not using the terminology of fuzzy set theory but admitting the parallel participation of players in several cooperation schemes

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(cf., [6]). The generalizations of the original model are further developed, e. g., in [3, 4], and they are discussed in [8, 9], too.

The following sections are motivated by some specific uncertainties connected with the interpretation of that model and with its "translation" into the reality of cooperative behavior. Namely, if each (fuzzy) coalition is considered to be a fuzzy subset of the set of all players, with membership function defined for all players (even if sometimes vanishing) then the real structure of cooperation in coalitions including the partition of the set of players into some "groups dealing the same interests" can become rather hidden and the conflict of motivations can be unclear. Some very brief comments on this were mentioned in [8] and partly in [9], too, and in [11] one of its possible versions is suggested, as well. From the formal point of view, there may appear some doubts on the sense of the concepts like disjointness of coalitions and, consequently the superadditivity, additivity, coalitional structure, and related concepts.

An attempt to handle these difficulties can be based on a modification of the formalism describing the concept of fuzzy coalition. The "traditional" fuzzy coalitions defined as fuzzy sets of players can be easily transformed into fuzzy classes of crisp coalitions. This transformation preserves the main advantages of the fuzzy sets of players and, moreover, it offers even more refined diversification of the cooperative bounds objectively existing in the game. Moreover, it offers a possibility to use the well-known properties of crisp coalitions (and deterministic TU-games) even for the applications in which the structure of cooperation is rather vague. In the main parts of the paper, we study three potentially possible versions of such model. More exactly, the eventual models of the considered game may be treated as certain mixtures of the two models of fuzzy coalitions – their representations by fuzzy sets of players or by fuzzy sets of crisp subcoalitions. Their analysis is focused on the mutual compatibility of these two approaches and on their proportions in one eventual model. The main results regard very elementary (and, consequently, fundamental) concepts of the cooperation theory, namely the disjointness of coalitions and the superadditivity. The following text represents a rather discussion paper contributing to the methodological analysis of the adequacy of particular approaches to the reality of cooperative behaviour as well as to the formal operability of the model.

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## 2. CRISP TU–GAME

The deterministic cooperative games with transferable utility (briefly, TU-games) are defined by a pair $(I, v)$ where $I$ is a (finite and non-empty) set of *players* and any its subset $K \subset I$ is called a *coalition*. The mapping $v : 2^I \to R$ is called a *characteristic function* of the game. For every $K \subset I$, $v(K)$ denotes the expected income of $K$. We suppose that for the empty coalition $\emptyset$, $v(\emptyset) = 0$.

Every vector $\boldsymbol{x}^K = (x_i)_{i\in K} \in R^K$, where $K \subset I$, such that

$$\sum\nolimits_{i\in K} x_i \leq v(K) \tag{1}$$

represents an *achievable* distribution of the total income of coalition $K$. Vectors

$$\boldsymbol{x} = (x_i)_{i\in I} \in R^I$$

will be called *imputations* and we say that an imputation $\boldsymbol{x}$ is *blocked* by coalition $K$ if the inequality (1) is strict.

The game is called *superadditive* iff for any pair of disjoint coalitions $K$, $K'$, $K \cap K' = \emptyset$

$$v(K \cup K') \geq v(K) + v(K'), \tag{2}$$

and it is called *convex* iff for any $K$, $K' \subset I$,

$$v(K \cup K') + v(K \cap L) \geq v(K) + v(K'). \tag{3}$$

For more details see, e. g., [5, 13].

## 3. FUZZY COALITIONS – CLASSICAL MODEL

As already explained in the heuristic introduction, it is not realistic to assume that each player participates in exactly one coalition which consumes all his potential "power". In fact, each of us parts his endeavor into cooperative activities in the frame of several groups sharing common interest in a social or economic process. One of these "groups" may be even the one-player coalition. This distribution of player's endeavor is modelled by the tools of fuzzy set theory – each coalition is considered to be a fuzzy subset of the set $I$. In the following sections, we denote for every set $X$ by $\mathcal{P}(X)$ the class of all crisp subsets of $X$ and by $\mathcal{F}(X)$ the class of all fuzzy subsets of $X$. It means that every crisp coalition $K$ is an element of $\mathcal{P}(I)$.

Every *fuzzy coalition* $L \in \mathcal{F}(I)$ is characterized by its membership function $\tau_L : I \to [0, 1]$ with the usual interpretation (cf. [1, 2, 3, 10, 9]). Evidently, every *crisp coalition* is a special case of fuzzy coalition with membership values in $\{0, 1\}$. To simplify the orientation in the next text, we denote by $K$ (eventually with indices) the crisp coalitions and by letters $L$, $J$, $M$ (with eventual indices) the generally fuzzy coalitions. Of course, $I$ still keeps to be a symbol of the crisp coalition of all players.

Without loss of generality, we suppose that $I = \{1, 2, \ldots, n\}$, i. e., there are $n$ players in the game. If we denote $N = 2^n - 1$ then the crisp coalitions may be labeled, $K_0, K_1, K_2, \ldots, K_N$ where $K_0 = \emptyset$.

