KYBERNETIKA — VOLUME *44* (2008), NUMBER 2, PAGES 151 – 170

MULTISTAGE STOCHASTIC PROGRAMS VIA AUTOREGRESSIVE SEQUENCES AND INDIVIDUAL PROBABILITY CONSTRAINTS

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The paper deals with a special case of multistage stochastic programming problems. In particular, the paper deals with multistage stochastic programs in which a random element follows an autoregressive sequence and constraint sets correspond to the individual probability constraints. The aim is to investigate a stability (considered with respect to a probability measures space) and empirical estimates. To achieve new results the Wasserstein metric determined by $\mathcal{L}_1$ norm and results of multiobjective optimization theory are employed.

Keywords: multistage stochastic programming problem, individual probability constraints, autoregressive sequence, Wasserstein metric, empirical estimates, multiobjective problems

AMS Subject Classification: 90C15

1. INTRODUCTION

It is well-known that multistage stochastic programming problems has been employed to determine optimal (or at least acceptable) solution in many applications. Let us recall some of them: Financial problems (see e. g. [3, 7]), melt control problem (see e. g. [3, 4]), power-station planning (see e. g. [23]), power scheduling and hydro-termal system control (see e. g. [24]), energy problems (see e. g. [31]), transportation and logistics problems (see e. g. [25]), unemployment problem (see e. g. [5, 17]). Some others problems can be found e. g. in [22] and [26].

From the mathematical point of view, the multistage stochastic programming problems belong to optimization problems depending on a probability measure. Usually, the operator of mathematical expectation appears in the objective function and, moreover, constraint set can depend on the probability measure also. The multistage stochastic programming problems correspond to applications (with an unneglected random element) that can be reasonably considered with respect to some finite "discrete" (say $(0, M)$; $M \geq 1$) time interval and simultaneously there exists a possibility to decompose them with respect to the individual time points. Moreover, a decision, at every individual time point say k, can depend only on the random elements realizations and the decisions to the time point $k-1$ (we say that it must be nonanticipative).

We focus on a special case of the multistage stochastic problems. In particular we focus on an analysis of the multistage stochastic programming problems in which the random element follows an autoregressive sequence and the constraint sets correspond to the individual probability constraints. It is known, that just a development of many economic characteristics follows autoregressive sequences (see in the financial mathematics e. g. the development of the price of market index or the price of bonds). From the mathematical point of view, this type of the development of the random characteristics gives a possibility to obtain a suitable properties of the individual "decomposed" problems.

A similar problems including also generally the Markov type of dependence have been investigated e. g. in [13, 15, 21]. However, this paper tries to present more detailed analysis of the problem. Moreover, the stability bounds (introduced in this paper) can be acceptable from the numerical point of view as well as they can be employed for empirical estimates investigation.

2. MATHEMATICAL DEFINITIONS

A few types of the multistage stochastic programming definitions are known from the stochastic programming literature. Let us recall two well-known approaches [2]: $(M+1)$-stage $(M \geq 1)$ stochastic programming problem is very often introduced as an optimization problem considered with respect to some abstract mathematical space (say $\mathcal{L}_p$ space, $p \geq 1$) or as a finite system of parametric (one-stage) optimization problems with an inner type of dependence. Employing the above mentioned second approach, we introduce $(M+1)$-stage stochastic programming problem as the problem:

Find
$$\varphi_{\mathcal{F}}(M) = \inf \left\{ \mathrm{E}_{F^{\xi^0}} g^0_{\mathcal{F}}(x^0, \xi^0) | \; x^0 \in \mathcal{K}^0 \right\}, \tag{2.1}$$

where the function $g^0_{\mathcal{F}}(x^0, z^0)$ is defined recursively

$$\begin{aligned} g^k_{\mathcal{F}}(\bar{x}^k, \bar{z}^k) \quad &= \quad \inf \left\{ \mathrm{E}_{F^{\xi^{k+1}|\bar{\xi}^k = \bar{z}^k}} \, g^{k+1}_{\mathcal{F}}(\bar{x}^{k+1}, \bar{\xi}^{k+1}) \, | x^{k+1} \in \mathcal{K}^{k+1}_{\mathcal{F}}(\bar{x}^k, \bar{z}^k) \right\}, \\ & \qquad k = 0, 1, \ldots, M-1, \\ g^M_{\mathcal{F}}(\bar{x}^M, \bar{z}^M) \quad &:= \quad g^M_0(\bar{x}^M, \bar{z}^M), \quad \mathcal{K}_0 := X^0. \end{aligned} \tag{2.2}$$

$\xi^j := \xi^j(\omega)$, $j = 0, 1, \ldots, M$ denotes an s-dimensional random vector defined on a probability space $(\Omega, \mathcal{S}, P)$; $F^{\xi^j}(z^j)$, $z^j \in \mathbb{R}^s$, $j = 0, 1 \ldots, M$ the distribution function of the ξ^j and $F^{\xi^k|\bar{\xi}^{k-1}}(z^k|\bar{z}^{k-1})$, $z^k \in \mathbb{R}^s$, $\bar{z}^{k-1} \in \mathbb{R}^{(k-1)s}$, $k = 1, \ldots, M$ the conditional distribution function (ξ^k conditioned by $\bar{\xi}^{k-1}$); $P_{F^{\xi^j}}$, $P_{F^{\xi^{k+1}|\bar{\xi}^k}}$, $j = 0, 1, \ldots, M$, $k = 0, 1, \ldots M-1$ the corresponding probability measures; $Z^j := Z_{F^{\xi^j}} \subset \mathbb{R}^s$, $j = 0, 1, \ldots, M$ the support of the probability measure $P_{F^{\xi^j}}$. Furthermore, the symbol $g^M_0(\bar{x}^M, \bar{z}^M)$ denotes a continuous function defined on $\mathbb{R}^{n(M+1)} \times \mathbb{R}^{s(M+1)}$; $X^k \subset \mathbb{R}^n$, $k = 0, 1, \ldots, M$ is a nonempty compact set; the symbol

$\mathcal{K}^{k+1}_{\mathcal{F}}(\bar{x}^k,\, \bar{z}^k) := \mathcal{K}^{k+1}_{F^{\xi^{k+1}|\bar{\xi}^k}}(\bar{x}^k,\, \bar{z}^k),\ k = 0,\, 1, \dots M-1$ denotes a multifunction mapping $\mathbb{R}^{n(k+1)} \times \mathbb{R}^{s(k+1)}$ into the space of subsets of $\mathbb{R}^n$. $\bar{\xi}^k (:= \bar{\xi}^k(\omega)) = [\xi^0, \dots \xi^k]$; $\bar{z}^k = [z^0, \dots z^k],\ z^j \in \mathbb{R}^s$; $\bar{x}^k = [x^0, \dots x^k],\ x^j \in \mathbb{R}^n$; $\bar{X}^k = X^0 \times X^1 \dots \times X^k$; $\bar{Z}^k := \bar{Z}^k_{\mathcal{F}} = Z_{F^{\xi^0}} \times Z_{F^{\xi^1}} \dots \times Z_{F^{\xi^k}},\ j = 0,\, 1,\, \dots, k,\ k = 0,\, 1, \dots M$. Symbols $\mathrm{E}_{F^{\xi^0}}$, $\mathrm{E}_{F^{\xi^{k+1}|\bar{\xi}^k = \bar{z}^k}},\ k = 0,\, 1, \dots M-1$ denote the operators of mathematical expectation corresponding to F^{ξ^0}, $F^{\xi^{k+1}|\bar{\xi}^k = \bar{z}^k},\ k = 0, \dots M-1$.

Evidently, the multistage stochastic programming problem (2.1), (2.2) depends essentially on a system of (generally) conditional distribution functions

$$\mathcal{F} = \{F^{\xi^0}(z^0),\, F^{\xi^k|\bar{\xi}^{k-1}}(z^k|\bar{z}^{k-1}),\, k = 1, \dots M\}. \tag{2.3}$$

Consequently, if we replace $\mathcal{F}$ by another system $\mathcal{G}$

$$\mathcal{G} = \{G^{\xi^0}(z^0),\, G^{\xi^k|\bar{\xi}^{k-1}}(z^k|\bar{z}^{k-1}),\, k = 1, \dots M\}, \tag{2.4}$$

we obtain another multistage stochastic programming problem with the optimal value denoted by $\varphi_{\mathcal{G}}(M)$. The aim of the paper will be, first, to investigate their relationship by the value

$$|\varphi_{\mathcal{F}}(M) - \varphi_{\mathcal{G}}(M)|.$$

Furthermore, the achieved results will be employed for the investigation of empirical estimates $\varphi_{\mathcal{F}_{\mathcal{N}}}(M)$ of $\varphi_{\mathcal{F}}(M)$. In particular, we shall try to investigate the probability properties of the value

$$|\varphi_{\mathcal{F}}(M) - \varphi_{\mathcal{F}_{\mathcal{N}}}(M)|$$

for the case when the system $\mathcal{F}$ is replaced by the corresponding system of empirical distribution functions.

