ON ENTROPIES FOR RANDOM PARTITIONS OF THE UNIT SEGMENT

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We prove the complete convergence of Shannon's, paired, genetic and α-entropy for random partitions of the unit segment. We also derive exact expressions for expectations and variances of the above entropies using special functions.

Keywords: paired entropy, genetic entropy, α-entropy, random partitions, complete convergence

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1. INTRODUCTION

Entropy is a measure of uncertainty or information. Shannon's entropy

$$H(p_0, \dots, p_k) = -\sum_{i=0}^{k} p_i \log p_i$$

(cf. [21]) for probabilities $p_0, \dots, p_k$, $\sum_{i=0}^{k} p_i = 1$, is the most common used measure of randomness. There are known many generalizations of this entropy (cf. [18]). Burbea and Rao [3] introduced ϕ-entropy defined as

$$D_k^{\phi} = \sum_{i=0}^{k} \phi(p_i)$$

where $(p_0, \dots, p_k)$ is a probability distribution and ϕ is a twice differentiable real function on $(0, 1)$. Special cases of ϕ-entropy are

- Shannon's entropy if $\phi(x) = -x \log x$,
- paired entropy if $\phi(x) = -x \log x - (1 - x) \log(1 - x)$,
- genetic entropy if $\phi(x) = x - x^2 - x^2(1 - x)^2$ (cf. [17]).

Menendez et al. [18] generalized ϕ-entropy and defined a family of (h, ϕ)-entropies,

$$D_k^{(h,\phi)} = h\left(\sum_{i=0}^{k} \phi(p_i)\right),$$

where $h : \mathbb{R} \to \mathbb{R}$ and $\phi : (0,1) \to \mathbb{R}$ are twice differentiable real functions. Examples of (h, ϕ)-entropies are as follows

- α degree entropy if $\phi(x) = x^\alpha$ and $h(x) = \frac{x-1}{2^{1-\alpha}-1}$, $\alpha > 0$, $\alpha \neq 1$ (cf. [14]),
- α order entropy if $\phi(x) = x^\alpha$ and $h(x) = \frac{\log x}{1-\alpha}$, $\alpha > 0$, $\alpha \neq 1$ (cf. [20]).

The Shannon entropy of spacings is the quantity

$$D_k^S = -\sum_{j=0}^{k} Y_{j,k} \log Y_{j,k},$$

where

$$Y_{j,k} = X_{j+1:k} - X_{j:k}, \quad 0 \le j \le k,$$

and $0 \le X_{1:k} \le \ldots \le X_{k:k} \le 1$, $X_{0:k} = 0$ and $X_{k+1:k} = 1$, are order statistics of a sample $(X_1, \ldots, X_n)$ from a distribution F. The asymptotic behaviour of Shannon's entropy D_k^S, when F is the uniform distribution, was studied in [7], [22] and [23]. Goldstein [7] proved that the sequence $(D_k^S - \log(k+1))$ converges to $\gamma - 1$ in probability as $k \to \infty$, where $\gamma = 0.577215\ldots$ is the Euler–Masheroni constant. Slud [23] showed that the convergence holds almost surely. Shao and Jimenez [22] proved that if the spacings come from a continuous distribution F then the almost sure convergence of D_k^S to $\gamma - 1$ characterizes uniform distribution among continuous distributions. Some related problems were investigated in Ekstörm [10], Hall [11], [12] and Misra [19]. In that papers limit theorems and tests of uniformity for sums of mth spacings were considered. We are interested in the complete convergence of Shannon's entropy of spacings D_k^S.

Recall that a sequence (X_n) converges completely to X $(X_n \xrightarrow{c} X)$ if for all $\varepsilon > 0$

$$\sum_{n=1}^{\infty} \Pr\left(|X_n - X| > \varepsilon\right) < \infty \quad \text{(cf. [16])}.$$

We consider also the complete convergence of paired, genetic and α-entropy of spacings. Namely, we investigate the asymptotic behaviour of D_k^S,

$$\begin{aligned} D_k^P &= -\sum_{j=0}^{k} \left(Y_{j,k} \log Y_{j,k} + (1 - Y_{j,k}) \log (1 - Y_{j,k})\right) \\ D_k^G &= \sum_{j=0}^{k} \left(Y_{j,k}(1 - Y_{j,k}) - Y_{j,k}^2 (1 - Y_{j,k})^2\right) \end{aligned}$$

and

$$D_k^\alpha = \frac{1}{2^{1-\alpha} - 1} \left(\sum_{j=0}^{k} Y_{j,k}^\alpha - 1 \right), \quad \alpha > 0, \quad \alpha \neq 1,$$

where $Y_{j,k}$ are uniform spacings, i. e. F is uniform distribution on $(0,1)$.

The paper is organized as follows. In Section 2 we present definitions and auxiliary results containing some formulae for sums and integrals. The explicit expressions for expectations and variances of the above entropies are given in Section 3. Finally, Section 4 is devoted to the complete convergence of D_k^S, D_k^P, D_k^G and D_k^α.

2. PRELIMINARIES

Denote by $H_k^{(r)}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, the harmonic number of order r, i. e.

$$H_k^{(r)} = \sum_{i=1}^{k} \frac{1}{i^r}, \quad r \geq 1$$

(cf. [9]). For simplicity we write $H_k := H_k^{(1)}$. For $r > 1$ we use Riemann ζ-function and Hurwitz generalized ζ-function defined, respectively, by

$$\zeta(r) = H_\infty^{(r)} = \sum_{i=1}^{\infty} \frac{1}{i^r}, \quad \zeta(r;q) = \zeta(r) - H_{q-1}^{(r)} = \sum_{i=0}^{\infty} \frac{1}{(i+q)^r}, \quad q \geq 1.$$

We also use the relation between the harmonic numbers, Psi (or Digamma) function

$$\psi(x) = \frac{\mathrm{d}}{\mathrm{d}x} \log \Gamma(x)$$

and the derivatives of Psi function (or Polygamma functions).
Here $\Gamma(x) = \int_0^\infty t^{x-1} \mathrm{e}^{-t} \mathrm{d}t$, $x > 0$, is the Gamma function. It is known that

$$H_k := H_k^{(1)} = \gamma + \psi(k+1) \tag{1}$$

(cf. [8]) and for $r \geq 2$

$$H_k^{(r)} = \frac{(-1)^r}{(r-1)!} \left(\psi^{(r-1)}(1) - \psi^{(r-1)}(k+1) \right)$$

as

$$\zeta(r;q) = \frac{(-1)^r}{(r-1)!} \psi^{(r-1)}(q).$$

By $B(x,y)$ we denote Beta function, i. e.

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} \mathrm{d}t, \quad x > 0, \quad y > 0. \tag{2}$$

$(\lambda)_r$ denotes the Pochhammer symbol (or the shifted factorial) defined by

$$(\lambda)_r := \frac{\Gamma(\lambda + r)}{\Gamma(r)} = \begin{cases} 1 & r = 0, \\ \lambda(\lambda+1)\ldots(\lambda+r-1) & r \in \mathbb{N}. \end{cases}$$

Also let $\beta(x)$ be the function defined by

$$\beta(x) := \frac{1}{2}\left[\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right)\right] = \sum_{j=0}^{\infty} \frac{(-1)^j}{j+x}, \quad x > 0,$$

and

$$\beta'(x) = -\sum_{j=0}^{\infty} \frac{(-1)^j}{(j+x)^2} \quad \text{(cf. [8])}.$$

We need the following lemmas.