The characteristic function of a TU-game with fuzzy coalitions is defined (see [1]) as a function $v : \mathcal{F}(I) \to R$ such that $v(K_0) = 0$. Its properties are investigated in numerous works (e. g., in [1, 2, 3, 4] and others). The extension of some basic concepts of the deterministic TU-games model to analogous games with fuzzy coalitions is quite inspirative.

There are some open topics which deserve attention. The roots of some of them can be found in a concealed but natural intuitive expectation that any TU-game with fuzzy coalitions would be interpreted as an extension of some crisp coalitional TU-game. It is to start from its elementary components, coalitions and characteristic function, and continue to more advanced basic concepts. This approach was (first time, as far as we know) open in [10] and [9], and its more thorough analysis is presented in the following sections of this paper. The analysis regards such elementary concept like the disjointness of fuzzy coalitions and its immediate consequences. On this basic level, some potentially admissible approaches to the fuzzy coalitions are illustrated.

## 4. FUZZY COALITIONS AS EXTENSION OF CRISP COOPERATION

The elementary attempt to connect fuzzy coalitions with their crisp counterparts was done in [8] and [9]. Keeping the notations introduced in the previous section and denoting by $\tau_0, \tau_1, \ldots, \tau_N$ the $\{0,1\}$ membership functions of the crisp coalitions $K_0, K_1, \ldots, K_N$, respectively, we say that a fuzzy coalition $L \in \mathcal{F}(I)$ is a *convex combination* of crisp coalitions $K_{j_1}, \ldots, K_{j_m}$ iff there exist real numbers $b_{j_1}, \ldots, b_{j_m} \in [0,1]$ such that $b_{j_1} + \cdots + b_{j_m} = 1$ and for every $i \in I$,

$$\tau_L(i) = b_{j_1}\tau_{j_1}(i) + \cdots + b_{j_m}\tau_{j_m}(i). \tag{4}$$

It is shown in [8] and [9] that for every fuzzy coalition $L$ there exists at least one set of crisp coalitions whose convex combination $L$ is. Note that the empty crisp coalition $K_0$ may be added to the set $\{K_{j_1}, \ldots, K_{j_m}\}$ which offers a possibility to use relatively wide spectrum of their coefficients $b_{j_1}, \ldots, b_{j_m}$ whose sum is less than 1 and complete it by $b_0$ without influencing the values $\tau_L(i)$. Moreover, there may generally exist more than one such groups. If $L$ is crisp, i.e., $L = K_j$ for some $j = 1, \ldots, N$, then there exists exactly one group, namely, the one-element group $\{K_j\}$ such that $L$ is "convex combination" of $\{K_j\}$ with coefficient $b_j = 1$.

The above facts offer an interesting interpretation. Namely, for any fuzzy coalition $L$ its membership function $\tau_L$ does not contain the complete information about the structure of cooperation of a player $i$ with other members of $L$. It specifies only, that he "invests" the intensity $\tau_L(i)$ of his total endeavor in the interests of coalition $L$. The representation of $L$ by a convex combination of several crisp coalitions shows that $L$ itself is a structure of a few crisp cooperating groups of players where, usually, each of them participates in the common goals of $L$ with only some part of its "power". The fact that there may exist more than one such convex combinations means that eventual choice of one of them brings some new information about the real existing structure of relations inside $L$. It appears to be useful to represent the fuzzy cooperation in the considered TU-game not only by fuzzy coalitions but also by some structure based on the convex combinations.

Before doing so, we attempt to describe also some relation between the values of characteristic function $v$ for crisp and fuzzy coalitions. If $L$ is a fuzzy coalition and $\{K_1, \ldots, K_m\}$ its representation by convex combination with coefficients $b_1, \ldots, b_m$ then it is correct to define the value

$$v(K_1, \ldots, K_m) = b_1 \cdot v(K_1) + \cdots + b_m \cdot v(K_m).$$

Then we may define the value of the characteristic function $v(L)$ for $L$ by

$$v(L) = \max\left(v(K_{j_1},\dots,K_{j_m}) : \{K_{j_1},\dots,K_{j_m}\} \text{ where } L \text{ is convex combination of } \{K_{j_1},\dots,K_{j_m}\}\right). \tag{5}$$

It is easy to see, due to our previous statements, that for crisp $L$, formula (5) gives exactly the value of characteristic function for the crisp coalition in the crisp TU-game which was extended by our fuzzified model.

Let us note that the representation of fuzzy coalitions by a convex combination of crisp coalitions is mentioned in [1], too.

The values of coefficients $b_{j_k}$, $k = 1,\dots,m$, in (4) have formal properties of values of membership function which fact justifies the following procedure.