To obtain new results we restrict our consideration to the special case when the following assumptions (mentioned already above) are fulfilled:

A.1 $\{\xi^k\}_{k=-\infty}^{\infty}$ follows a (generally) nonlinear autoregressive sequence

$$\xi^k = H(\xi^{k-1}) + \varepsilon^k, \tag{2.5}$$

where $\xi^0,\, \varepsilon^k,\ k = 1,\, 2,\, \dots$ are stochastically independent; $\varepsilon^k,\ k = 1, \dots$ identically distributed. $H := (H_1, \dots H_s)$ is a Lipschitz vector function defined on $\mathbb{R}^s$. We denote the distribution function corresponding to $\varepsilon^1 = (\varepsilon^1_1, \dots \varepsilon^1_s)$ by the symbol F^{ε} and suppose the realization ξ^0 to be known,

A.2 there exist functions $f^{k+1}_{i,\,j},\ i = 1, \dots s,\ j = 1, \dots k+1,\ k = 0, \dots M-1$ defined on $\mathbb{R}^n$ and $\alpha_i \in (0,\, 1),\ i = 1, \dots s,\ \bar{\alpha} = (\alpha_1, \dots \alpha_s)$ such that

$$\begin{aligned} &\mathcal{K}^{k+1}_{\mathcal{F}}(\bar{x}^k,\, \bar{z}^k) \quad (:= \mathcal{K}^{k+1}_{\mathcal{F}}(\bar{x}^k,\, \bar{z}^k;\, \bar{\alpha})) \\ &= \bigcap_{i=1}^{s} \left\{ x^{k+1} \in X^{k+1} : P_{F^{\xi^{k+1}|\bar{\xi}^k = \bar{z}^k}} \left\{ \sum_{j=1}^{k+1} f^{k+1}_{i,\,j}(x^j) \le \xi^{k+1}_i \right\} \ge \alpha_i \right\}, \\ &\xi^{k+1} = (\xi^{k+1}_1, \dots \xi^{k+1}_s). \end{aligned} \tag{2.6}$$

Under the assumption A.1 the system $\mathcal{F}$ is determined by F^{ξ^0} and F^{ε}. Consequently, if we replace these two probability distribution functions by another G^{ξ^0} and G^{ε} we obtain also another system $\mathcal{G}$.

Remark. Evidently, we consider special types of "underlying" problems with a random element. In particular, we consider the problems in which the random coefficients can appear only on the right hand sides of the constraints and in the objective function.

3. PROBLEM ANALYSIS

Evidently, the problem (2.1) is a "classical" one-stage stochastic programming problem, the problems (2.2) are (generally) parametric one-stage stochastic programming problems. Moreover, these problems are mostly parametric recourse problems. Consequently, to be the multistage stochastic programming problem (2.1), (2.2) well defined it is necessary to be finite a.s. the optimal values of the inner problems (2.2). Of course, to this end, it is necessary to be "individual" constraint sets nonempty a.s. There are well known (from the stochastic programming literature) sufficient assumptions guaranteeing property in the linear case (fixed complete recourse matrices) or generally relatively complete recourse constraints (for more details see [1]). In this contribution we try to extend the sufficient assumptions guaranteeing this property. To this end we employ the approach introduced in [19], where a linear case example (of recourse problem) is introduced in which the matrix is not complete recourse, however evidently our conditions are fulfilled. There achieved results are based on the theory of the multiobjective deterministic optimization (see e. g. [6, 8]). This approach can be very suitable, especially, in a linear case. Namely there a modified simplex algorithm (for linear parametric programming) can be employed to verify that constraint set is nonempty. In a general case a parametric convex programming algorithms have to be employed; an approximate approach based on Lipschitz property of the corresponding "weight" function (and bounded assumption) can be also employed (for more details see Lemma 1). Evidently, then the fulfilling of the constraints with probability α' very "near" to α can be guaranteed. Moreover, this approach together with assumptions A.1, A.2 can guarantee the existence of finite individual objective functions.

To analyze the properties of $\mathcal{K}^{k+1}_{\mathcal{F}}(\bar{x}^k,\, \bar{z}^k)$, $k = 0, \dots M-1$ let us, first, consider "deterministic" constraint sets corresponding to the assumption A.3:

A.3 there exist continuous functions $f_i^{k+1}(x^{k+1})$, $h_i(\bar{x}^k,\, \bar{z}^k)$, $i = 1, \dots s$, $k = 0, \dots M-1$ defined on X^{k+1} and $\bar{X}^k \times \bar{Z}^k$ such that

$$\begin{aligned} & \mathcal{K}^{k+1}_{\mathcal{F}}(\bar{x}^k,\, \bar{z}^k) := \mathcal{K}^{k+1}(\bar{x}^k,\, \bar{z}^k) \\ = \quad & \left\{x^{k+1} \in X^{k+1} : f_i^{k+1}(x^{k+1}) \le h_i^{k+1}(\bar{x}^k,\, \bar{z}^k),\, i = 1, \dots s\right\}, \quad (3.1) \\ & \bar{x}^k \in \bar{X}^k,\, \bar{z}^k \in \bar{Z}^k. \end{aligned}$$

Evidently, it is easy to see that under the assumption A.3 (for $k = 0, \dots M-1$) the following implications is valid.

$$\begin{aligned} & \mathcal{K}^{k+1}(\bar{x}^k,\, \bar{z}^k) \quad \text{is nonempty for} \quad \bar{x}^k \in \bar{X}^k,\ \bar{z}^k \in \bar{Z}^k \\ \Longrightarrow\ & \mathcal{K}^{k+1}(\bar{x}^k(1),\, \bar{z}^k(1)) \quad \text{is nonempty for every} \quad \bar{x}^k(1) \in \bar{X}^k,\, \bar{z}^k(1) \in \bar{Z}^k \\ & \text{such that} \quad h_i(\bar{x}^k,\, \bar{z}^k) \,\le\, h_i(\bar{x}^k(1),\, \bar{z}^k(1)),\ i = 1, \dots s. \end{aligned}$$

Moreover, it follows from the multiobjective optimization theory that if $\bar{X}^k$, $\bar{Z}^k$ are compact sets and $\mathcal{K}_E^{k+1}(\bar{x}^k,\, z^k)$ denotes the set of efficient points of the multiobjective problem:

Find

$$\min h_i^{k+1}(\bar{x}^k,\, \bar{z}^k),\ i = 1, \dots s \quad \text{subject to} \quad \bar{x}^k \in \bar{X}^k,\, z^k \in Z^k, \tag{3.2}$$

then

$$\begin{aligned} & \mathcal{K}^{k+1}(\bar{x}^k,\, \bar{z}^k) \quad \text{is nonempty for} \quad (\bar{x}^k,\, z^k) \in \mathcal{K}_E^{k+1}(\bar{x}^k,\, z^k) \\ \Longrightarrow\ & \mathcal{K}^{k+1}(\bar{x}^k,\, \bar{z}^k) \quad \text{is nonempty for} \quad \bar{x}^k \in \bar{X}^k,\, \bar{z}^k \in \bar{Z}^k. \end{aligned} \tag{3.3}$$

(For the definition of the efficient points see e. g. [6] or the Section 4.)

Let us now return to the case corresponding to the assumptions A.1, A.2. Evidently, if we define quantiles $k_{F_i^{\xi^{k+1}|\bar{\xi}^k=\bar{z}^k}}(\alpha_i),\ k_{F_i^{\varepsilon}}(\alpha_i),\ \alpha_i \in (0,\, 1),\ i = 1, \dots, s,$ $k = 0,\, 1, \dots, M-1$ by

$$\begin{aligned} k_{F_i^{\xi^{k+1}|\bar{\xi}^k=\bar{z}^k}}(\alpha_i) &= \sup_{z_i^{k+1}\in\mathbb{R}^1} \left\{ z_i^{k+1} : P_{F_i^{\xi^{k+1}|\bar{\xi}^k=\bar{z}^k}} \left\{ z_i^{k+1} \le \xi_i^{k+1} \right\} \ge \alpha_i \right\}, \\ k_{F_i^{\varepsilon}}(\alpha_i) &= \sup_{z_i\in\mathbb{R}^1} \left\{ z_i : P_{F_i^{\varepsilon}}\{z_i \le \varepsilon_i\} \ge \alpha_i \right\}, \end{aligned}$$

then under the assumptions A.1, A.2 we can obtain

$$k_{F_i^{\varepsilon}}(\alpha_i) = k_{F_i^{\xi^{k+1}|\bar{\xi}^k=\bar{z}^k}}(\alpha_i) - H_i(z^k).$$

(Symbols $F_i^{\xi^{k+1}|\bar{\xi}^k=\bar{z}^k}$, F_i^{ε}, $i = 1, \dots s$ denote one-dimensional marginal distribution functions corresponding to $F^{\xi^{k+1}|\bar{\xi}^k=\bar{z}^k}$ and F^{ε}, $\bar{z}^k \in \bar{Z}^k$.)

According to the last relation we can (under the assumptions A.1, A.2) obtain that

$$\begin{aligned} \mathcal{K}_{\mathcal{F}}^{k+1}(\bar{x}^k,\, \bar{z}^k) &= \bigcap_{i=1}^{s} \left\{ x^{k+1} \in X^{k+1} : \sum_{j=1}^{k+1} f_{i,\,j}^{k+1}(x^j) \le k_{F_i^{\xi^{k+1}|\xi^k=z^k}}(\alpha_i) \right\} \\ &= \bigcap_{i=1}^{s} \left\{ x^{k+1} \in X^{k+1} : \sum_{j=1}^{k+1} f_{i,\,j}^{k+1}(x^j) \le k_{F_i^{\varepsilon}}(\alpha_i) + H_i(z^k) \right\}. \end{aligned} \tag{3.4}$$

Consequently, setting for $\bar{\alpha} = (\alpha_1, \dots, \alpha_s)$, $h_i^{k+1}(\bar{x}^k, \bar{z}^k)$, $i = 1, \dots, s,\ k = 0, \dots, M-1$ by

$$h_i^{k+1}(\bar{x}^k,\, \bar{z}^k) := h_i^{k+1}(\bar{x}^k,\, \bar{z}^k,\, k_{F_i^{\varepsilon}}(\alpha_i)) := k_{F_i^{\varepsilon}}(\alpha_i) + H_i(z^k) - \sum_{j=1}^{k} f_{i,\,j}^{k+1}(x^j), \tag{3.5}$$

we obtain “classical deterministic nonlinear” constraint sets (3.1) in the form

$$\mathcal{K}^{k+1}_{\mathcal{F}}(\bar{x}^k,\, \bar{z}^k) = \bigcap_{i=1}^{s} \left\{ x^{k+1} \in X^{k+1} : \, f^{k+1}_{i,\,k+1}(x^{k+1}) \le h^{k+1}_i(\bar{x}^k,\, \bar{z}^k) \right\}. \tag{3.6}$$

Remark. Evidently for every $k = 0, \dots M-1$, $\sum_{j=1}^{k} f^{k+1}_{i,\,j}(x^j)$ can be replaced by some continuous function $f^{k+1}_i(\bar{x}^k)$ defined on $\bar{X}^k$.