Lemma 1. The following summation formulae hold true

$$\sum_{j=1}^{k}\binom{k}{j}\frac{(-1)^{j+1}}{j}=\psi(k+1)+\gamma \quad \text{(cf. [8], 0.155.4)}, \tag{3}$$

$$\sum_{j=0}^{k}\binom{k}{j}(-1)^{j}H_{j+r}=-B(k,r+1), \quad r\in\mathbb{N} \quad \text{(cf. [25])}, \tag{4}$$

$$\sum_{j=1}^{k+2}\frac{(-1)^{j-1}}{j^2}=\frac{\pi^2}{12}+(-1)^k\beta'(k+3) \quad \text{(cf. [13], 5.12.50)}, \tag{5}$$

$$\sum_{j=1}^{k+1}\frac{H_j}{j}=\frac{1}{2}\left(\psi(k+2)+\gamma\right)^2+\frac{1}{2}\left(\frac{\pi^2}{6}-\psi'(k+2)\right) \quad \text{(cf. [9], 6.71)}, \tag{6}$$

$$\sum_{j=1}^{\infty}\frac{1}{j(k+1+j)^2}=\frac{\psi(k+2)+\gamma}{(k+1)^2}-\frac{\psi'(k+2)}{k+1}, \quad k\in\mathbb{N} \quad \text{(cf. [13], 6.1.82)}. \tag{7}$$

Lemma 2.

$$\sum_{j=0}^{k}\binom{k+1}{j}\frac{(-1)^j}{(k+1-j)(j+1)}=\frac{(-1)^k\left(\psi(k+3)+\gamma\right)}{k+2}+\frac{1}{(k+2)^2}, \tag{8}$$

$$\sum_{j=0}^{k+1}\binom{k+2}{j}\frac{(-1)^jH_j}{k+2-j}=(-1)^{k+1}\left((\gamma+\psi(k+3))^2-\psi'(k+3)\right)+2\beta'(k+3), \tag{9}$$

$$\begin{aligned}\sum_{j=0}^{k-2}\binom{k-2}{j}\frac{(-1)^j}{(k-j)^2(j+1)(j+2)}&=\frac{(-1)^k\left(\gamma+\psi(k+3)-1\right)}{k(k^2-1)(k+2)}\\&\quad+\frac{1}{(k^2-1)(k+1)}-\frac{1}{k(k-1)(k+2)^2},\end{aligned} \tag{10}$$

$$\begin{aligned}\sum_{j=0}^{k-2}\binom{k-2}{j}\frac{(-1)^jH_{j+2}}{(k-j)^2(j+1)(j+2)}&=-\frac{2(k+2)\beta'(k+3)+1}{k(k^2-1)(k+2)^2}+\frac{1}{(k-1)(k+1)^2}\\&\quad+\frac{(-1)^k\left((\gamma+\psi(k+3))^2-(\gamma+\psi(k+3))-\psi'(k+3)\right)}{k(k^2-1)(k+2)}.\end{aligned} \tag{11}$$

Proof. To prove (8) we use

$$\frac{1}{(k+1-j)(j+1)}=\frac{1}{k+2}\left(\frac{1}{k+1-j}+\frac{1}{j+1}\right)$$

and next by (3) and (1)

$$\sum_{j=0}^{k}\binom{k+1}{j}\frac{(-1)^j}{(k+1-j)(j+1)}=\frac{1}{k+2}\sum_{j=0}^{k}\binom{k+1}{j}\frac{(-1)^j}{k+1-j}$$
$$+\frac{1}{k+2}\sum_{j=0}^{k+1}\binom{k+1}{j}\frac{(-1)^j}{j+1}+\frac{(-1)^k}{(k+2)^2}$$
$$=\frac{(-1)^k}{k+2}\sum_{j=1}^{k+1}\binom{k+1}{j}\frac{(-1)^{j+1}}{j}+\frac{1+(-1)^k}{(k+2)^2}=\frac{(-1)^k\left(\psi(k+3)+\gamma\right)}{k+2}+\frac{1}{(k+2)^2}.$$

Now we prove (9). Let

$$S_k:=\sum_{j=0}^{k}\binom{k+1}{j}\frac{(-1)^jH_j}{k+1-j}.$$

Using

$$\binom{k+2}{j}=\binom{k+1}{j}+\binom{k+1}{j-1},\quad j=0,\dots,k+1,\quad \binom{k}{-1}=0,$$

we get

$$S_{k+1}=\sum_{j=0}^{k+1}\binom{k+1}{j}\frac{(-1)^jH_j}{k+2-j}+\sum_{j=0}^{k+1}\binom{k+1}{j-1}\frac{(-1)^jH_j}{k+2-j}$$
$$=\frac{\sum_{j=0}^{k+2}\binom{k+2}{j}(-1)^jH_j}{k+2}-\frac{(-1)^kH_{k+2}}{k+2}-\sum_{j=0}^{k}\frac{\binom{k+1}{j}(-1)^j}{(j+1)(k+1-j)}-S_k.$$

Then by (4) and (8) we get the recurrence relation

$$S_{k+1}=-\frac{2}{(k+2)^2}-\frac{2(-1)^kH_{k+2}}{k+2}-S_k,\quad k=0,1,\dots$$

where $S_0=0$. Hence

$$S_{k+1}=2(-1)^k\sum_{j=1}^{k+2}\frac{(-1)^{j-1}}{j^2}-2(-1)^k\sum_{j=1}^{k+2}\frac{H_j}{j},$$

which by (6) and (5) gives (9).

Now consider (10). We see that

$$\sum_{j=0}^{k-2}\binom{k-2}{j}\frac{(-1)^j}{(j+1)(j+2)(k-j)^2}=\frac{1}{k(k^2-1)(k+2)}\sum_{j=2}^{k+1}\binom{k+2}{j}(-1)^j\frac{k+1-j}{k+2-j},$$

and changing the order of summation we have

$$\sum_{j=0}^{k-2}\binom{k-2}{j}\frac{(-1)^j}{(k-j)^2(j+1)(j+2)}=\frac{(-1)^k}{k(k^2-1)(k+2)}\sum_{j=1}^{k}\binom{k+2}{j}(-1)^j\frac{j-1}{j}$$

$$=\frac{(-1)^k}{k(k^2-1)(k+2)}\left(\sum_{j=0}^{k+2}\binom{k+2}{j}(-1)^j-1+\sum_{j=1}^{k+2}\binom{k+2}{j}\frac{(-1)^{j+1}}{j}\right)$$

$$+\frac{1}{(k-1)(k+1)^2}-\frac{1}{k(k-1)(k+2)^2}.$$

Finally using (3) we get (10).