As we denote by $\mathcal{P}(I)$ the set of all crisp coalitions, $\mathcal{P}(I) = \{K_0, K_1,\dots,K_N\}$, then every fuzzy coalition $L$ can be characterized by a fuzzy subset $\mathcal{L}$ of $\mathcal{P}(I)$ with membership function $\beta_{\mathcal{L}} : \mathcal{P}(I) \to [0,1]$. Namely, if $L$ is a convex combination of $\{K_{j_1},\dots,K_{j_m}\}$ with coefficients $b_{j_1},\dots,b_{j_m}$ then it is possible to introduce a fuzzy set $\mathcal{L} \in \mathcal{F}(\mathcal{P}(I))$ with membership function $\beta_{\mathcal{L}} : \mathcal{P}(I) \to [0,1]$ such that we put for every $K \in \mathcal{P}(I)$

$$\begin{aligned} \beta_{\mathcal{L}}(K) &= b_{j_k} \quad \text{if } K = K_{j_k} \text{ for some } k = 1,\dots,m, \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{6}$$

If $L \in \mathcal{F}(I)$ with $\tau_L$, and $\mathcal{L} \in \mathcal{F}(\mathcal{P}(I))$ with $\beta_{\mathcal{L}}$ are connected via the above construction then we say that $\mathcal{L}$ *reflects the cooperation* in $L$.

The above discussion of results presented in [9, 10] immediately admits the possibility that some fuzzy coalitions in the sense of fuzzy subsets of $I$ can be characterized by more than one fuzzy subsets of $\mathcal{P}(I)$. It means that the definition of fuzzy coalitions as fuzzy subset of the set $\mathcal{P}(I)$ is more sophisticated than the traditional one based on fuzzy subset of $I$. These observations can be formulated in the following statements.

**Lemma 1.** If $L$ is a fuzzy coalition with membership function $\tau_L : I \to [0,1]$ then there exists at least one $\mathcal{L} \in \mathcal{F}(\mathcal{P}(I))$ with membership function $\beta_{\mathcal{L}} : \mathcal{P}(I) \to [0,1]$ such that $L$ is convex combination of $\mathcal{P}(I)$ with coefficients $\beta_{\mathcal{L}}(K)$, $K \in \mathcal{P}(I)$.

Proof. The statement follows from the above construction. □

**Remark 1.** There may exist more than one fuzzy subsets of $\mathcal{P}(I)$ reflecting the cooperation in a fuzzy coalition $L \in \mathcal{F}(I)$.

**Example 1.** Let $n = 4$ and let $L$ be fuzzy coalition such that $\tau_L(i) = 1/2$ for all $i \in I$. Then $L$ may be identified with, e. g., the following fuzzy subsets $\mathcal{L}$ of $\mathcal{P}(I)$:

$$\begin{aligned}
&\beta_{\mathcal{L}} : && \beta_{\mathcal{L}}(\{1,2\}) = 1/2, && \beta_{\mathcal{L}}(\{3,4\}) = 1/2, && \beta_{\mathcal{L}}(K) = 0 \text{ otherwise},\\
&\beta'_{\mathcal{L}} : && \beta'_{\mathcal{L}}(\{1,3\}) = 1/2, && \beta'_{\mathcal{L}}(\{2,4\}) = 1/2, && \beta'_{\mathcal{L}}(K) = 0 \text{ otherwise},\\
&\beta''_{\mathcal{L}} : && \beta''_{\mathcal{L}}(I) = 1/2, && \beta''_{\mathcal{L}}(\{K_0\}) = 1/2, && \beta''_{\mathcal{L}}(K) = 0 \text{ otherwise},\\
&\beta^*_{\mathcal{L}} : && \beta^*_{L}(\{1,2,3\}) = 1/2, && \beta^*_{\mathcal{L}}(\{4\}) = 1/2, && \beta^*_{\mathcal{L}}(K) = 0 \text{ otherwise},
\end{aligned}$$

and many others. □

**Lemma 2.** A fuzzy subset $\mathcal{L}$ of $\mathcal{P}(I)$ with membership function $\beta_{\mathcal{L}}$ reflects the cooperation in a fuzzy coalition $L \in \mathcal{F}(I)$ iff

$$\sum_{K\in\mathcal{P}(I)} \beta_{\mathcal{L}}(K) = 1.$$

P r o o f . The statement follows from the definition of convex combination (cf. (4)) and from definitoric formula (6), immediately. □

**Remark 2.** The previous results mean that there exist fuzzy subsets of $\mathcal{P}(I)$ which do not correspond to any fuzzy subset of $I$.

Remarks 1 and 2 show that the mutual correspondence between two concepts of fuzzy coalitions represented by fuzzy subsets of $I$ (with $\tau_L$) and fuzzy subsets of $\mathcal{P}(I)$ (with $\beta_{\mathcal{L}}$) is very weak and it regards (in a very limited sense) only the cases in which

$$\beta_{\mathcal{L}}(K_0) + \beta_{\mathcal{L}}(K_1) + \cdots + \beta_{\mathcal{L}}(K_N) = 1.$$