4. SOME DEFINITIONS AND AUXILIARY ASSERTIONS

4.1. Multiobjective deterministic optimization

A multiobjective deterministic optimization problem can be introduced as the problem:

Find

$$\min h^*_i(v), \;\; i = 1, \dots s \quad \text{subject to } v \in \mathcal{K}. \tag{4.1}$$

h^*_i, $i = 1, \dots s$ are functions defined on $\mathbb{R}^{n_1}$, $\mathcal{K} \subset \mathbb{R}^{n_1}$ is a nonempty set.

Definition 1. (Geoffrion [8]) The vector v^* is an efficient solution of the problem (4.1) if and only if $v^* \in \mathcal{K}$ and if there exists no $v \in \mathcal{K}$ such that $h^*_i(v) \le h^*_i(v^*)$ for $i = 1, \dots s$ and such that for at least one i_0 one has $h^*_{i_0}(v) < h^*_{i_0}(v^*)$. We denote the set of efficient points of the problem (4.1) by the symbol $\mathcal{K}_E$.

First, let us consider a special case when

- i.1 there exist deterministic vectors $d^i \in \mathbb{R}^{n_1}$, $d_i := d^i(1 \times n_1)$, $i = 1, \dots s$, such that $h^*_i(v) = d^i v$, $i = 1, \dots s$, $v \in \mathcal{K}$, $v := v\,(n_1 \times 1)$,
- i.2 $\mathcal{K} = \{v \in \mathbb{R}^{n_1} : \, Av = b, \, v \ge 0\}$, where $A := A(\bar{m} \times n_1)$, $b := b(\bar{m} \times 1)$ are a deterministic matrix and a deterministic vector.

We recall the theorem of Issermann 1974 (for more details see e. g. [6]).

Theorem 1. Let the assumptions i.1 and i.2 be fulfilled. A feasible $v^* \in \mathcal{K}$ is an efficient solution of the problem (4.1) if and only if there exists a $\lambda = (\lambda_1, \dots, \lambda_s) \in \mathbb{R}^s$, $\lambda_i > 0$, $i = 1, \dots s$ such that

$$\sum_{i=1}^{s} \lambda_i d^i v^* \; \le \; \sum_{i=1}^{s} \lambda_i d^i v \quad \text{for every} \quad v \in \mathcal{K}.$$

Evidently, the assumptions i.1, i.2 correspond to many applications (see e. g. [32]). However, very often the assumption of the linear constraints is not fulfilled. If at least the corresponding functions are convex ones, we can obtain (from the numerical point of view at least approximately) also acceptable conditions determining efficient points. To this end, first, we recall the definition of properly efficient points.

Definition 2. (Geoffrion [8]) The vector v^* is a properly efficient solution of the multiobjective optimization problem (4.1) if and only if it is efficient and if there exists a scalar $M > 0$ such that for each i and each $v \in \mathcal{K}$ satisfying $h_i^*(v) < h_i^*(v^*)$ there exists at least one j such that $h_j^*(v^*) < h_j^*(v)$ and

$$\frac{h_i^*(v^*) - h_i^*(v)}{h_j^*(v) - h_j^*(v^*)} \le M. \tag{4.2}$$

We denote the set of properly efficient points of problem (4.1) by the symbol $\mathcal{K}_{PE}$.

To recall the next auxiliary assertion we define the set Λ by the relation:

$$\Lambda = \left\{ \lambda \in \mathbb{R}^s : \lambda = (\lambda_1, \dots \lambda_s), \, \lambda_i \in \langle 0, 1 \rangle, \, i = 1, \dots s, \, \sum_{i=1}^{s} \lambda_i = 1 \right\}.$$

Proposition 1. (Geoffrion [8]) Let $\mathcal{K}$ be a convex set and let h_i^*, $i = 1, \dots s$ be convex functions on $\mathcal{K}$. Then v^* is a properly efficient solution of problem (4.1) if and only if v^* is optimal in

$$\begin{aligned} &\min_{v \in \mathcal{K}} h^{*,\lambda}(v) \quad \text{for some} \quad \lambda \in \Lambda \quad \text{with} \quad \lambda_i > 0, \, i = 1, \dots, s \\ &\text{and} \quad h^{*,\lambda}(v) = \sum_{i=1}^{s} \lambda_i h_i^*(v). \end{aligned} \tag{4.3}$$

If we denote by the symbols $h^*(\mathcal{K}_E)$, $h^*(\mathcal{K}_{PE}) \subset \mathbb{R}^s$ the image of $\mathcal{K}_E$, $\mathcal{K}_{PE} \subset \mathbb{R}^{n_1}$ obtained by the vector function $h^* = (h_1^*, \dots h_s^*)$, then the implication

$$\begin{aligned} &\mathcal{K} \quad \text{closed and convex}, \; h_i^*, \, i = 1, \dots s \text{ continuous and convex on} \quad \mathcal{K} \\ &\qquad \Longrightarrow h^*(\mathcal{K}_{PE}) \subset h^*(\mathcal{K}_E) \subset \bar{h}^*(\mathcal{K}_{PE}) \end{aligned} \tag{4.4}$$

has been recalled in [8]. The symbol $\bar{h}^*(\mathcal{K}_{PE})$ denotes a closure of $h^*(\mathcal{K}_{PE})$.

Proposition 1 and the relation (4.4) are very suitable for a determination (or at least "estimation") of efficient and properly efficient points as well as their function value. However, to this end it is necessary to be $\mathcal{K}$ a convex set and h_i^*, $i = 1, \dots s$ convex functions. If these assumptions are not fulfilled, then corresponding conditions are more complicated (for more details see e. g. [6]).

Completed this part, we recall an auxiliary assertion that can be very easy proven.

Lemma 1. Let $\mathcal{K} \subset \mathbb{R}^{n_1}$ be a nonempty set, h_i^*, $i = 1, \dots, s$ be functions defined on $\mathbb{R}^{n_1}$. Let, moreover, the function $h^{*,\lambda}$ be defined by the relation (4.3). We obtain.

1. If $h_i^*, i = 1, \dots, s$ are Lipschitz functions on $\mathcal{K}$ with the Lipschitz constants $\bar{L}_i$, then $h^{*,\lambda}$, $\lambda \in \Lambda$ is a Lipschitz function on $\mathcal{K}$ with a Lipschitz constant not greater then $\sum_{i=1}^{s} \bar{L}_i$.

2. If $h_i^*, i = 1, \dots, s$ are bounded functions on $\mathcal{K}$, ($|h_i^*(v)| \le \bar{M}$, $v \in \mathcal{K}$, $i = 1, \dots, s$, $\bar{M} > 0$), then for every v, $h^{*,\lambda}$ is a Lipschitz function on Λ with a Lipschitz constant not greater then $s\bar{M}$.

4.2. One-stage stochastic programming problems

To recall results achieved for one-stage stochastic programming problem let $\bar{g}_0(x, z)$ be a function defined on $\mathbb{R}^n \times \mathbb{R}^s$, $\xi\, (:= \xi(\omega)) = [\xi_1, \ldots, \xi_s]$, $\eta\, (:= \eta(\omega)) = [\eta_1, \ldots, \eta_s]$ be s-dimensional random vectors defined on $(\Omega, \mathcal{S}, P)$. We denote by F, G; P_F, P_G; Z_F, Z_G the distribution functions, probability measures and probability measures supports corresponding to ξ and η; by F_i, G_i, $i = 1, \ldots, s$ one-dimensional marginal distribution functions corresponding to F and G. Let, moreover $X \subset \mathbb{R}^n$ be a nonempty ("deterministic") set, X_F, $X_G \subset \mathbb{R}^n$ be nonempty sets depending generally on F and G.

A rather general "classical" one-stage stochastic programming problem can be introduced in the form:

Find
$$\varphi(F) = \inf\{\mathrm{E}_F \bar{g}_0(x, \xi) \,|\, x \in X_F\}. \tag{4.5}$$

To recall the definition of the Wasserstein metric $d^s_{W_1}$, let

$$\mathcal{M}_{\cdot}(\mathbb{R}^s) = \left\{ \nu \in \mathcal{P}(\mathbb{R}^s) : \int_{\mathbb{R}^s} \|z\| \, \nu(\mathrm{d}z) < \infty \right\},$$

where $\mathcal{P}(\mathbb{R}^s)$ denotes the set of all Borel probability measures on $\mathbb{R}^s$, $s \ge 1$, $\|\cdot\|$ denotes a "suitable" norm in $\mathbb{R}^s$. We denote by $\|\cdot\|^i_s$, $i = 1, 2$ the norm corresponding to the space $\mathcal{L}_i$, $i = 1, 2$ in $\mathbb{R}^s$. If the Wasserstein metric is determined by $\|\cdot\|^1_s$, then we denote $\mathcal{M}_{\cdot}(\mathbb{R}^s) := \mathcal{M}_1(\mathbb{R}^s)$ and it is possible to employ the approach of [30]. The following assertion has been proven in [16].