Now we prove (11). Applying the same evaluations as above we see that

$$\sum_{j=0}^{k-2}\binom{k-2}{j}\frac{(-1)^j H_{j+2}}{(j+1)(j+2)(k-j)^2}=\frac{1}{k(k^2-1)(k+2)}\sum_{j=0}^{k-2}\binom{k+2}{j+2}(-1)^j H_{j+2}\frac{k-1-j}{k-j}$$

$$=\frac{1}{k(k^2-1)(k+2)}\left(\sum_{j=1}^{k+2}\binom{k+2}{j}(-1)^j H_j-(-1)^k H_{k+2}-\sum_{j=1}^{k+1}\binom{k+2}{j}\frac{(-1)^j H_j}{k+2-j}\right)$$

$$+\frac{1}{(k-1)(k+1)^2},$$

which after using (4) and (9) gives (11). □

Corollary 1.

$$\sum_{j=0}^{k-2}\binom{k-2}{j}\frac{(-1)^j\,(H_{j+2}-1)}{(j+1)(j+2)(k-j)^2}=(-1)^k\frac{\left((\gamma+\psi(k+3)-1)^2-\psi'(k+3)\right)}{k(k^2-1)(k+2)}$$
$$-\frac{2\beta'(k+3)}{k(k^2-1)(k+2)}+\frac{1}{(k^2-1)(k+2)^2}. \tag{12}$$

Lemma 3.

$$\int_0^1 x^{p-1}\left(\log\frac{1}{x}\right)^q \mathrm{d}x=\frac{1}{p^{q+1}}\Gamma(q+1),\quad p>0,\quad q>-1\quad \text{(cf. [8], 3.653.2)}, \tag{13}$$

and for $p>1$, $q>1$

$$\int_0^1 x^{p-1}(1-x)^{q-1}\log\frac{1}{x}\,\mathrm{d}x=B(p,q)(\psi(p+q)-\psi(p))\quad \text{(cf. [8], 3.628.1)}, \tag{14}$$

$$\int_0^1 x^p\log\frac{1}{x}\log\frac{1}{1-x}\,\mathrm{d}x=\frac{\psi(p+2)+\gamma}{(p+1)^2}-\frac{\psi'(p+2)}{p+1}, \tag{15}$$

for $r \in \mathbb{N}$ and $q \in \mathbb{N}$

$$\int_0^1 x^{r-1}\Big(\log \frac{1}{1-x}\Big)^q \mathrm{d}x$$
$$= \frac{q!}{r} \sum_{a_1+2a_2+\ldots+qa_q=q} \prod_{i=1}^{q} \frac{(-1)^{ia_i}(\psi^{(i-1)}(1)-\psi^{(i-1)}(r+1))}{a_i!(i!)^{a_i}} \quad \text{(cf. [2])},$$

where the summation is over all $a_i \in \mathbb{N}$, $i = 1, \ldots, q$, and $c(j,q)$ are the unsigned Stirling numbers of the first kind. In particular

$$\int_0^1 x^{r-1}\left(\log \frac{1}{1-x}\right)^2 \mathrm{d}x = \frac{(\psi(r+1)+\gamma)^2 + \frac{\pi^2}{6} - \psi'(r+1)}{r}. \tag{16}$$

P r o o f. We prove (15). Using the expansion

$$\log \frac{1}{1-x} = \sum_{j=1}^{\infty} \frac{x^j}{j}, \quad |x| < 1,$$

and (13) we get

$$\int_0^1 x^r \log \frac{1}{x} \log \frac{1}{1-x} \,\mathrm{d}x = \sum_{j=1}^{\infty} \frac{1}{j(j+r+1)^2},$$

which by (7) proves (15). □

3. MEAN AND VARIANCE OF ENTROPY FOR RANDOM PARTITIONS

In this section we present formulae for the expectation and variance of Shannon's, paired, genetic and α-entropy of spacings. In what follows we use Darling's (cf. [4]) moment formulae for the statistic $\sum_{j=0}^{k} f(Y_{j,k})$, which are given by

$$\mathrm{E}\left(\sum_{j=0}^{k} f(Y_{j,k})\right) = k(k+1)\int_0^1 (1-x)^{k-1} f(x)\,\mathrm{d}x, \tag{17}$$

$$\mathrm{E}\left(\sum_{j=0}^{k} f(Y_{j,k})\right)^2 = k(k+1)\int_0^1 (1-x)^{k-1} f^2(x)\,\mathrm{d}x + k^2(k^2-1)\int_0^1\int_0^{1-x} (1-x-y)^{k-2} f(x)f(y)\,\mathrm{d}y\,\mathrm{d}x, \tag{18}$$

and

$$\mathrm{E}\left(\sum_{j=0}^{k} f(Y_{j,k})\sum_{j=0}^{k} g(Y_{j,k})\right) = k(k+1)\int_0^1 (1-x)^{k-1} f(x)g(x)\,\mathrm{d}x + k^2(k^2-1)\int_0^1\int_0^{1-x} (1-x-y)^{k-2} f(x)g(y)\,\mathrm{d}y\,\mathrm{d}x, \tag{19}$$

for any real functions f and g such that the above moments exist.

First we consider D_k^S. The results regarding the moments of Shanon's entropy of spacings are partially known (cf. [7, 23]) but we establish explicit formulae in terms of Polygamma functions.

Proposition 1. The expectation and the variance of D_k^S are given by

$$\mathrm{E}D_k^S = \psi(k+2) + \gamma - 1, \tag{20}$$

$$\operatorname{var} D_k^S = \frac{\pi^2 - 6}{3(k+2)} - \zeta(2, k+2). \tag{21}$$

P r o o f. Using (17) and (14) with $f(x) = -x \log x$ we get

$$\begin{aligned}\mathrm{E}D_k^S &= k(k+1)\int_0^1 x(1-x)^{k-1}\log\frac{1}{x}\,\mathrm{d}x\\ &= k(k+1)B(2,k)\left(\psi(k+2)-\psi(2)\right) \quad \text{(cf. [7, 23])}\end{aligned}$$

which gives (20).

Now by (18) we have

$$\begin{aligned}\mathrm{E}\left(D_k^S\right)^2 &= k(k+1)\int_0^1 (1-x)^{k-1}x^2\log^2 x\,\mathrm{d}x\\ &\quad + k^2(k^2-1)\int_0^1\int_0^{1-x}(1-x-y)^{k-2}x\log xy\log y\,\mathrm{d}y\,\mathrm{d}x,\end{aligned}$$

and substituting $y = (1-x)t$ in the second integral we get

$$\begin{aligned}\mathrm{E}\left(D_k^S\right)^2 &= k(k+1)\int_0^1 \left(x^{k+1}\log^2\frac{1}{1-x} - 2x^k\log^2\frac{1}{1-x} + x^{k-1}\log^2\frac{1}{1-x}\right)\mathrm{d}x\\ &+ k^2(k^2-1)\int_0^1 t(1-t)^{k-2}\,\mathrm{d}t\left(\int_0^1 x^k\log\frac{1}{x}\log\frac{1}{1-x}\,\mathrm{d}x - \int_0^1 x^{k+1}\log\frac{1}{x}\log\frac{1}{1-x}\,\mathrm{d}x\right)\\ &+ k^2(k^2-1)\int_0^1 x(1-x)^k\log\frac{1}{x}\,\mathrm{d}x\int_0^1 t(1-t)^{k-2}\log\frac{1}{t}\,\mathrm{d}t,\end{aligned}$$