If we focus our attention on a fuzzy coalition $L$ and on the relations of one player $i \in I$ to his partners in $L$, then the classical concept of fuzzy coalition offers only one simple information. Namely, the value $\tau_L(i)$ showing which deal of his capacity player $i$ contributes to the activity of $L$. On the other hand, the paradigm that the fuzziness of a coalition means, in fact, that it itself is a combination of partial "internal" and homogenous groups more or less contributing to $L$, opens a possibility to analyze the relations of $i$ to other partners in $L$ in a much more sophisticated way. For every player $i \in I$ and every fuzzy coalition $L \in \mathcal{F}(I)$ for which a fuzzy set $\mathcal{L} \in \mathcal{F}(\mathcal{P}(I))$ and $\beta_{\mathcal{L}}$ were constructed by the above method, we define a mapping ${}^{(i)}\beta_{\mathcal{L}} : \mathcal{P}(I) \to [0,1]$ which we call a *structure of contacts* of $i$ in characterization $\beta_{\mathcal{L}}$, where for any $K \in \mathcal{P}(I)$

$$\begin{aligned} {}^{(i)}\beta_{\mathcal{L}}(K) &= \beta_{\mathcal{L}}(K) \quad \text{if } i \in K, \\ &= 0 \qquad\quad\ \text{if } i \notin K. \end{aligned} \tag{7}$$

**Lemma 3.** For any fuzzy coalition $L$, $\mathcal{L} \in \mathcal{F}(\mathcal{P}(I))$, with $\beta_{\mathcal{L}}$ reflecting the cooperation in $L$, and player $i \in I$,

$$\sum_{K\in\mathcal{P}(I)} {}^{(i)}\beta_{\mathcal{L}}(K) = \tau_L(i).$$

P r o o f . The equality follows from (4), (6) and (7), immediately. □

The previous results show that there exists certain relation between the conception of fuzzy coalition as a fuzzy set from $\mathcal{F}(I)$ and as a fuzzy subset from $\mathcal{F}(\mathcal{P}(I))$.

This relation is illustrated by Lemma 3, e. g. On the other hand, the relation is not very tight – there is no one-to-one correspondence between both types of fuzzy coalitions. Among other consequences it means that the method of extension of the characteristic function $v$ used in Section 3 cannot be simply transmitted to the $\mathcal{F}(\mathcal{P}(I))$. Nevertheless, it is possible to define other extensions (one of them is suggested in [11]) and the assumption that function $v$ was extended from $\mathcal{P}(I)$ to $\mathcal{F}(\mathcal{P}(I))$ is fully justified.

### 4.1. Fuzzy superadditivity

The concept of superadditivity, however simple it is, can be easily used for illustration of differences between both approaches to the fuzziness of coalitions. If we wish to respect the analogy between TU-games with crisp and fuzzy coalitions reflecting the fact that fuzzy coalitions extend the set of their crisp counterparts, we are to consider the fact that the superadditivity is closely connected with the disjointness of the relevant coalitions. If those coalitions are fuzzy (with different degree of membership) then the concept of disjointness itself may become rather uncertain. In this subsection, we present a sequence of three models of the disjointness of fuzzy coalitions and consequent concepts of superadditivity.

The paradigm due to which the superadditivity of a fuzzy game is to be fuzzy is not dogmatic. E. g., Aubin in [1] suggest its definition as a crisp property. In our symbols, by an inequality

$$v(L+J) \geq v(L) + v(J)$$

where $L,\ J \in \mathcal{F}(I)$, and $\tau_L(i) + \tau_J(i) \leq 1$, $\tau_{L+J}(i) = \tau_L(i) + \tau_J(i)$, $i \in I$.

The three models combine, more or less, both approaches to the concept of fuzzy coalition and they differ in their mutual proportions.

#### 4.1.a. Fuzzy subsets of $I$

The superadditivity in the TU-games becomes rather vague if the fuzzy coalitions are considered. The dogmatic view on the disjointness as strict separation of two coalitions seems to be too strong if the eventual intersection contains only players with some negligible participation in the coalitions. We would not forget that, formally, all membership functions $\tau_L$ of fuzzy coalitions are defined over the complete set $I$. From this point of view, the *disjointness* becomes a fuzzy relation defined over the Cartesian product $\mathcal{F}(I) \times \mathcal{F}(I)$. Its membership function will be denoted $\overline{\delta} : \mathcal{F}(I) \times \mathcal{F}(I) \to [0,1]$. Now, we may choose one of the following approaches. In the first one, the conception of disjointness (and then also superadditivity) can be based on the traditional perception of fuzzy coalition as fuzzy subset of $I$, with $\tau_L$. Then we may proceed as follows. We define its value for any pair of fuzzy coalitions $L,\ M$ by

$$\overline{\delta}(L,M) = 1 - \max_{i \in I} \left( \min(\tau_L(i), \tau_M(i)) \right). \tag{8}$$

**Remark 3.** With respect to Lemma 3, it is easy to see that

$$\overline{\delta}(L, M) = 1 - \max_{i \in I} \left( \min \left( \sum\nolimits_{K \in \mathcal{P}(I)} {}^{(i)}\beta_{\mathcal{L}}(K), \sum\nolimits_{K \in \mathcal{P}(I)} {}^{(i)}\beta_{\mathcal{M}}(K) \right) \right),$$

where $\mathcal{L}$ with $\beta_{\mathcal{L}}$ and $\mathcal{M}$ with $\beta_{\mathcal{M}}$ are some of the fuzzy sets from $\mathcal{F}(\mathcal{P}(I))$ which reflect the cooperation in $L$ and $M$, respectively, due to (6). Lemma 3 implies that the last equality is independent of the choice of actual $\mathcal{L}$ and $\mathcal{M}$ among those which reflect the cooperation in $L$ and $M$.