Proposition 2. (Kaňková and Houda [16]) Let P_F, $P_G \in \mathcal{M}_1(\mathbb{R}^s)$. If for every $x \in X$, $\bar{g}_0$ is a Lipschitz (with respect to $\mathcal{L}_1$ norm) function of $z \in \mathbb{R}^s$, $z = (z_1, \ldots z_s)$, the Lipschitz constant L is not depending on $x \in X$. If, moreover, for every $x \in X$ a finite $\mathrm{E}_F \bar{g}_0(x, \xi)$, $\mathrm{E}_G \bar{g}_0(x, \eta)$ exist, then

$$|\mathrm{E}_F \bar{g}_0(x, \xi) - \mathrm{E}_G \bar{g}_0(x, \eta)| \le L \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| \, \mathrm{d}z_i \quad \text{for} \quad x \in X.$$

The following lemma follows from the triangular inequality.

Lemma 2. Let X_F, X_G, $X \subset \mathbb{R}^n$ be nonempty, compact sets, $\bar{g}_0$ be a uniformly continuous function defined on $X \times \mathbb{R}^s$; X_F, $X_G \subset X$. If, moreover, for every $x \in X$ a finite $\mathrm{E}_F \bar{g}_0(x, \xi)$, $\mathrm{E}_G \bar{g}_0(x, \eta)$ exist, then

$$\begin{aligned} &\left| \inf_{x \in X_F} \mathrm{E}_F \bar{g}_0(x, \xi) - \inf_{x \in X_G} \mathrm{E}_G \bar{g}_0(x, \eta) \right| \\ \le\ &\left| \inf_{x \in X_F} \mathrm{E}_F \bar{g}_0(x, \xi) - \inf_{x \in X_F} \mathrm{E}_G \bar{g}_0(x, \eta) \right| + \left| \inf_{x \in X_F} \mathrm{E}_G \bar{g}_0(x, \eta) - \inf_{x \in X_G} \mathrm{E}_G \bar{g}_0(x, \eta) \right|. \end{aligned}$$

Evidently, the assertion of Proposition 2 can be employed for the investigation of empirical estimates. To this end, let F^N denote empirical distribution function determined by random sample $\{\xi^i\}_{i=1}^N$ corresponding to the distribution function F.

Proposition 3. (Kaňková [20]) Let $t > 0$, $P_F \in \mathcal{M}_1(\mathbb{R}^s)$, $\{\xi^i\}_{i=1}^{\infty}$ be an independent sequence of s-dimensional random vectors with a common distribution function F. If

1. F^N is determined by $\{\xi^i\}_{i=1}^{N}$, $N = 1, 2, \ldots$,
2. P_{F_i}, $i = 1, \ldots s$ are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^1$ (we denote by f_i the probability densities corresponding to F_i),
3. there exist constants $C_1, C_2 > 0$ and $T > 0$ such that

$$f_i(z_i) \le C_1 \exp\{-C_2|z_i|\} \quad \text{for} \quad z_i \in (-\infty, -T) \cup (T, \infty), \quad i = 1, \ldots s,$$

then for $i \in \{1, \ldots s\}$, $t > 0$ and $\beta \in (0, \frac{1}{2})$ it holds that

$$P\left\{N^{\beta} \int_{-\infty}^{\infty} |F_i(z_i) - F_i^N(z_i)| \, \mathrm{d}z_i > t\right\} \underset{(N \to \infty)}{\longrightarrow} 0.$$

To introduce the next implication we assume:

A.4 there exist constants ϑ_i, $i = 1, \ldots s$ and a surroundings $U_i(k_{F_i}(\alpha_i))$ of $k_{F_i}(\alpha_i)$ such that $f_i(z_i) > \vartheta_i$ for $z_i \in U_i(k_{F_i}(\alpha_i))$,

$$k_{F_i}(\alpha_i) = \sup_{z_i \in \mathbb{R}^1} \{z_i : P_{F_i}\{z_i \le \xi_i\} \ge \alpha_i\}.$$

The following implication follows (for $i = i, \ldots s$) from results presented in [29]:

$$\begin{aligned} \text{A.4} \quad \Longrightarrow \quad & P\{N^{\beta}|k_{F_i}(\alpha_i) - k_{F_i^N}(\alpha_i)| > t\} \underset{(N \to \infty)}{\longrightarrow} 0 \\ & \text{for every} \quad t > 0, \quad \beta \in (0, \tfrac{1}{2}). \end{aligned} \tag{4.6}$$

5. MAIN RESULTS

Let the assumptions A.1, A.2 be fulfilled. Evidently, the system $\mathcal{F}$ is under the assumption A.1 determined by the distribution functions F^{ξ^0} and F^{ε}. Consequently, if we can assume that the realization ξ^0 is known, then $\mathcal{F}$ is determined by F^{ε}.

To introduce new results of this paper, let us first for $k = 0, \ldots M-1$ define deterministic multiobjective problems:

Find

$$\begin{aligned} & \min h_i^{k+1}(\bar{x}^k, \bar{z}^k), \; i = 1, \ldots s \quad \text{subject to} \quad \bar{x}^k \in \bar{X}^k, \, \bar{z}^k \in \bar{Z}^k \\ & \text{with} \quad h_i^{k+1} \quad \text{defined by} \quad (3.5). \end{aligned} \tag{5.1}$$

If we define $G^{k+1,\lambda}(\bar{x}^k, \bar{z}^k)$, $\mathcal{K}_{\mathcal{F}}^{k+1,\Lambda}(\bar{X}^k, \bar{Z}^k)$, $k = 0, \ldots M-1$ by

$$G^{k+1,\lambda}(\bar{x}^k, \bar{z}^k) = \sum_{i=1}^{s} \lambda_i h_i^{k+1}(\bar{x}^k, \bar{z}^k), \; \bar{x}^k \in \bar{X}^k, \, \bar{z}^k \in \bar{Z}^k, \quad \lambda \in \Lambda, \tag{5.2}$$

$$\mathcal{K}_{\mathcal{F}}^{k+1,\,\Lambda}(\bar{X}^k,\,\bar{Z}^k) = \begin{aligned}[t] &\Big\{\bar{x}^k \in \bar{X}^k,\, \bar{z}^k \in \bar{Z}^k : G^{k+1,\,\lambda}(\bar{x}^k,\,\bar{z}^k) \\ &= \min\{G^{k+1,\,\lambda}(\bar{x}^k,\,\bar{z}^k) : \bar{x}^k \in \bar{X}^k,\, \bar{z}^k \in \bar{Z}^k \\ &\quad \text{for some} \quad \lambda \in \Lambda,\, \lambda_i > 0,\, i = 1, \dots s\Big\} \end{aligned} \tag{5.3}$$

and $\mathcal{K}_{\mathcal{F},\,h}^{k+1}(u)$, $u \in \mathbb{R}^s$ by

$$\mathcal{K}_{\mathcal{F},\,h}^{k+1}(u) = \Big\{x^{k+1} \in X^{k+1} : f_{i,\,k+1}^{k+1}(x^{k+1}) \le u_i,\, i = 1, \dots, s\Big\} \;\; u = (u_1, \dots, u_s),$$

then evidently

$$\mathcal{K}_{\mathcal{F}}^{k+1}(\bar{x}^k,\,\bar{z}^k) \;=\; \mathcal{K}_{\mathcal{F},\,h}^{k+1}(h(\bar{x}^k,\,\bar{z}^k)), \quad \bar{x}^k \in \bar{X}^k,\, \bar{z}^k \in \bar{Z}^k.$$

If, furthermore,

$$\mathcal{H}_{\mathcal{F}}^{k+1,\,\Lambda}(\bar{X}^k,\,\bar{Z}^k) = \begin{aligned}[t] &\Big\{u \in \mathbb{R}^s : u = (u_1, \dots u_s),\, u_i = h_i^{k+1}(\bar{x}^k,\,\bar{z}^k), \\ &i = 1, \dots s \text{ for some } (\bar{x}^k,\,\bar{z}^k) \in \mathcal{K}_{\mathcal{F}}^{k+1,\,\Lambda}(\bar{X}^k,\,\bar{Z}^k)\Big\}, \end{aligned} \tag{5.4}$$

the symbol $\bar{\mathcal{K}}_{\mathcal{F}}^{k+1,\,\Lambda}(\bar{X}^k,\,\bar{Z}^k)$, $\bar{\mathcal{H}}_{\mathcal{F}}^{k+1,\,\Lambda}(\bar{X}^k\,\bar{Z}^k)$ denote closures of $\mathcal{K}_{\mathcal{F}}^{k+1,\,\Lambda}(\bar{X}^k,\,\bar{Z}^k)$, $\mathcal{H}_{\mathcal{F}}^{k+1,\,\Lambda}(\bar{X}^k\,\bar{Z}^k)$, then we can introduce stability results. However, first we make the following remark.

Remark.

1. If, $h_i(\bar{x}^k,\,\bar{z}^k)$, $i = 1, \dots s$ are linear functions on $\bar{X}^k \times \bar{Z}^k$ and simultaneously $\bar{X}^k$, $\bar{Z}^k$ are defined by a system of linear inequalities, then it follows from Theorem 1 that $\mathcal{K}_{\mathcal{F}}^{k+1,\,\Lambda}(\bar{X}^k,\,\bar{Z}^k)$ is a set of efficient points of the problem (5.1).

2. If, $h_i(\bar{x}^k,\,\bar{z}^k)$, $i = 1, \dots s$ are convex functions on convex sets, then it follows from Proposition 1 that $\mathcal{K}_{\mathcal{F}}^{k+1,\,\Lambda}(\bar{X}^k,\,\bar{Z}^k)$ is a set of properly efficient points of the problem (5.1).