Therefore by (2), (16) and (14) we obtain

$$\mathrm{E}\left(D_k^S\right)^2 = \left(\gamma + \psi(k+2) - 1\right)^2 - \zeta(2, k+2) + \frac{\pi^2 - 6}{3(k+2)},$$

which with (20) leads to (21). □

Since

$$\zeta(2, k+2) \geq \frac{1}{(k+2)(k+3)} + \frac{1}{(k+3)(k+4)} + \frac{1}{(k+4)(k+5)} + \ldots = \frac{1}{k+2}$$

we get

Corollary 2.
$$\operatorname{var} D_k^S \le \frac{\pi^2-9}{3(k+2)}.$$

For paired entropy D_k^P we have

Proposition 2. The expectation and the variance of D_k^P are given by

$$\mathrm{E}D_k^P = \psi(k+1)+\gamma, \tag{22}$$

$$\operatorname{var} D_k^P = \frac{\pi^2-6}{3(k+2)} - \zeta(2,k+2) - \frac{k}{k+2}\zeta\left(2,\frac{k+2}{2}\right) + \frac{k(2k+3)}{(k+1)^2(k+2)}. \tag{23}$$

Proof. Write $\overline{D}_k^S = -\sum_{j=0}^{k}(1-Y_{j,k})\log(1-Y_{j,k})$. Then using (17) with $f(x) = -(1-x)\log(1-x)$, we have

$$\mathrm{E}\overline{D}_k^S = k(k+1)\int_0^1 (1-x)^k \log\frac{1}{1-x}\,\mathrm{d}x = \frac{k}{k+1}. \tag{24}$$

Since $\mathrm{E}D_k^P = \mathrm{E}D_k^S + \mathrm{E}\overline{D}_k^S$ then by (20) we get (22).

Now using (18) we have

$$\mathrm{E}\left(\overline{D}_k^S\right)^2 = k(k+1)\int_0^1 (1-x)^{k+1}\log^2(1-x)\,\mathrm{d}x + k^2(k^2-1)$$
$$\cdot \int_0^1\int_0^{1-x}(1-x-y)^{k-2}(1-x)\log(1-x)(1-y)\log(1-y)\,\mathrm{d}y\,\mathrm{d}x := A_k + B_k,$$

say. By (13)

$$A_k := k(k+1)\int_0^1 (1-x)^{k+1}\log^2(1-x)\,\mathrm{d}x = \frac{2k(k+1)}{(k+2)^3}.$$

Next setting $t = 1-y$ in B_k we have

$$B_k = k^2(k^2-1)\left(\int_0^1\int_0^1 (t-x)^{k-2}(1-x)\log(1-x)t\log t\,\mathrm{d}t\,\mathrm{d}x\right.$$
$$\left. - \int_0^1\int_0^x (t-x)^{k-2}(1-x)\log(1-x)t\log t\,\mathrm{d}t\,\mathrm{d}x\right).$$

Then using the binomial expansion for $(t-x)^{k-2}$ in the first integral and substituting $t = zx$ in the second we get

$$B_k = k^2(k^2-1)\sum_{j=0}^{k-2}\binom{k-2}{j}(-1)^j\int_0^1 x^j(1-x)\log\frac{1}{1-x}\,\mathrm{d}x\int_0^1 t^{k-j-1}\log\frac{1}{t}\,\mathrm{d}t$$
$$-(-1)^k k^2(k^2-1)\left(\int_0^1 x^k(1-x)\log(1-x)\log x\,\mathrm{d}x\int_0^1 z(1-z)^{k-2}\,\mathrm{d}z\right.$$
$$\left. + \int_0^1 x(1-x)^k\log\frac{1}{x}\,\mathrm{d}x\int_0^1 z(1-z)^{k-2}\log\frac{1}{z}\,\mathrm{d}z\right).$$

Applying (2), (14) and (15) we see that

$$B_k = k^2(k^2-1)\sum_{j=0}^{k-2}\binom{k-2}{j}\frac{(-1)^j\,(H_{j+2}-1)}{(j+1)(j+2)(k-j)^2} - \frac{(-1)^k k}{k+2}\Big((\gamma+\psi(k+3)-1)^2-\psi'(k+3)\Big).$$

Hence by (12) we get

$$B_k = \frac{k^2}{(k+2)^2} - \frac{2k}{k+2}\beta'(k+3).$$

Therefore
$$\mathrm{E}\left(\overline{D}_k^S\right)^2 = -\frac{2k}{k+2}\beta'(k+3) + \frac{k(k^2+4k+2)}{(k+2)^3},$$

which by (24) and the equality $\beta'(k+3) = -\beta'(k+2) - \frac{1}{(k+2)^2}$ gives

$$\operatorname{var}\overline{D}_k^S = \frac{2k}{k+2}\beta'(k+2) + \frac{k}{(k+1)^2(k+2)}.$$

Now using (19) with $f(x) = -(1-x)\log(1-x)$ and $g(x) = -x\log x$ we get

$$\mathrm{E}\overline{D}_k^S D_k^S = k(k+1)\int_0^1 (1-x)^k x\log x\log(1-x)\,\mathrm{d}x + k^2(k^2-1)\int_0^1\int_0^{1-x}(1-x-y)^{k-2}(1-x)\log(1-x)y\log y\,\mathrm{d}y\mathrm{d}x := C_k + E_k,$$

say. For C_k we have

$$C_k := k(k+1)\int_0^1 (1-x)^k x\log x\log(1-x)\,\mathrm{d}x = k(k+1)\int_0^1 (x^k\log x\log(1-x) - x^{k+1}\log x\log(1-x))\,\mathrm{d}x,$$

which by (15) gives

$$C_k = \frac{k(2k+3)}{(k+1)(k+2)^2}(\gamma+\psi(k+2)) - \frac{2k(k+1)}{(k+2)^3} - \frac{k}{k+2}\zeta(2,k+2).$$

Substituting $y = (1-x)t$ in E_k we get

$$E_k = k^2(k^2-1)\left(\int_0^1 (1-x)^{k+1}\log\frac{1}{1-x}\,\mathrm{d}x\int_0^1 t(1-t)^{k-2}\log\frac{1}{t}\,\mathrm{d}t + \int_0^1 (1-x)^{k+1}\log^2\frac{1}{1-x}\,\mathrm{d}z\int_0^1 t(1-t)^{k-2}\,\mathrm{d}t\right),$$

which by (13) and (14) gives

$$E_k = \frac{k(k+1)}{(k+2)^2}(\gamma+\psi(k+2)) - \frac{k(k^2+2k+2)}{(k+2)^3}.$$

Therefore

$$\mathrm{E}\overline{D}_k^S D_k^S = \frac{k}{k+1}\left(\gamma+\psi(k+2)\right) - \frac{k}{k+2}\zeta(2,k+2) - \frac{k}{k+2},$$

and

$$\operatorname{cov}\,(\overline{D}_k^S, D_k^S) = -\frac{k}{k+2}\zeta(2,k+2) + \frac{k}{(k+1)(k+2)}.$$