**Lemma 4.** If $\mathcal{L}$ with $\beta_{\mathcal{L}}$ and $\mathcal{M}$ with $\beta_{\mathcal{M}}$ reflect the cooperation in $L$, $M \in \mathcal{F}(I)$, respectively, and if for some $K \in \mathcal{P}(I)$, $\beta_{\mathcal{L}}(K) > 0$ and $\beta_{\mathcal{M}}(K) > 0$ then, evidently, $\overline{\delta}(L, M) < 1$.

Proof. The statement follows from Remark 3 immediately. □

**Theorem 1.** If the coalitions $L$, $M$ are crisp, i. e., $L$, $M \in \mathcal{P}(I)$, then $\overline{\delta}(L, M) = 1$ if $L \cap M = \emptyset$ and $\overline{\delta}(L, M) = 0$ if $L \cap M \neq \emptyset$ in the deterministic sense.

Proof. The statement follows from (8) and from the fact that for crisp coalitions $L$, $M$ the values $\tau_L(i)$ and $\tau_M(i)$ belong to $\{0, 1\}$. □

Now, it is natural to define the superadditivity as a fuzzy property defined on the class of all TU-games $(I, v)$ over the set of players $I$. Its membership function is denoted by $\overline{\sigma}_I$ with values $\overline{\sigma}_I(v) \in [0, 1]$ depending on the fuzzy disjointness of fuzzy coalitions for which the classical inequality (2) if fulfilled, for the extension of the characteristic function $v$ defined by (5). More exactly, we define

$$\overline{\sigma}_I(v) = 1 - \max\left(\overline{\delta}(L, M) : L,\, M \in \mathcal{F}(I),\, v(L \cup M) < v(L) + v(M)\right), \tag{9}$$

where the union $L \cup M$ means that

$$\tau_{L \cup M}(i) = \max\left(\tau_L(i), \tau_M(i)\right). \tag{10}$$

**Lemma 5.** It is easy to verify that for the game in which only crisp coalitions are admissible the above definition of superadditivity coincides with the deterministic one recollected in Section 2, formula (2).

Proof. The statement follows from Lemma 4 and from (9), immediately. □

4.1.b. Fuzzy subsets of $\mathcal{P}(I)$ – respecting also $I$

The above approach to the superadditivity respects the classical model of fuzzy coalition (see, e. g. [1]) with its advantages and discrepancies where the latter ones consist namely in very weak connections between the crisp and fuzzy coalitions. If we prefer to respect the paradigm that fuzzy coalitions are, rather than some independent objects, extensions of the crisp ones with more complex structure of cooperative relations, then it is desirable to reconsider the above model and to modify it. We have done something similar in the previous paragraphs of Section 4, namely by introducing the concepts of characterization of fuzzy coalition, structure of contacts and, especially, in formula (5) where the tight relation between crisp and fuzzy coalitions is stressed.

If we accept the principle that fuzzy coalition is not to be described as fuzzy subset of $I$ but as a fuzzy class of crisp subcoalitions than its impact on the concept of disjointness (and other concepts derived from it) is quite significant.

If the cooperation in a fuzzy coalition $L$ is identified with $\mathcal{L}$ and $\beta_{\mathcal{L}}$ reflecting its cooperation then also the disjointness may be understood as a relation between the crisp subcoalitions with positive membership $\beta_{\mathcal{L}}(\cdot)$. The disjointness keeps being a fuzzy relation between fuzzy coalitions. We denote its membership function by $\delta : \mathcal{F}(I) \times \mathcal{F}(I) \to [0,1]$ but now, it is defined by

$$\delta(L,M) = 1 - \max_{i \in I} \left( \max_{K,K' \in \mathcal{P}(I)} \left( \min \left( {}^{(i)}\beta_{\mathcal{L}}(K), {}^{(i)}\beta_{\mathcal{M}}(K') \right) \right) \right), \quad L,\, M \in \mathcal{F}(I), \tag{11}$$

where $\mathcal{L}$ with $\beta_{\mathcal{L}}$ and $\mathcal{M}$ with $\beta_{\mathcal{M}}$ reflect the cooperation in $L$ and $M$, respectively, and ${}^{(i)}\beta_{\mathcal{L}}$, ${}^{(i)}\beta_{\mathcal{M}}$ are defined by (7).

Let us note that this formulation represents a hybrid approach to the phenomenon of disjointness in the sense that it is formally based on the fuzzy subsets of $\mathcal{P}(I)$ but, as a consequence of the application of the structures of contacts ${}^{(i)}\beta_{\mathcal{L}}$, it does not contradict with the traditional interpretation of coalitions as (may be fuzzy) subsets of $I$.

**Theorem 2.** If $L$, $M$ are crisp then $\delta(L,M) = 1$ iff $L \cap M = \emptyset$ and $\delta(L,M) = 0$ iff $L \cap M \neq \emptyset$.