5.1. Stability results

Theorem 2. Let the assumptions A.1, A.2 be fulfilled, $k \in \{0, \dots M-1\}$, $\bar{X}^k$, $\bar{Z}^k$ be nonempty convex, closed sets. Let, moreover, F^ε, G^ε be two s-dimensional distribution functions determining the systems $\mathcal{F}$ and $\mathcal{G}$, $\alpha_i \in (0,\,1)$, $i = 1, \dots s$, $\bar{\alpha} = (\alpha_1, \dots \alpha_s)$. If

1. there exist finite $\mathrm{E}_{F^\varepsilon}\varepsilon$, $\mathrm{E}_{G^\varepsilon}\varepsilon$,

2. a. H_i, $i = 1, \dots s$ are convex, continuous Lipschitz functions on Z^k,
   b. $f_{i,\,j}^{k+1}$, $i = 1, \dots s$, $j = 1, \dots,$ k are concave, continuous Lipschitz functions on $\bar{X}^k$,

c. $h_i^{k+1}(\bar{x}^k,\, \bar{z}^k) = k_{F_i^\varepsilon}(\alpha_i) + H_i(z^k) - \sum_{j=1}^{k} f_{i,\,j}^{k+1}(x^j),\ i = 1, \dots s,$

3. at least one of the following assumptions holds

a. $\mathcal{K}_{\mathcal{F},\,h}^{k+1}(u)$ is a nonempty set for every $u \in \bar{\mathcal{H}}_{\mathcal{F}}^{k+1,\,\Lambda}(\bar{X}^k,\, \bar{Z}^k)$;
$\bar{\mathcal{H}}_{\mathcal{F}}^{k+1,\,\Lambda}(\bar{X}^k,\, \bar{Z}^k)$ is a compact set,

b. X^k, Z^k, $k = 1, \dots, M$ are nonempty, convex, compact sets, and, moreover, $\mathcal{K}_{\mathcal{F},\,h}^{k+1}(\bar{x}^k,\, \bar{z}^k)$ is a nonempty set for every $(\bar{x}^k, \bar{z}^k) \in \bar{\mathcal{K}}_{\mathcal{F}}^{k+1,\,\Lambda}(\bar{X}^k, \bar{Z}^k)$,

4. $g_0^M(\bar{x}^M,\, \bar{z}^M)$ is a Lipschitz function on $\bar{X}^M \times \bar{Z}^M$,

5. there exists a constant $C_k > 0$ such that for every $\bar{x}^k(i) \in \bar{X}^k$, $\bar{z}^k(i) \in \bar{Z}^k$, $i = 1,\, 2,\ h^{k+1} = (h_1^{k+1}, \dots h_s^{k+1})$ it holds that

$$\begin{aligned} &\Delta \left[\mathcal{K}_{\mathcal{F}}^{k+1}(\bar{x}^k(1),\, \bar{z}^k(1),\, k_{F^\varepsilon}(\bar{\alpha})),\, \mathcal{K}_{\mathcal{F}}^{k+1}(\bar{x}^k(2),\, \bar{z}^k(2),\, k_{G^\varepsilon}(\bar{\alpha}))\right] \\ \leq\ & C_k \left\| h^{k+1}(\bar{x}^k(1),\, \bar{z}^k(1),\, k_{F^\varepsilon}(\bar{\alpha})) - h^{k+1}(\bar{x}^k(2),\, \bar{z}^k(2),\, k_{G^\varepsilon}(\bar{\alpha})) \right\|_s^2, \end{aligned}$$

$(k_{F^\varepsilon}(\bar{\alpha}) = (k_{F_1^\varepsilon}(\alpha_1), \dots k_{F_s^\varepsilon}(\alpha_s)),\ k_{G^\varepsilon}(\bar{\alpha}) = (k_{G_1^\varepsilon}(\alpha_1), \dots k_{G_s^\varepsilon}(\alpha_s))),$

then there exists constants $C_{W_1}^i$, $C_K^i > 0$, $i = 1, \dots s$ such that

$$\begin{aligned} &|\varphi_{\mathcal{F}}(M) - \varphi_{\mathcal{G}}(M)| \\ \leq\ & \sum_{i=1}^{s} C_{W_1}^i \int_{\mathbb{R}^1} |F_i^\varepsilon(z_i) - G_i^\varepsilon(z_i)|\, \mathrm{d}z_i + \sum_{i=1}^{s} C_K^i |k_{F_i^\varepsilon}(\alpha_i) - k_{G_i^\varepsilon}(\alpha_i)|. \end{aligned} \tag{5.5}$$

The proof of Theorem 2 is given in the Appendix.

Remark. It follows from the results of [9] or [11] that the assumption 5 of Theorem 2 is fulfilled (of course under some additional assumptions) if e. g.

- H_i, $i = 1, \dots s$ are Lipschitz functions on Z^k, $k = 1, \dots M$,
- $f_{i,\,j}^{k+1}$, $i = 1, \dots s$, $j = 1, \dots k$ are Lipschitz functions on $\bar{X}^j$,

and, moreover, one of the following situation happen: $f_{i,\,k+1}^{k+1}$, $i = 1, \dots s$ are linear, convex or differentiable functions with the gradients fulfilling some special properties (for more details see e. g. [9, 11]).

Theorem 2 introduces stability results under the assumptions that $f_{i,\,j}^{k+1}$, $i = 1, \dots s$, $j = 1, \dots k$ are concave functions and H_i, $i = 1, \dots s$ convex functions. Moreover, to this assertion the assumption 5 has to be verified. The situation is more simple in the special case when the problem (5.1) is a multiobjective linear programming problem. In this special case, the assumption 3 can be replaced by one that can be verified by modified simplex algorithm (for more details see e. g. [6]). To introduce the corresponding theorem we assume:

i.3 H_i, $i = 1, \dots s$, $k = 1, \dots, M$ are linear functions on $\bar{Z}^k$,

i.4 $f_{i,\,j}^{k+1}$, $i = 1, \dots s$, $j = 1, \dots k$, $k = 1, \dots M - 1$ are linear functions on X^j.

Theorem 3. Let the assumptions A.1, A.2 be fulfilled, $k \in \{0, \dots M-1\}$, $\bar{X}^k$, $\bar{Z}^k$ be nonempty convex, closed polyhedral sets. Let, moreover, F^ε, G^ε be two s-dimensional distribution functions determining the systems $\mathcal{F}$ and $\mathcal{G}$, $\alpha_i \in (0, 1)$, $i = 1, \dots s$, $\bar{\alpha} = (\alpha_1, \dots \alpha_s)$. If

1. the assumptions 1, 4 and 5 of Theorem 2 are fulfilled,
2. the assumptions i.3 and i.4 are fulfilled,
3. $h_i^{k+1}(\bar{x}^k, \bar{z}^k) = k_{F_i^\varepsilon}(\alpha_i) + H_i(z^k) - \sum\limits_{j=1}^{k} f_{i,j}^{k+1}(x^j), \ i = 1, \dots s,$
4. $\mathcal{K}_{\mathcal{F}}^{k+1}(\bar{x}^k, \bar{z}^k)$ is a nonempty set for every $(\bar{x}^k, \bar{z}^k) \in \mathcal{K}_{\mathcal{F}}^{k+1, \Lambda}(\bar{X}^k, \bar{Z}^k)$, $\mathcal{K}_{\mathcal{F}}^{k+1, \Lambda}(\bar{X}^k, \bar{Z}^k)$ is a compact set,

then there exists constants $\bar{C}_{W_1}^i$, $\bar{C}_K^i > 0$, $i = 1, \dots s$ such that

$$\begin{aligned} & |\varphi_{\mathcal{F}}(M) - \varphi_{\mathcal{G}}(M)| \\ \leq \ & \sum_{i=1}^{s} \bar{C}_{W_1}^i \int_{\mathbb{R}^1} |F_i^\varepsilon(z_i) - G_i^\varepsilon(z_i)| \, \mathrm{d}z_i + \sum_{i=1}^{s} \bar{C}_K^i |k_{F_i^\varepsilon}(\alpha_i) - k_{G_i^\varepsilon}(\alpha_i)|. \end{aligned} \tag{5.6}$$

The proof of Theorem 3 is given in the Appendix.

Remark. If we compare the assumptions of Theorem 2 and Theorem 3 we can see that the assumptions 2, 3 (Theorem 2) are replaced by more simple assumptions 2, 3 and 4 (Theorem 3). This assumptions can be verified by a more simple way; this possibility is guaranteed by Theorem 1.

5.2. Empirical estimates results

To study the empirical estimates $\varphi_{\mathcal{F}_N}(M)$, $N = 1, 2, \dots$ of the optimal value of $\varphi_{\mathcal{F}}(M)$, let $\{\varepsilon^i\}_{i=1}^{\infty}$ be a sequence of independent s-dimensional random vectors with common distribution function F^ε. We denote by the symbol F_N^ε empirical distribution function determined by $\{\varepsilon^i\}_{i=1}^{N}$ and the corresponding marginal empirical distribution functions by the symbols $F_N^{\varepsilon_i}$, $i = 1, \dots s$, $N = 1, \dots$. Employing the assertions of the last subsection we can obtain.