Hence

$$\begin{aligned}\operatorname{var} D_k^P &= \operatorname{var} \overline{D}_k^S + \sigma^2 D_k^S + 2\operatorname{cov}\left(\overline{D}_k^S, D_k^S\right)\\ &= \frac{\pi^2-6}{3(k+2)} - \zeta(2,k+2) - \frac{2k}{k+2}\left(\zeta(2,k+2)-\beta'(k+2)\right) + \frac{k(2k+3)}{(k+1)^2(k+2)}\end{aligned}$$

which proves (23) after using the equality

$$\zeta(2,k+2) - \beta'(k+2) = \sum_{j=0}^{\infty}\frac{1+(-1)^j}{(j+k+2)^2} = \frac{1}{2}\zeta\left(2,\frac{k+2}{2}\right).$$ □

Corollary 3. $$\operatorname{var} D_k^P \le \frac{\pi^2-7}{3(k+2)}.$$

P r o o f. Since $-\zeta(2,k+2) \le -\frac{1}{k+2}$ and $-\zeta\left(2,\frac{k+2}{2}\right) \le -\frac{2}{k+2}$ then

$$\operatorname{var} D_k^P \le \frac{\pi^2-7}{3(k+2)} - \frac{(2k-1)(k^2-1)+3}{3(k+1)^2(k+2)^2},$$

which proves the required assertion. □

Remark 1. For $k \ge 7$ $$\operatorname{var} D_k^P \le \frac{\pi^2-8}{3(k+2)}.$$

Now we present the moments of D_k^G.

Proposition 3. The expectation and the variance of D_n^G are given by

$$\mathrm{E}D_k^G = \frac{k(k^2+5k+10)}{(k+2)_3}, \tag{25}$$

$$\operatorname{var} D_k^G = \frac{16k\left(k^6+15k^5+103k^4+435k^3+1282k^2+2700k+3240\right)}{(k+2)_3(k+2)_7}. \tag{26}$$

P r o o f. Using (17) with $f(x) = x(1-x) - x^2(1-x)^2$ write $\mathrm{E}D_k^G = \mathrm{E}\hat{D}_k^G + \mathrm{E}\overline{D}_k^G$, where

$$\mathrm{E}\hat{D}_k^G := k(k+1)\int_0^1 x(1-x)^k\,\mathrm{d}x = k(k+1)B(2,k+1) = \frac{k}{k+2}$$

and

$$\mathrm{E}\overline{D}_k^G := -k(k+1)\int_0^1 x^2(1-x)^{k+1}\,\mathrm{d}x = -k(k+1)B(3,k+2) = -\frac{2k(k+1)}{(k+2)_3},$$

which gives (25).

Next using (18) with $f(x) = x(1-x)$

$$\mathrm{E}\left(\hat{D}_k^G\right)^2 = k(k+1)\int_0^1 x^2(1-x)^{k+1}\,\mathrm{d}x + k^2(k^2-1)\int_0^1\int_0^{1-x}(1-x-y)^{k-2}x(1-x)y(1-y)\,\mathrm{d}y\,\mathrm{d}x.$$

Making the substitution $y = (1-x)t$ in the second integral we get

$$\mathrm{E}\left(\hat{D}_k^G\right)^2 = k(k+1)\int_0^1 x^2(1-x)^{k+1}\,\mathrm{d}x + k^2(k^2-1) \cdot \int_0^1\int_0^1 x(1-x)^{k+1}t(1-t)^{k-2}(1-(1-x)t)\,\mathrm{d}t\,\mathrm{d}x = \frac{k(k^2+5k+2)}{(k+2)_3}.$$

Similarly, from (18) and (19), we get

$$\mathrm{E}\left(\overline{D}_k^G\right)^2 = \frac{4k(k^3+18k^2+59k+18)}{(k+3)_6} \quad \text{and} \quad \mathrm{E}\,\hat{D}_k^G\overline{D}_k^G = -\frac{2k(k^2+8k+3)}{(k+3)_4}.$$

Hence

$$\operatorname{var}\hat{D}_k^G = \frac{4k}{(k+2)(k+2)_3},$$

$$\operatorname{var}\overline{D}_k^G = \frac{4k(k^6-3k^5-59k^4+147k^3+1714k^2+2520k+864)}{(k+2)_3(k+2)_7}$$

and

$$\operatorname{cov}\left(\hat{D}_k^G,\overline{D}_k^G\right) = \frac{4k(k^2-7k-6)}{(k+2)(k+2)_5}.$$

Finally (26) we obtain from

$$\operatorname{var}D_k^G = \operatorname{var}\hat{D}_k^G + \operatorname{var}\overline{D}_k^G + 2\operatorname{cov}\left(\hat{D}_k^G,\overline{D}_k^G\right). \qquad \square$$

Corollary 4.

$$\operatorname{var}D_k^G \le \frac{16}{k^3}$$

For α-entropy D_k^α we get

Proposition 4. The expectation and the variance of D_k^α are given by

$$\mathrm{E}D_k^\alpha = \frac{1}{2^{1-\alpha}-1}\left(k(k+1)B(\alpha+1,k)-1\right), \tag{27}$$

$$\operatorname{var} D_k^\alpha = \frac{k(k+1)}{(2^{1-\alpha}-1)^2}\left(B(2\alpha+1,k)(1+\alpha k B(\alpha+1,\alpha)) - k(k+1)B^2(\alpha+1,k)\right). \quad (28)$$

P r o o f. Using (17) and (18) we get

$$\mathrm{E}\sum_{j=0}^{k} Y_{j,k}^\alpha = k(k+1)\int_0^1 (1-x)^{k-1}x^\alpha \, \mathrm{d}x$$

and

$$\mathrm{E}\left(\sum_{j=0}^{k} Y_{j,k}^\alpha\right)^2 = k(k+1)\int_0^1 (1-x)^{k-1}x^{2\alpha}\, \mathrm{d}x + k^2(k^2-1)\int_0^1\int_0^{1-x}(1-x-y)^{k-2}x^\alpha y^\alpha \, \mathrm{d}y\mathrm{d}x,$$

which gives (27) and (28). □

Corollary 5.

$$\mathrm{E}D_k^\alpha \sim \frac{1}{2^{1-\alpha}-1}\left(\frac{\Gamma(\alpha+1)}{k^{\alpha-1}} - 1\right), \quad (29)$$

$$\operatorname{var} D_k^\alpha \sim \frac{\Gamma(2\alpha+1)-\left(\alpha^2+1\right)\Gamma^2(\alpha+1)}{(2^{1-\alpha}-1)^2 k^{2\alpha-1}} \quad \text{(cf. [4])}. \quad (30)$$

P r o o f. Using the formula

$$\frac{\Gamma(k)}{\Gamma(k+\beta)} = \frac{1}{k^\beta} - \frac{\beta(\beta-1)}{2k^{\beta+1}} + o\left(\frac{1}{k^{\beta+1}}\right), \quad \beta \ge 0, \quad k \to \infty$$

(cf. [4], [24], p. 67, 3.31) in (27) and (28) we get the desired assertions. □

4. ASYMPTOTIC PROPERTIES

Let $U_0, U_1, \dots, U_k$ be exponential distributed random variables with mean 1. It is known that

$$Y_{j,k} \overset{d}{=} \frac{U_j}{\sum_{i=0}^k U_i} \quad \text{for} \quad 0 \le j \le k \quad (31)$$

(cf. [6]), and the equality holds in distribution. Slud in [23] established the rate of the almost sure convergence of the sequence $(D_k^S - \log(k+1))$. Using the above representation and the law of iterated logarithm he proved that

$$\log(k+1) - D_k^S + \gamma - 1 = O\left((\log\log k/k)^{\frac{1}{2}}\right) \quad \text{a.s.,} \quad k \to \infty.$$

We prove the complete convergence of that sequence.