P r o o f . The statement follows from (11). If $L$, $M$ are crisp, i. e., $L,\, M \in \mathcal{P}(I)$ then, due to the first paragraph of Section 4, $\beta_{\mathcal{L}}(L) = 1$, $\beta_{\mathcal{M}}(M) = 1$ and $\beta_{\mathcal{L}}(K) = 0$, $\beta_{\mathcal{M}}(K) = 0$ otherwise. Moreover, for all $i \in L$, ${}^{(i)}\beta_{\mathcal{L}}(L) = 1$ and for all $i \in M$, ${}^{(i)}\beta_{\mathcal{M}}(M) = 1$ and the values of ${}^{(i)}\beta_{\mathcal{L}}(\cdot)$ and ${}^{(i)}\beta_{\mathcal{M}}(\cdot)$ vanish in other cases. It means that for disjoint $L$, $M$, always at least one of the values ${}^{(i)}\beta_{\mathcal{L}}(K)$, ${}^{(i)}\beta_{\mathcal{M}}(K)$ for any $K \in \mathcal{P}(I)$ and any $i \in I$ is equal to 0 and, consequently, $\delta(L,M) = 1$. On the other hand, if there exists $i \in L \cap M$ then ${}^{(i)}\beta_{\mathcal{L}}(K) = {}^{(i)}\beta_{\mathcal{M}}(K) = 1$ for $K = L$ and $K' = M$ and, consequently, $\delta(L,M) = 0$. □

Then it is easy to modify formula (9) by means of modifying the condition of disjointness and to define the *fuzzy superadditivity* as a fuzzy property of the TU-games over the set of players $I$. We denote its membership function $\sigma_I$ and define it for a game $(I, v)$ by

$$\sigma_I(v) = 1 - \max\left(\delta(L,M) : L,\, M \in \mathcal{F}(I),\, v(L \cup M) < v(L) + v(M)\right), \tag{12}$$

where the union $L \cup M$ is a fuzzy coalition with its own fuzzy subset $\mathcal{L} \cup \mathcal{M} \in \mathcal{F}(\mathcal{P}(I))$ and $\beta_{\mathcal{L} \cup \mathcal{M}} : \mathcal{P}(I) \to [0,1]$, reflecting the cooperation in $L \cup M$, practically independent of $\beta_{\mathcal{L}}$ and $\beta_{\mathcal{M}}$, and $v$ is the extension of the characteristic function from $\mathcal{P}(I)$ to $\mathcal{F}(I)$, again, analogously to (9) and with respect to (5).

**Remark 4.** If $L$, $M \in \mathcal{F}(I)$ in (12) and $\mathcal{L}$, $\mathcal{M} \in \mathcal{F}(\mathcal{P}(I))$ reflecting their cooperation are such that $\min(\beta_{\mathcal{L}}(K), \beta_{\mathcal{M}}(K)) = 0$ for all $K \in \mathcal{P}(I)$ then it is possible (and, perhaps, natural) to use special expression

$$\beta_{\mathcal{L}\cup\mathcal{M}}(K) = \max\left(\beta_{\mathcal{L}}(K), \beta_{\mathcal{M}}(K)\right)/\,2, \quad K \in \mathcal{P}(I).$$

In accordance with (5), then

$$v(L \cup M) \geq \sum\nolimits_{K\in\mathcal{P}(I)} \beta_{\mathcal{L}\cup\mathcal{M}}(K) \cdot v(K).$$

**Remark 5.** Analogously to the previous case, it is easy to verify that for a TU-game with only crisp coalitions the previous definition of fuzzy superadditivity corresponds with the classical deterministic one (cf. Lemma 5 and (2)).

4.1.c. Exclusively fuzzy subsets of $\mathcal{P}(I)$

The last approach to the disjointness (and, consequently, superadditivity) of fuzzy coalitions follows consequently from their representation by fuzzy subsets of the set $\mathcal{P}(I)$. It means that their *fuzzy disjointness* keeps being a fuzzy relation, i. e., fuzzy subset of $\mathcal{F}(\mathcal{P}(I)) \times \mathcal{F}(\mathcal{P}(I))$, with membership function $\delta^* : \mathcal{F}(\mathcal{P}(I)) \times \mathcal{F}(\mathcal{P}(I)) \to [0, 1]$, defined by

$$\delta^*(\mathcal{L}, \mathcal{M}) = 1 - \max_{K\in\mathcal{P}(I)} \left[\min\left(\beta_{\mathcal{L}}(K), \beta_{\mathcal{M}}(K)\right)\right] \tag{13}$$

for $\mathcal{L}$, $\mathcal{M} \in \mathcal{F}(\mathcal{P}(I))$.

This definition of fuzzy disjointness essentially differs from the classical and intuitively accepted one. Let us note, for illustration, that fuzzy coalition of 4 players described in Example 1 as fuzzy subset of $I$, has several different representations by fuzzy subsets of $\mathcal{P}(I)$. Many of them are completely disjoint in the sense of (13) i. e. $\delta^*(\cdot, \cdot) = 1$, even if they represent the same fuzzy coalition in the sense of Section 3, i. e., the values of $\overline{\delta}(\cdot, \cdot)$ and $\delta(\cdot, \cdot)$ are equal to 0.