Theorem 4. Let the assumptions A.1, A.2 be fulfilled, $k \in \{0, \dots M-1\}$, $\bar{X}^k$, $\bar{Z}^k$ be nonempty convex, closed sets. Let, moreover, F^ε be a distribution function determining the system $\mathcal{F}$, $\alpha_i \in (0, 1)$, $i = 1, \dots s$, $\bar{\alpha} = (\alpha_1, \dots \alpha_s)$. If

1. the assumptions 2, 3, 4 and 5 (for $k_{G^\varepsilon}(\bar{\alpha}) \in U(k_{F^\varepsilon}(\bar{\alpha}))$; $U(k_{F^\varepsilon}(\bar{\alpha}))$ a surroundings of $k_{F^\varepsilon}(\bar{\alpha})$ of Theorem 2) are fulfilled,
2. $\{\varepsilon^i\}_{i=1}^{\infty}$ is a sequence of independent s-dimensional random vectors with common distribution function F^ε,

3. $P_{F_i^\varepsilon}$, $i = 1, \dots s$ are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^1$ (we denote by f_i^ε, $i = 1, \dots s$ the corresponding probability densities),

4. there exist constants C_1, $C_2 > 0$ and $T > 0$ such that

$$f_i^\varepsilon(z_i) \le C_1 \exp\{-C_2|z_i|\} \quad \text{for} \quad z_i \in (-\infty, -T) \cup (T, \infty), \quad i = 1, \dots s,$$

5. the assumption A.4 is fulfilled,

then

$$P\{N^\beta|\varphi_{\mathcal{F}}(M) - \varphi_{\mathcal{F}_N}(M)| > t\} \underset{(N \to \infty)}{\longrightarrow} 0 \quad \text{for } \beta \in (0, \tfrac{1}{2}).$$

Proof. Since the existence of finite $\mathrm{E}_{F^\varepsilon}\varepsilon$ follows from the assumption 4, we can see that the assertion of Theorem 4 follows from the assertions of Theorem 2, Proposition 3 and the relation (4.6). □

Theorem 5. Let the assumptions A.1, A.2 be fulfilled, $k \in \{0, \dots M-1\}$, $\bar{X}^k$, $\bar{Z}^k$ be nonempty convex, compact polyhedral sets. Let, moreover, F^ε is a distribution function determining the system $\mathcal{F}$, $\alpha_i \in (0, 1)$, $i = 1, \dots s$, $\bar{\alpha} = (\alpha_1, \dots \alpha_s)$. If

1. the assumptions 1, 2, 3 and 4 (for $k_{G^\varepsilon}(\bar{\alpha}) \in U(k_{F^\varepsilon}(\bar{\alpha}))$, ;$U(k_{F^\varepsilon}(\bar{\alpha}))$, a surroundings of $k_{F^\varepsilon}(\bar{\alpha})$ of Theorem 3 are fulfilled,

2. $\{\varepsilon^i\}_{i=1}^{\infty}$ is a sequence of independent s-dimensional random vectors with common distribution function F^ε,

3. $P_{F_i^\varepsilon}$, $i = 1, \dots s$ are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^1$. (We denote by f_i^ε, $i = 1, \dots s$ the corresponding probability densities),

4. there exist constants C_1, $C_2 > 0$ and $T > 0$ such that

$$f_i^\varepsilon(z_i) \le C_1 \exp\{-C_2|z_i|\} \quad \text{for} \quad z_i \in (-\infty, -T) \cup (T, \infty), \quad i = 1, \dots s,$$

5. the assumption A.4 is fulfilled,

then

$$P\{N^\beta|\varphi_{\mathcal{F}}(M) - \varphi_{\mathcal{F}_N}(M)| > t\} \underset{(N \to \infty)}{\longrightarrow} 0 \quad \text{for } \beta \in (0, \tfrac{1}{2}).$$

Proof. Since the existence of finite $\mathrm{E}_{F^\varepsilon}\varepsilon$ follows from the assumption 4, we can see that the assertion of Theorem 5 follows from assertions of Proposition 3, Theorem 1, Theorem 3 and the relation (4.6). □

APPENDIX

To prove the assertions of the last section we have to deal with individual objective functions. However, first, we prove some assertions dealing with individual constraint sets. To this end we generalize the results of [19].

Part 1. Constraint sets

Let us for $k \in \{0, 1, \dots M-1\}$ consider the problem (5.1). If we denote by the symbol $\mathcal{K}^{k+1}_{\mathcal{F},E}(\bar{X}^k, \bar{Z}^k)$, $\mathcal{K}^{k+1}_{\mathcal{F},PE}(\bar{X}^k, \bar{Z}^k) \subset \mathbb{R}^{kn} \times \mathbb{R}^{ks}$ the sets of efficient and properly efficient points $(\bar{x}^k, \bar{z}^k)$ of the problem (5.1), then it follows from the relation (3.3) that for compact sets $\bar{X}^k$, $\bar{Z}^k$

$$\begin{aligned} &\mathcal{K}^{k+1}_{\mathcal{F}}(\bar{x}^k, \bar{z}^k) \quad \text{is a nonempty set for} \quad (\bar{x}^k, \bar{z}^k) \in \mathcal{K}^{k+1}_{\mathcal{F},E}(\bar{X}^k, \bar{Z}^k) \\ \Longrightarrow\ &\mathcal{K}^{k+1}_{\mathcal{F}}(\bar{x}^k, \bar{z}^k) \quad \text{is a nonempty set for} \quad (\bar{x}^k, \bar{z}^k) \in \bar{X}^k \times \bar{Z}^k. \end{aligned} \tag{S.1}$$

Furthermore, if $h^{k+1}_i(\bar{x}^k, \bar{z}^k)$ are convex functions on convex sets $\bar{X}^k$, $\bar{Z}^k$, then it follows from Proposition 1 for $G^{k+1,\lambda}(\bar{x}^k, \bar{z}^k)$, $\lambda \in \Lambda$ introduced by (5.2) that

$$\begin{aligned} &(\bar{x}^k, \bar{z}^k) \quad \text{is a solution of the problem} \quad \min\{G^{k+1,\lambda}(\bar{x}^k, \bar{z}^k): \\ &\bar{x}^k \in \bar{X}^k, \bar{z}^k \in \bar{Z}^k\} \quad \text{for some} \quad \lambda \in \Lambda,\ \lambda_i > 0,\ i = 1, \dots s \\ \Longleftrightarrow\ &(\bar{x}^k, \bar{z}^k) \in \mathcal{K}^{k+1}_{\mathcal{F},PE}(\bar{X}^k, \bar{Z}^k). \end{aligned} \tag{S.2}$$

To obtain a relationship between (S.1) and (S.2) we employ the relation (4.4). To this end let $\mathcal{K}^{k+1,\Lambda}_{\mathcal{F}}(\bar{X}^k, \bar{Z}^k)$ be defined by the relation (5.3) and let $\mathcal{H}^{k+1,\Lambda}_{\mathcal{F}}(\bar{X}^k, \bar{Z}^k)$ be defined by the relation (5.4). If the assumptions of Theorem 2 are fulfilled, then according to Proposition 1 also

$$\begin{aligned} \mathcal{H}^{k+1,\Lambda}_{\mathcal{F}}(\bar{X}^k\, \bar{Z}^k) = \ &\Big\{u \in \mathbb{R}^s : u = (u_1, \dots u_s),\ u_i = h^{k+1}_i(\bar{x}^k, \bar{z}^k), \\ &i = 1, \dots s \quad \text{for some} \quad (\bar{x}^k, \bar{z}^k) \in \mathcal{K}^{k+1}_{\mathcal{F},PE}(\bar{X}^k, \bar{Z}^k)\Big\}. \end{aligned} \tag{S.3}$$

We denote by the symbols $\bar{\mathcal{K}}^{k+1,\Lambda}_{\mathcal{F}}(\bar{X}^k, \bar{Z}^k)$, $\bar{\mathcal{H}}^{k+1,\Lambda}_{\mathcal{F}}(\bar{X}^k, \bar{Z}^k)$ and $\bar{\mathcal{K}}^{k+1}_{\mathcal{F},PE}(\bar{X}^k, \bar{Z}^k)$ closures of $\mathcal{K}^{k+1,\Lambda}_{\mathcal{F}}(\bar{X}^k, \bar{Z}^k)$, $\mathcal{H}^{k+1,\Lambda}_{\mathcal{F}}(\bar{X}^k, \bar{Z}^k)$ and $\mathcal{K}^{k+1}_{\mathcal{F},PE}(\bar{X}^k, \bar{Z}^k)$. Evidently, $\bar{\mathcal{K}}^{k+1,\Lambda}_{\mathcal{F}}(\bar{X}^k, \bar{Z}^k) = \bar{\mathcal{K}}^{k+1}_{\mathcal{F},PE}(\bar{X}^k, \bar{Z}^k)$ and, moreover, if $\bar{X}^k$, $\bar{Z}^k$ are compact, convex sets and h^{k+1}_i, $i = 1, \dots s$ continuous, convex functions on $\bar{X}^k \times \bar{Z}^k$, then

$$\begin{aligned} &\bar{\mathcal{H}}^{k+1,\Lambda}_{\mathcal{F}}(\bar{X}^k, \bar{Z}^k) = \{u \in \mathbb{R}^s : u_i = h^{k+1}_i(\bar{x}^k, \bar{z}^k),\ i = 1, \dots s \\ &\text{for some} \quad \bar{x}^k \in \bar{X}^k,\ \bar{z}^k \in \bar{Z}^k; \quad (\bar{x}^k, \bar{z}^k) \in \bar{\mathcal{K}}^{k+1,\Lambda}_{\mathcal{F}}(\bar{X}^k, \bar{Z}^k)\}. \end{aligned}$$

Consequently, if, $\bar{X}^k$, $\bar{Z}^k$ are compact convex sets, h^{k+1}_i, $i = 1, \dots s$ convex, continuous functions on $\bar{X}^k \times \bar{Z}^k$, then

$$\begin{aligned} &\mathcal{K}^{k+1}_{\mathcal{F}}(\bar{x}^k, \bar{z}^k) \text{ is a nonempty set for every } (\bar{x}^k, \bar{z}^k) \in \bar{\mathcal{K}}^{k+1,\Lambda}_{\mathcal{F}}(\bar{X}^k, \bar{Z}^k) \\ \Longrightarrow\ &\mathcal{K}^{k+1}_{\mathcal{F}}(\bar{x}^k, \bar{z}^k) \text{ is a nonempty set for every } \bar{x}^k \in \bar{X}^k,\ \bar{z}^k \in \bar{Z}^k. \end{aligned} \tag{S.4}$$

We have proven the following auxiliary assertion.