Theorem 1. $$D_k^S - \log(k+1) \xrightarrow{c} \gamma - 1, \quad k \to \infty.$$

Proof. Let $\varepsilon > 0$. Using (31) we see that

$$\Pr\left(\left|D_k^S - \log(k+1) - \gamma + 1\right| > \varepsilon\right) = \Pr\left(\left|\log \frac{\sum_{j=0}^k U_j}{k+1} + \frac{\sum_{j=0}^k U_j \log \frac{1}{U_j}}{\sum_{j=0}^k U_j} - \gamma + 1\right| > \varepsilon\right)$$
$$\leq \Pr\left(\left|\log \frac{\sum_{j=0}^k U_j}{k+1}\right| > \frac{\varepsilon}{2}\right) + \Pr\left(\left|\frac{\sum_{j=0}^k U_j \log \frac{1}{U_j}}{k+1} - \gamma + 1\right| > \frac{\varepsilon \sum_{j=0}^k U_j}{4(k+1)}\right)$$
$$+ \Pr\left(\left|\frac{1}{k+1}\sum_{j=0}^k U_j - 1\right| > \frac{\varepsilon}{4(1-\gamma)} \frac{1}{k+1}\sum_{j=0}^k U_j\right).$$

Now let $0 < \delta < 1$. Then

$$\Pr\left(\left|\frac{1}{k+1}\sum_{j=0}^k U_j - 1\right| > \frac{\varepsilon}{4(1-\gamma)} \frac{1}{k+1}\sum_{j=0}^k U_j\right)$$
$$\leq \Pr\left(\left|\frac{1}{k+1}\sum_{j=0}^k U_j - 1\right| > \frac{\varepsilon}{4(1-\gamma)} \frac{1}{k+1}\sum_{j=0}^k U_j, \left|\frac{1}{k+1}\sum_{j=0}^k U_j - 1\right| \leq \delta\right)$$
$$+ \Pr\left(\left|\frac{1}{k+1}\sum_{j=0}^k U_j - 1\right| > \frac{\varepsilon}{4(1-\gamma)} \frac{1}{k+1}\sum_{j=0}^k U_j, \left|\frac{1}{k+1}\sum_{j=0}^k U_j - 1\right| > \delta\right)$$
$$\leq \Pr\left(\left|\frac{1}{k+1}\sum_{j=0}^k U_j - 1\right| > \frac{\varepsilon(1-\delta)}{4(1-\gamma)}\right) + \Pr\left(\left|\frac{1}{k+1}\sum_{j=0}^k U_j - 1\right| > \delta\right).$$

Similarly

$$\Pr\left(\left|\frac{1}{k+1}\sum_{j=0}^k U_j \log\frac{1}{U_j} - \gamma + 1\right| > \frac{\varepsilon}{4} \frac{1}{k+1}\sum_{j=0}^k U_j\right)$$
$$\leq \Pr\left(\left|\frac{1}{k+1}\sum_{j=0}^k U_j \log\frac{1}{U_j} - \gamma + 1\right| > \frac{\varepsilon(1-\delta)}{4}\right) + \Pr\left(\left|\frac{1}{k+1}\sum_{j=0}^k U_j - 1\right| > \delta\right).$$

Also by the Theorem of Hsu and Robbins (cf. [5], [16])

$$\sum_{k=1}^\infty \Pr\left(\left|\frac{1}{k+1}\sum_{j=0}^k U_j \log\frac{1}{U_j} - \gamma + 1\right| > \frac{\varepsilon(1-\delta)}{4}\right) < \infty,$$

and

$$\sum_{k=1}^\infty \Pr\left(\left|\frac{1}{k+1}\sum_{j=0}^k U_j - 1\right| > \delta\right) < \infty.$$

Moreover, we see that

$$\sum_{k=1}^{\infty} \Pr\left(\left|\log \frac{\sum_{j=0}^{k} U_j}{k+1}\right| > \frac{\varepsilon}{2}\right) \leq \sum_{k=1}^{\infty} \Pr\left(\left|\frac{\sum_{j=0}^{k} U_j}{k+1} - 1\right| > e^{\frac{\varepsilon}{2}} - 1\right) + \sum_{k=1}^{\infty} \Pr\left(\left|\frac{\sum_{j=0}^{k} U_j}{k+1} - 1\right| > 1 - e^{-\frac{\varepsilon}{2}}\right) < \infty,$$

which ends the proof. □

Additional information can be obtained from Heyde's theorem [15], who proved that

$$\lim_{\varepsilon \to 0} \varepsilon^2 \sum_{n=1}^{\infty} \Pr(|S_n - n\mu| > n\varepsilon) = \sigma^2, \tag{32}$$

where S_n is the sum of n i.i.d. random variables with mean μ and variance var. Here we obtain

Corollary 6.

$$\lim_{\varepsilon \to 0} \varepsilon^2 \sum_{k=1}^{\infty} \Pr\left(\left|D_k^S - \log(k+1) - \gamma + 1\right| > \varepsilon\right) \leq 18+16\left((1-\gamma)^2 + (2-\gamma)^2 + \frac{\pi^2}{3}\right).$$

P r o o f. Letting $\delta = \varepsilon$ in the inequalities of Theorem 1 we get

$$\begin{aligned} \Pr\left(\left|D_k^S - \log(k+1) - \gamma + 1\right| > \varepsilon\right) &\leq 2 \Pr\left(\left|\frac{1}{k+1}\sum_{j=0}^{k} U_j - 1\right| > \varepsilon\right) \\ &+ \Pr\left(\left|\frac{1}{k+1}\sum_{j=0}^{k} U_j - 1\right| > \frac{\varepsilon(1-\varepsilon)}{4(1-\gamma)}\right) + \Pr\left(\left|\frac{1}{k+1}\sum_{j=0}^{k} U_j \log\frac{1}{U_j} - \gamma + 1\right| > \frac{\varepsilon(1-\varepsilon)}{4}\right) \\ &+ \Pr\left(\left|\frac{1}{k+1}\sum_{j=0}^{k} U_j - 1\right| > e^{\frac{\varepsilon}{2}} - 1\right) + \Pr\left(\left|\frac{1}{k+1}\sum_{j=0}^{k} U_j - 1\right| > 1 - e^{-\frac{\varepsilon}{2}}\right). \end{aligned}$$

which by (32) gives

$$\lim_{\varepsilon \to 0} \varepsilon^2 \sum_{k=1}^{\infty} \Pr\left(\left|D_k^S - \log(k+1) - \gamma + 1\right| > \varepsilon\right) \leq (2 + 16(1-\gamma)^2)\operatorname{var} U_1 + 16\operatorname{var}(U_1 \log U_1).$$

and after using $\operatorname{var} U_1 = 1$ and $\operatorname{var}(U_1 \log U_1) = (2-\gamma)^2 + \frac{\pi^2}{3} + 1$ we complete the proof. □