For this consequent acceptation of fuzzy coalitions from $\mathcal{F}(\mathcal{P}(I))$, also their union and intersection gains completely different sense. Namely, for $\mathcal{L}$, $\mathcal{M} \in \mathcal{F}(\mathcal{P}(I))$, $\mathcal{L} \cup \mathcal{M}$ and $\mathcal{L} \cap \mathcal{M}$ are from $\mathcal{F}(\mathcal{P}(I))$, too, and for any $K \in \mathcal{P}(I)$,

$$\beta_{\mathcal{L}\cup\mathcal{M}}(K) = \max\left(\beta_{\mathcal{L}}(K), \beta_{\mathcal{M}}(K)\right), \quad \beta_{\mathcal{L}\cap\mathcal{M}}(K) = \min\left(\beta_{\mathcal{L}}(K), \beta_{\mathcal{M}}(K)\right). \tag{14}$$

Except very special degenerated cases, the sums

$$\sum\nolimits_{K\in\mathcal{P}(I)} \beta_{\mathcal{L}\cup\mathcal{M}}(K) \quad \text{and} \quad \sum\nolimits_{K\in\mathcal{P}(I)} \beta_{\mathcal{L}\cap\mathcal{M}}(K) \tag{15}$$

are not equal to 1. With respect to Lemma 2 it means that they have no counterparts in the class $\mathcal{F}(I)$. Consequently, in this model, we have definitely left the environment of fuzzy coalitions extending the class of subsets of $I$ by its fuzzy subsets.

It is worth mentioning that for crisp $K$, $K'$ such that there exists $i \in I$ for which $i \in K$ and $i \in K'$ (the crisp coalitions are not disjoint in the classical sense), but

$K \neq K'$, the intersection (16) is empty and $\mathcal{L}$, $\mathcal{M}$, where $\beta_{\mathcal{L}}(K) = 1$, $\beta_{\mathcal{M}}(K') = 1$, and $\beta_{\mathcal{L}}(\cdot)$, $\beta_{\mathcal{M}}(\cdot)$ vanish for other coalitions are fuzzy disjoint in the sense that $\delta^*(K, K') = 0$.

Meanwhile the disjointness is a property of the inter-coalitional relation, the superadditivity is to respect also specific properties of the characteristic function $v$. Till now, we have considered $v$ as a mapping $v : \mathcal{P}(I) \to R$ extended to $v : \mathcal{F}(I) \to R$ by means of (5). In this subsection, where we consider fuzzy subsets of $\mathcal{P}(I)$, i. e., fuzzy sets from $\mathcal{F}(\mathcal{P}(I))$ for the main representation of coalitional cooperation, it is desirable to extend $v$ also on the mapping $v : \mathcal{F}(\mathcal{P}(I)) \to R$.

Let us stress the fact that the consistency of this extension with the original characteristic function $v : \mathcal{P}(I) \to R$ is desirable.

Let us consider a fuzzy set of crisp coalitions $\mathcal{L} \in \mathcal{F}(\mathcal{P}(I))$ with membership function $\beta_{\mathcal{L}} : \mathcal{P}(I) \to [0, 1]$. Then we define the value

$$v(\mathcal{L}) = \max \{v(K) \cdot \beta_{\mathcal{L}}(K) : K \in \mathcal{P}(I)\} . \tag{16}$$

**Remark 6.** It is easy to see that if the fuzzy set $\mathcal{L}$ is formed by a single possible crisp coalition $K \in \mathcal{P}(I)$, where $\beta_{\mathcal{L}}(K) = 1$, $\beta_{\mathcal{L}}(K') = 0$ for $K' \neq K$, $K' \in \mathcal{P}(I)$, then evidently $v(\mathcal{L}) = v(K)$.

Let us note that (16) is not the single possibility of extension of $v$ on the set $\mathcal{F}(\mathcal{P}(I))$. The alternative approach, extending $v$ into a fuzzy function, is considered in [12].

Even in this model the definition of *fuzzy superadditivity* preserves the classical pattern, and it is defined as a fuzzy property of TU-games with membership function $\sigma^*_I$ such that for any $(I, v)$ the value $\sigma^*_I(v)$ denoting the possibility that $(I, v)$ is superadditive is defined by

$$\sigma^*_I(v) = 1 - \max (\delta^*(\mathcal{L}, \mathcal{M}) : \mathcal{L}, \mathcal{M} \in \mathcal{F}(\mathcal{P}(I)), v(\mathcal{L} \cup \mathcal{M}) < v(\mathcal{L}) + v(\mathcal{M})) . \tag{17}$$

In this formula, the consequent and formally pure definition of union by (15) enables to define the value of $v(\mathcal{L} \cup \mathcal{M})$ (rather similarly to (5)) by the formula used in Remark 4.