Proposition S.1. Let $\alpha_i \in (0,\, 1)$, $\bar{\alpha} = (\alpha_1, \ldots \alpha_s)$. If for $k \in \{1, \ldots M-1\}$,

1. X^j, $j = 1, \ldots k$, Z^k are nonempty convex, compact sets,
2. a. H_i, $i = 1, \ldots s$ are convex functions on Z^k,
   b. $f_{i,\,j}^{k+1}$, $i = 1, \ldots s$, $j = 1, \ldots,\ k$ are concave functions on $\bar{X}^k$,
   c. $h_i^{k+1}(\bar{x}^k,\, \bar{z}^k) = k_{F_i^{\varepsilon}}(\alpha_i) + H_i(z^k) - \sum\limits_{j=1}^{k} f_{i,\,j}^{k+1}(x^j),\ i = 1, \ldots s,$
3. $\bar{\mathcal{K}}_{\mathcal{F}}^{k+1,\,\Lambda}(\bar{X}^k,\, \bar{Z}^k)$ is a closure of $\mathcal{K}_{\mathcal{F}}^{k+1,\,\Lambda}(\bar{X}^k,\, \bar{Z}^k)$ defined by the relation (5.3),

then

$$\begin{aligned} & \mathcal{K}_{\mathcal{F}}^{k+1}(\bar{x}^k,\, \bar{z}^k) \quad \text{is a nonempty set for every} \quad (\bar{x}^k,\, \bar{z}^k) \in \bar{\mathcal{K}}_{\mathcal{F}}^{k+1,\,\Lambda}(\bar{X}^k,\, \bar{Z}^k) \\ \Longrightarrow\ & \mathcal{K}_{\mathcal{F}}^{k+1}(\bar{x}^k\, \bar{z}^k) \quad \text{is a nonempty set for every} \quad \bar{x}^k \in \bar{X}^k,\, \bar{z}^k \in \bar{Z}^k. \end{aligned}$$

Evidently, the compact property of the set $\bar{X}^k$, $\bar{Z}^k$ can be replaced by the compact property of the set $\bar{\mathcal{H}}_{\mathcal{F}}^{k+1,\,\Lambda}(\bar{X}^k,\, \bar{Z}^k)$ (defined by the relation (5.4)).

Proposition S.1 introduces sufficient assumptions under which the constraint sets corresponding to the inner problems in (2.2) are nonempty. It can be rather complicated to verify exactly the assumptions. The exception is only the linear case.

Proposition S.2. Let $\alpha_i \in (0,\, 1)$, $\bar{\alpha} = (\alpha_1, \ldots \alpha_s)$. If for $k \in \{1, \ldots M-1\}$,

1. X^j, $j = 1, \ldots k$, Z^k are nonempty convex, compact, polyhedral sets,
2. the assumptions i.3, i.4 are fulfilled,

then

$$\begin{aligned} & \mathcal{K}_{\mathcal{F}}^{k+1}(\bar{x}^k,\, \bar{z}^k) \quad \text{is a nonempty set for every} \quad (\bar{x}^k,\, \bar{z}^k) \in \mathcal{K}_{\mathcal{F}}^{k+1,\,\Lambda}(\bar{X}^k,\, \bar{Z}^k) \\ \Longrightarrow\ & \mathcal{K}_{\mathcal{F}}^{k+1}(\bar{x}^k,\, \bar{z}^k) \quad \text{is a nonempty set for every} \quad \bar{x}^k \in \bar{X}^k,\, \bar{z}^k \in \bar{Z}^k. \end{aligned}$$

P r o o f. The proof of Proposition S.2 follows immediately from the assertion of Theorem 1. □

Evidently, the problem:

Find

$$\min\{G^{k+1,\,\lambda}(\bar{x}^k,\, \bar{z}^k) : \bar{x}^k \in \bar{X}^k,\, \bar{z}^k \in \bar{Z}^k\} \quad \text{for} \quad \lambda \in \Lambda,\, \lambda_i > 0,\, i = 1, \ldots s$$

is (under the assumptions of Proposition S.2) a problem of linear parametric programming. A modification of the well known simplex algorithm to solve this problem can be found in [6].

Part 2. Objective functions

First, let us generalize and modify the assertion introduced in [14].

Proposition S.3. Let the assumption A.1 be fulfilled, $k \in \{0, \dots M-1\}$. If

1. $P_{F^{\varepsilon}}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^s$. (We denote by f^{ε} the corresponding probability density),
2. a. $g_{\mathcal{F}}^{k+1}(\bar{x}^{k+1},\, \bar{z}^{k+1})$ is a Lipschitz function (with respect to $\mathcal{L}_2$ norm) on $\bar{X}^{k+1} \times \bar{Z}^{k+1}$,
   b. there exists a finite $\mathrm{E}_{F^{\varepsilon}}\varepsilon$,

then

$$\mathrm{E}_{F^{\xi^{k+1}|\bar{\xi}^k = \bar{z}^k}}\, g_{\mathcal{F}}^{k+1}(\bar{x}^{k+1},\, \bar{\xi}^{k+1})$$

is a Lipschitz function (with respect to $\mathcal{L}_2$ norm) on $\bar{X}^{k+1} \times \bar{Z}^k$.

Proof. First, since evidently $F^{\xi^{k+1}|\bar{\xi}^k = \bar{z}^k}(z^{k+1}|\bar{z}^k) = F^{\varepsilon}(z^{k+1} - H(z^k))$ we can see that

$$\begin{aligned}
& \mathrm{E}_{F^{\xi^{k+1}|\bar{\xi}^k = \bar{z}^k}}\, g_{\mathcal{F}}^{k+1}(\bar{x}^{k+1},\, \bar{\xi}^{k+1}) \\
= & \int_{Z_{F^{\xi^{k+1}|\bar{\xi}^k = \bar{z}^k}}} g_{\mathcal{F}}^{k+1}(\bar{x}^{k+1},\, \bar{z}^{k+1})\, \mathrm{d}F^{\xi^{k+1}|\bar{\xi}^k = \bar{z}^k}(z^{k+1}|\bar{z}^k) \\
= & \int_{Z_{F^{\xi^{k+1}|\bar{\xi}^k = \bar{z}^k}}} g_{\mathcal{F}}^{k+1}(\bar{x}^{k+1},\, \bar{z}^{k+1})\, f^{\varepsilon}(z^{k+1} - H(z^k))\, \mathrm{d}z^{k+1} \\
= & \int_{Z_{F^{\varepsilon}}} g_{\mathcal{F}}^{k+1}(\bar{x}^{k+1},\, (\bar{z}^k,\, u + H(z^k)) f^{\varepsilon}(u)\, \mathrm{d}u, \quad \bar{z}^{k+1} = (\bar{z}^k,\, z^{k+1}),
\end{aligned}$$

where $Z_{F^{\xi^{k+1}|\bar{\xi}^k = \bar{z}^k}}$ denotes the probability measure support corresponding to $F^{\xi^{k+1}|\bar{\xi}^k = \bar{z}^k}$.

Now already it follows from the assumption 2 and elementary properties of integral that $g_{\mathcal{F}}^{k+1}(\bar{x}^{k+1},\, (\bar{z}^k,\, u + H(z^k))$ is a Lipschitz function (w.r.t. $\mathcal{L}_2$ norm) on $\bar{X}^{k+1} \times \bar{Z}^k$. □

Furthermore, we recall and modify the assertion of [9] (see also [11]).

Lemma S.1. Let $k \in \{0, \dots M-1\}$, $\bar{X}^{k+1}$, $\bar{Z}^{k+1}$ be nonempty, compact sets. Let, moreover, the assumption A.1 be fulfilled. If

1. the assumptions of Proposition S.3 are fulfilled.
2. $\mathcal{K}_{\mathcal{F}}^{k+1}(\bar{x}^k,\, \bar{z}^k)$ is a multifunction mapping $\bar{X}^k \times \bar{Z}^k$ into the space of nonempty, closed subsets of $\mathbb{R}^n$ such that for every $\bar{x}^k(i) \in \bar{X}^k$, $\bar{z}^k(i) \in \bar{Z}^k$, $h^{k+1} = (h_1^{k+1},\, \dots,\, h_s^{k+1})$ it holds for a constant $C_k > 0$ that

$$\begin{aligned}
& \Delta[\mathcal{K}_{\mathcal{F}}^{k+1}(\bar{x}^k(1),\, \bar{z}^k(1)\, k_{F^{\varepsilon}}(\bar{\alpha})),\, \mathcal{K}_{\mathcal{F}}^{k+1}(\bar{x}^k(2),\, \bar{z}^k(2),\, k_{G^{\varepsilon}}(\bar{\alpha}))] \\
\leq \; & C_k\, \| h^{k+1}(\bar{x}^k(1),\, \bar{z}^k(1),\, k_{F^{\varepsilon}}(\bar{\alpha})) - h^{k+1}(\bar{x}^k(2),\, \bar{z}^k(2),\, k_{F^{\varepsilon}}(\bar{\alpha}))\|_s^2,
\end{aligned}$$

3. for every $\bar{x}^k \in \bar{X}^k$, $\bar{z}^k \in \bar{Z}^k$,

$$\inf \left\{ \mathrm{E}_{F^{\xi^{k+1}|\bar{\xi}^k=\bar{z}^k}} \ g_{\mathcal{F}}^{k+1}(\bar{x}^{k+1}, \, \bar{\xi}^{k+1}) \, | x^{k+1} \in \mathcal{K}_{\mathcal{F}}^{k+1}(\bar{x}^k, \, \bar{z}^k) \right\} > -\infty,$$

then $g_{\mathcal{F}}^k(\bar{x}^k, \, \bar{z}^k)$ is a Lipschitz function (with respect to $\mathcal{L}_2$ norm) on $\bar{X}^k \times \bar{Z}^k$. ($\Delta[\cdot, \, \cdot]$ the Hausdorff distance, for the definition see e.g. [28].)