Theorem 2. $$D_k^P - \log(k+1) \xrightarrow{c} \gamma, \quad k \to \infty.$$

Proof. Since $D_k^P = D_k^S + \overline{D}_k^S$ then it is enough to show $\overline{D}_k^S \xrightarrow{c} 1$. Using the inequality
$$(1-x)x \le (1-x)\log\frac{1}{1-x} \le x, \quad x < 1,$$
we have
$$\sum_{j=0}^{k} Y_{j,k} - \sum_{j=0}^{k} Y_{j,k}^2 \le \overline{D}_k^S = -\sum_{j=0}^{k}(1-Y_{j,k})\log(1-Y_{j,k}) \le \sum_{j=0}^{k} Y_{j,k}.$$
Hence for any given $\varepsilon > 0$
$$\sum_{k=1}^{\infty} \Pr\left(\left|\overline{D}_k^S - 1\right| > \varepsilon\right) \le \sum_{k=1}^{\infty} \Pr\left(\sum_{j=0}^{k} Y_{j,k}^2 > \varepsilon\right) \le \frac{16}{\varepsilon^2}\sum_{k=1}^{\infty}\frac{1}{k^3} = \frac{16\zeta(3)}{\varepsilon^2} < \infty,$$
which ends the proof. □

Now taking into account that
$$\Pr\left(\left|D_k^P - \log(k+1) - \gamma\right| > 2\varepsilon\right) = \Pr\left(\left|D_k^S - \log(k+1) - \gamma + \overline{D}_k^S\right| > 2\varepsilon\right)$$
$$\le \Pr\left(\left|D_k^S - \log(k+1) - \gamma + 1\right| > \varepsilon\right) + \Pr\left(\left|\overline{D}_k^S - 1\right| > \varepsilon\right)$$
we get

Corollary 7.
$$\lim_{\varepsilon\to 0}\varepsilon^2\sum_{k=1}^{\infty}\Pr\left(\left|D_k^P - \log(k+1) - \gamma\right| > \varepsilon\right) \le 72 + 64\left((1-\gamma)^2 + (2-\gamma)^2 + \frac{\pi^2}{3} + \zeta(3)\right).$$

Theorem 3. $$D_k^G \xrightarrow{c} 1, \quad k \to \infty.$$

Proof. Let $\varepsilon > 0$. If $k \to \infty$ then $\mathrm{E}D_k^G \to 1$ and by Chebyshev's inequality and (26)
$$\sum_{k=1}^{\infty}\Pr\left(\left|D_k^G - \mathrm{E}D_k^G\right| > \varepsilon\right) \le \sum_{k=1}^{\infty}\frac{\operatorname{var} D_k^G}{\varepsilon^2} \le \frac{16}{\varepsilon^2}\sum_{k=1}^{\infty}\frac{1}{k^3} = \frac{16\zeta(3)}{\varepsilon^2} < \infty,$$
which implies the theorem. □

Remark 2. Note that by (26)
$$\lim_{\varepsilon\to 0}\varepsilon^2\sum_{k=1}^{\infty}k\Pr\left(\left|D_k^G - \mathrm{E}D_k^G\right| > \varepsilon\right) \le \sum_{k=1}^{\infty}\frac{16}{k^2} = \frac{8\pi^2}{3}$$

In the proof of complete convergence of α-entropy we use the following theorem of Baum and Katz [1].

Theorem 4. (cf. Baum and Katz [1]) Let $\frac{1}{2} < \alpha \le 1$ and $\{X_k, k \ge 1\}$ be the i.i.d. random variables. If $\mathrm{E}|X_k|^{\frac{2}{\alpha}} < \infty$, $\mathrm{E}X_k = \mu$ and $S_n = X_1 + \ldots + X_n$ then for all $\varepsilon > 0$

$$\sum_{n=1}^{\infty} \Pr(|S_n - n\mu| > n^{\alpha}\varepsilon) < \infty.$$

Theorem 5. For $\alpha > \frac{1}{2}$

$$D_k^{\alpha} - (k+1)^{1-\alpha}\frac{\Gamma(\alpha+1)}{2^{1-\alpha}-1} \xrightarrow{c} \frac{1}{1-2^{1-\alpha}}, \quad k \to \infty.$$

P r o o f. Let $\varepsilon > 0$. If $\alpha > 1$ then by (29) $\mathrm{E}D_k^{\alpha} \to \dfrac{1}{1-2^{1-\alpha}}$, $k \to \infty$. Using Chebyshev's inequality and (30)

$$\sum_{k=1}^{\infty} \Pr\left(|D_k^{\alpha} - \mathrm{E}D_k^{\alpha}| > \varepsilon\right) \le \sum_{k=1}^{\infty} \frac{\operatorname{var} D_k^{\alpha}}{\varepsilon^2} \le \sum_{k=1}^{\infty} \frac{C}{k^{2\alpha-1}} < \infty,$$

which implies the theorem.

Now let $\alpha \in \left(\frac{1}{2}, 1\right)$. Using (31) we see that

$$\Pr\left(\left|D_k^{\alpha} - (k+1)^{1-\alpha}\frac{\Gamma(\alpha+1)}{2^{1-\alpha}-1} - \frac{1}{1-2^{1-\alpha}}\right| > \varepsilon\right) = \Pr\left((k+1)^{1-\alpha}\left|\frac{1}{k+1}\sum_{j=0}^{k} U_j^{\alpha}\right.\right.$$
$$\left.\left. -\Gamma(\alpha+1)\left(\frac{1}{k+1}\sum_{j=0}^{k} U_j\right)^{\alpha}\right| > \varepsilon\left(2^{1-\alpha}-1\right)\left(\frac{1}{k+1}\sum_{j=0}^{k} U_j\right)^{\alpha}\right).$$

Now let $0 < \delta < 1$ and $\varepsilon_1 = \varepsilon\left(2^{1-\alpha}-1\right)$. Then

$$\Pr\left((k+1)^{1-\alpha}\left|\frac{1}{k+1}\sum_{j=0}^{k} U_j^{\alpha} - \Gamma(\alpha+1)\left(\frac{1}{k+1}\sum_{j=0}^{k} U_j\right)^{\alpha}\right| > \varepsilon_1\left(\frac{1}{k+1}\sum_{j=0}^{k} U_j\right)^{\alpha}\right)$$
$$\le \Pr\left(\left|\sum_{j=0}^{k}\left(U_j^{\alpha} - \Gamma(\alpha+1)\right)\right| > \varepsilon_1\frac{1-\delta}{2}(k+1)^{\alpha}\right) + \Pr\left(\left|\left(\frac{1}{k+1}\sum_{j=0}^{k} U_j\right)^{\alpha} - 1\right| > \delta\right)$$
$$+ \Pr\left((k+1)^{1-\alpha}\left|\left(\frac{1}{k+1}\sum_{j=0}^{k} U_j\right)^{\alpha} - 1\right| > \frac{\varepsilon_1(1-\delta)}{2\Gamma(\alpha+1)}\right).$$