The procedure described in this subsection has some evident discrepancies. Their roots consist in the fact that this model of fuzzy coalition as a fuzzy subset of $\mathcal{P}(I)$ separates that notion from the natural demand due to which even the fuzzy coalition is to be a set of players, in some sense extending the crisp coalition model. It means that it is to characterize the distribution of each player's endeavor among the coalitions in which he participates. This demand is not respected. For example, even if $\mathcal{L}$, $\mathcal{M} \in \mathcal{F}(\mathcal{P}(I))$ fulfil the statement of Lemma 3 and for some $i \in I$

$$\sum_{K \in \mathcal{P}(I)} {}^{(i)}\beta_{\mathcal{L}}(K) \leq 1, \quad \sum_{K \in \mathcal{P}(I)} {}^{(i)}\beta_{\mathcal{M}}(K) \leq 1$$

the union of both fuzzy coalitions need not respect that limitation, and then

$$\sum_{K \in \mathcal{P}(I)} {}^{(i)}\beta_{\mathcal{L} \cup \mathcal{M}}(K) > 1,$$

in such case, player $i \in I$ distributes more of his "energy" than he disposes with.

Nevertheless, the approach used in [11] follows from 4.1.c with some modifications reflecting the individual motivation of particular players, and the monotonicity of the pay-off function for the fuzzy coalitions.

### 4.2. Convexity of fuzzy coalitions

If we are to fuzzify the concept of convexity (cf. (3)), the situation is rather simpler, as the convexity is not conditioned by anything like disjointness. On the other hand, the problems regarding the unrealistic analytical properties of $\mathcal{L} \cup \mathcal{M}$ and $\mathcal{L} \cap \mathcal{M}$ for $\mathcal{L}$, $\mathcal{M} \in \mathcal{F}(\mathcal{P}(I))$ meant in Subsection 4.1.c, keep significant even for the concept of convexity. As there is no fuzzy disjointness, it is possible to treat the convexity of TU-game with fuzzy coalitions analogously to the method used in 4.1.a and to compare it with analogy to 4.1.c.

Then we say that TU-game with fuzzy coalitions from $\mathcal{F}(I)$ is convex if for every $L$, $M \in \mathcal{F}(I)$

$$v(L \cup M) + v(L \cap M) \geq v(L) + v(M), \tag{18}$$

where

$$\tau_{L\cup M}(i) = \max\left(\tau_L(i), \tau_M(i)\right), \quad \tau_{L\cap M}(i) = \min\left(\tau_L(i), \tau_M(i)\right). \tag{19}$$

**Remark 7.** It can be easily seen that $L \cup M$ and $L \cap M$ are fuzzy coalitions from $\mathcal{F}(I)$, it means that (5) can be used for them, and the above formula (18) is correct.

The above definition (18), (19) is a weakening of more sophisticated concept of convex game introduced in [2]. Namely, the Butnariu's definition better reflects the intuitively demanded monotonicity of marginal contributions of any coalition. In fact, for a great majority of applications, the version suggested in [2] is more realistic.

The alternative approach based on the assumption that $\mathcal{L}$, $\mathcal{M} \in \mathcal{F}(\mathcal{P}(I))$ is connected with the discrepancies analogous to those mentioned in Subsection 4.1.c. They follow from the concepts of $\mathcal{L} \cup \mathcal{M}$ and $\mathcal{L} \cap \mathcal{M}$ given by (16) and their paradoxical properties.

Nevertheless, these discrepancies were overcome in [11], where they were compensed by some other significant modifications of the model. In this paper, we limit our attention to the (more complicated) concept of superadditivity (depending on the disjointness) which sufficiently illustrates the differences between the presented models.

## 5. CONCLUSION

The definition of fuzzy coalition as a fuzzy subset of the class of all crisp coalitions is, itself, formally correct, and it can be closely connected with the fuzzy coalitions defined as fuzzy sets of players. Anyhow, their further processing closely analogous to the processing of the fuzzy coalitions from $\mathcal{F}(I)$ leads to some paradoxes, mostly following from the attempts to manage the concepts of union and intersection of such fuzzy coalitions.

The acceptance of the alternative model of fuzzy coalition given here and in [11] does not mean that its further development can follow without alternatives. We have tested three of them on the very basic concept of superadditivity. It is obvious that all of them are in certain limits possible but each of them is connected with formal problems demanding other and more essential interventions in the model. The methodological principles presented here in Subsection 4.1.c were further developed in [11] and the results are quite optimistic. They appear to be an adequate reflection of the realistic cooperation with vague participation in coalitions.

Anyhow, the definition of the fuzzy coalitions as fuzzy subsets of the class $\mathcal{P}(I)$ appears inspirative and perspective. It effectively extends the existing model and brings its new interpretations, and it also offers a qualitatively new view at the structure of fuzziness in cooperative behaviour. Hence, it appears to be an interesting topic of the further development of the theory of TU-games with fuzzy coalitions.

The above paper is consequently formulated in the terms of "classical" fuzzy set (and fuzzy quantities) theory, not using the formalism of triangular norms and co-norms. Some of the definitions and results can be essentially extended if the concepts of the $t$-norms theory are used, e. g., in (8) and (18), instead of the classical tools.

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