P r o o f. Since the assumptions of Proposition S.3 are fulfilled we can see that $\mathrm{E}_{F^{\xi^{k+1}|\bar{\xi}^k=\bar{z}^k}} \ g_{\mathcal{F}}^{k+1}(\bar{x}^{k+1}, \, \bar{\xi}^{k+1})$ is a Lipschitz function on $\bar{X}^{k+1} \times \bar{Z}^k$. The assertion of Lemma 3 follows then from the assertions of [11] (Lemma 1). □

Furthermore, it follows from the assumption 2 of Lemma 3 and the relation (3.5) that under the assumptions A.1, A.2 for $\bar{x}^k \in \bar{X}^k$, $\bar{z}^k \in \bar{Z}^k$ there exists a constant C_1 such that

$$\Delta \left[\mathcal{K}_{\mathcal{F}}^{k+1}(\bar{x}^k, \bar{z}^k, k_{F^\varepsilon}(\bar{\alpha})), \mathcal{K}_{\mathcal{G}}^{k+1}(\bar{x}^k, \bar{z}^k, k_{G^\varepsilon}(\bar{\alpha})) \right] \le C_1 \sum_{i=1}^{s} \left| k_{F_i^\varepsilon}(\alpha_i) - k_{G_i^\varepsilon}(\alpha_i) \right|. \quad \text{(S.5)}$$

P r o o f o f T h e o r e m 2. First, it follows from Lemma 3 that (under the assumptions) for $k = 0, \ldots M-1$, $g_{\mathcal{F}}^{k+1}(\bar{x}^{k+1}, \, \bar{z}^{k+1})$ are Lipschitz functions on $\bar{X}^{k+1} \times \bar{Z}^{k+1}$. Furthermore, it follows from the assertion of Proposition S.3 that $\mathrm{E}_{F^{\xi^{k+1}|\bar{\xi}^k=\bar{z}^k}} \ g_{\mathcal{F}}^{k+1}(\bar{x}^{k+1}, \, \bar{\xi}^{k+1})$ is a Lipschitz function (with respect to $\mathcal{L}_2$ norm) on $\bar{X}^{k+1} \times \bar{Z}^k$. However, then according to the assertions of Proposition 2 there exists a constant L_k such that

$$\begin{aligned} & |\mathrm{E}_{F^{\xi^{k+1}|\bar{\xi}^k=\bar{z}^k}} \, g_{\mathcal{F}}^{k+1}(\bar{x}^{k+1}, \, \bar{z}^{k+1}) - \mathrm{E}_{G^{\xi^{k+1}|\bar{\xi}^k=\bar{z}^k}} \, g_{\mathcal{F}}^{k+1}(\bar{x}^{k+1}, \, \bar{z}^{k+1})| \\ \le \; & L_k \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i^{\xi^{k+1}|\bar{\xi}^k=\bar{z}^k}(z_i^{k+1}|\bar{z}^k) \; - \; G_i^{\xi^{k+1}|\bar{\xi}^k=\bar{z}^k}(z_i^{k+1}|\bar{z}^k)| \, \mathrm{d}z_i^{k+1} \\ & \text{for} \quad \bar{x}^{k+1} \in \bar{X}^{k+1}, \, \bar{z}^k \in \bar{Z}^k. \end{aligned}$$

Employing the assertions of Lemma 2 and the relation (S.5) we can see that there exist constants $C_{W_1}^{i,\,k}$, $C_K^{i,\,k}$, $i = 1, \ldots s$ such that for $\bar{x}^k \in \bar{X}^k$, $\bar{z}^k \in \bar{Z}^k$

$$\begin{aligned} & |g_{\mathcal{F}}^k(\bar{x}^k, \, \bar{z}^k) - g_{\mathcal{G}}^k(\bar{x}^k, \, \bar{z}^k)| \\ \le \; & \sum_{i=1}^{s} C_{W_1}^{i,\,k} \int_{\mathbb{R}^1} |F_i^\varepsilon(z_i) - G_i^\varepsilon(z_i)| \, \mathrm{d}z_i + \sum_{i=1}^{s} C_K^{i,\,k} |k_{F_i^\varepsilon}(\alpha_i) - k_{G_i^\varepsilon}(\alpha_i)|. \end{aligned} \qquad \text{(S.6)}$$

and, furthermore, employing the inequality that has been proven (for $x^0 \in X^0$) in [12] (see also [15])

$$
\begin{aligned}
& |\mathrm{E}_{F^{\xi^0}} g^0_F(x^0, \xi^0) - \mathrm{E}_{G^{\xi^0}} g^0_G(x^0, \xi^0)| \\
\le\ & |\mathrm{E}_{F^{\xi^0}} \inf_{x^1 \in \mathcal{K}^1_{\mathcal{F}}(x^0, \xi^0)} \mathrm{E}_{F^{\xi^1|\xi^0}} g^1_F(\bar{x}^1, \bar{\xi}^1) \\
-\ & \mathrm{E}_{G^{\xi^0}} \inf_{x^1 \in \mathcal{K}^1_{\mathcal{F}}(x^0, \xi^0)} \mathrm{E}_{F^{\xi^1|\xi^0}} g^1_F(\bar{x}^1, \bar{\xi}^1)| \\
+\ & |\mathrm{E}_{G^{\xi^0}} \inf_{x^1 \in \mathcal{K}^1_{\mathcal{F}}(x^0, \xi^0)} \mathrm{E}_{F^{\xi^1|\xi^0}} \inf_{x^2 \in \mathcal{K}^2_{\mathcal{F}}(\bar{x}^1, \bar{\xi}^1)} \mathrm{E}_{F^{\xi^2|\bar{\xi}^1}} g^2_F(\bar{x}^2, \bar{\xi}^2) \\
-\ & \mathrm{E}_{G^{\xi^0}} \inf_{x^1 \in \mathcal{K}^1_{\mathcal{G}}(x^0, \xi^0)} \mathrm{E}_{G^{\xi^1|\xi^0}} \inf_{x^2 \in \mathcal{K}^2_{\mathcal{F}}(\bar{x}^1, \bar{\xi}^1)} \mathrm{E}_{F^{\xi^2|\bar{\xi}^1}} g^3_F(\bar{x}^2, \bar{\xi}^2)| \\
\vdots\ & \\
& |\mathrm{E}_{G^{\xi^0}} \inf_{x^1 \in \mathcal{K}^1_{\mathcal{G}}(x^0, \xi^0)} \ldots\ldots \inf_{x^{M-1} \in \mathcal{K}^{M-1}_{\mathcal{G}}(\bar{x}^{M-2}, \bar{\xi}^{M-2})} \mathrm{E}_{G^{\xi^{M-1}|\bar{\xi}^{M-2}}} \\
& \inf_{x^M \in \mathcal{K}^M_{\mathcal{F}}(\bar{x}^{M-1}, \bar{\xi}^{M-1})} \mathrm{E}_{F^{\xi^M|\bar{\xi}^{M-1}}} g^M_F(\bar{x}^M, \bar{\xi}^M) - \\
& \mathrm{E}_{G^{\xi^0}} \inf_{x^1 \in \mathcal{K}^1_{\mathcal{G}}(x^0, \xi^0)} \ldots\ldots \inf_{x^{M-1}_{\mathcal{G}} \in \mathcal{K}^{M-1}_{\mathcal{G}}(\bar{x}^{M-2}, \bar{\xi}^{M-2})} \mathrm{E}_{G^{\xi^{M-1}|\bar{\xi}^{M-2}}} \\
& \inf_{x^M \in \mathcal{K}^M_{\mathcal{G}}(\bar{x}^{M-1}, \bar{\xi}^{M-1})} \mathrm{E}_{G^{\xi^M|\bar{\xi}^{M-1}}} g^M_G(\bar{x}^M, \bar{\xi}^M)|.
\end{aligned}
\tag{S.7}
$$

Evidently, if $\bar{X}^k$, $\bar{Z}^k$, $k = 1, \ldots, M-1$ are not compact sets, then we have replace the set $\mathcal{K}^{k+1,\Lambda}_{\mathcal{F}}(\bar{X}^k, \bar{Z}^k$ by the set $\mathcal{H}^{k+1,\Lambda}_{\mathcal{F},h}(\bar{X}^k, \bar{Z}^k$ and assume that the set $\mathcal{H}^{k+1,\Lambda}_{\mathcal{F},h}(\bar{X}^k, \bar{Z}^k$ is a compact one. We can see that the assertion of Theorem 2 is valid. □

P r o o f o f T h e o r e m 3. Employing the assertion of Proposition S.2 instead of Proposition S.1 in the complete proof of Theorem 2 we can obtain the assertion of Theorem 3. □

ACKNOWLEDGEMENT

This work was partially supported by the Czech Science Foundation under Grants 402/08/0107, 402/07/1113 and 402/06/1417.

(Received October 30, 2007.)

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