Using Theorem 4 we see that

$$\sum_{k=1}^{\infty} \Pr\left(\left|\sum_{j=0}^{k}\left(U_j^{\alpha} - \Gamma(\alpha+1)\right)\right| > \frac{\varepsilon_1(1-\delta)}{2}(k+1)^{\alpha}\right) < \infty,$$

and by Theorem of Hsu and Robbins (cf. [5, 16])

$$\sum_{k=1}^{\infty}\Pr\left(\left|\left(\frac{1}{k+1}\sum_{j=0}^{k}U_j\right)^{\alpha}-1\right|>\delta\right)\le\sum_{k=1}^{\infty}\Pr\left(\left|\frac{1}{k+1}\sum_{j=0}^{k}U_j-1\right|>(\delta+1)^{\frac{1}{\alpha}}-1\right)$$
$$+\sum_{k=1}^{\infty}\Pr\left(\left|\frac{1}{k+1}\sum_{j=0}^{k}U_j-1\right|>1-(1-\delta)^{\frac{1}{\alpha}}\right)<\infty.$$

Now let $\varepsilon_2=\frac{\varepsilon_1(1-\delta)}{2\Gamma(\alpha+1)}$. Then

$$\Pr\left((k+1)^{1-\alpha}\left|\left(\frac{1}{k+1}\sum_{j=0}^{k}U_j\right)^{\alpha}-1\right|>\varepsilon_2\right)\le\Pr\left(\left|\sum_{j=0}^{k}(U_j-1)\right|>(k+1)\right.$$
$$\left.\cdot\left(\left(1+\frac{\varepsilon_2}{(k+1)^{1-\alpha}}\right)^{\frac{1}{\alpha}}-1\right)\right)+\Pr\left(\left|\sum_{j=0}^{k}(U_j-1)\right|>(k+1)\left(1-\left(1-\frac{\varepsilon_2}{(k+1)^{1-\alpha}}\right)^{\frac{1}{\alpha}}\right)\right).$$

Since $\left(\left(1+\frac{\varepsilon_2}{(k+1)^{1-\alpha}}\right)^{\frac{1}{\alpha}}-1\right)\sim\frac{\varepsilon_2}{\alpha}(k+1)^{\alpha-1}$ and $\left(1-\left(1-\frac{\varepsilon_2}{(k+1)^{1-\alpha}}\right)^{\frac{1}{\alpha}}\right)\sim\frac{\varepsilon_2}{\alpha}(k+1)^{\alpha-1}$ then

$$\sum_{k=1}^{\infty}\Pr\left((k+1)^{1-\alpha}\left|\left(\frac{1}{k+1}\sum_{j=0}^{k}U_j\right)^{\alpha}-1\right|>\varepsilon_2\right)$$
$$\le\sum_{k=1}^{\infty}\Pr\left(\left|\sum_{j=0}^{k}(U_j-1)\right|>\frac{\varepsilon_2}{\alpha}(k+1)^{\alpha}\right)+\sum_{k=1}^{\infty}\Pr\left(\left|\sum_{j=0}^{k}(U_j-1)\right|>\frac{\varepsilon_2}{\alpha}(k+1)^{\alpha}\right)<\infty,$$

by Theorem 4. The proof is complete. □

Moreover for $0<\alpha\le\frac{1}{2}$ we get the following statement

Theorem 6. If $0<\alpha\le\frac{1}{2}$ then

$$k^{\alpha-1}D_k^{\alpha}\xrightarrow{a.s.}\frac{\Gamma(\alpha+1)}{2^{1-\alpha}-1},\qquad k\to\infty. \tag{33}$$

Proof. We see that $k^{\alpha-1}\mathrm{E}D_k^{\alpha}\sim\dfrac{\Gamma(\alpha+1)}{2^{1-\alpha}-1}$. Now by Chebyshev's inequality and (30)

$$\Pr\left(\left|k^{\alpha-1}D_k^{\alpha}-k^{\alpha-1}\mathrm{E}D_k^{\alpha}\right|>\varepsilon\right)\le\frac{\operatorname{var}D_k^{\alpha}k^{2\alpha-2}}{\varepsilon^2}\le\frac{C}{k}.$$

But for every k there exists an integer $m = m(k)$ with $m^2 < k \le (m+1)^2$. Hence $0 < k - m^2 \le 2m$ and $k \to \infty$ implies $m \to \infty$. Moreover

$$\sum_{m=1}^{\infty} \Pr\left(\left|m^{2(\alpha-1)} D^{\alpha}_{m^2} - m^{2(\alpha-1)} E D^{\alpha}_{m^2}\right| > \varepsilon\right) \le \sum_{m=1}^{\infty} \frac{C}{m^2} < \infty,$$

which gives

$$m^{2(\alpha-1)} D^{\alpha}_{m^2} \xrightarrow{c} \frac{\Gamma(\alpha+1)}{2^{1-\alpha}-1}, \quad m \to \infty. \tag{34}$$

Since under the constraint $\sum_{j=1}^{k} x_i = 1$, $x_i \ge 0$, we have $\sum_{j=1}^{k} x_i^{\alpha} \le \frac{1}{k^{\alpha-1}}$, then

$$\left|k^{\alpha-1} D^{\alpha}_k - m^{2(\alpha-1)} D^{\alpha}_{m^2}\right| = \left|\left(k^{\alpha-1} - m^{2(\alpha-1)}\right) \sum_{j=0}^{m^2} Y^{\alpha}_{j,k} - k^{\alpha-1} \sum_{j=m^2+1}^{k} Y^{\alpha}_{j,k}\right|$$

$$\le \frac{k^{1-\alpha} - m^{2(1-\alpha)}}{k^{1-\alpha}} + \frac{\left(k-m^2\right)^{1-\alpha}}{k^{1-\alpha}} \le \left(1+\frac{1}{m}\right)^{2(1-\alpha)} - 1 + \left(\frac{2}{m}\right)^{1-\alpha} \quad \text{a.s..}$$

Therefore we get

$$\left|k^{\alpha-1} D^{\alpha}_k - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}-1}\right| \le \left|k^{\alpha-1} D^{\alpha}_k - m^{2(\alpha-1)} D^{\alpha}_{m^2}\right| + \left|m^{2(\alpha-1)} D^{\alpha}_{m^2} - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}-1}\right|$$

$$\le \left(1+\frac{1}{m}\right)^{2(1-\alpha)} - 1 + \left(\frac{2}{m}\right)^{1-\alpha} + \left|m^{2(\alpha-1)} D^{\alpha}_{m^2} - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}-1}\right|,$$

which by (34) implies (33). □

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