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HOPF BIFURCATION ANALYSIS OF SOME HYPERCHAOTIC SYSTEMS WITH TIME–DELAY CONTROLLERS

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A four-dimensional hyperchaotic Lü system with multiple time-delay controllers is considered in this paper. Based on the theory of Hopf bifurcation in delay system, we obtain a simple relationship between the parameters when the system has a periodic solution. Numerical simulations show that the assumption is a rational condition, choosing parameter in the determined region can control hyperchaotic Lü system well, the chaotic state is transformed to the periodic orbit. Finally, we consider the differences between the analysis of the hyperchaotic Lorenz system, hyperchaotic Chen system and hyperchaotic Lü system.

Keywords: Hopf bifurcation, periodic solution, multiple time-delays and parameters, hyperchaotic Lü system, hyperchaotic Chen system, hyperchaotic Lorenz system

AMS Subject Classification: O415

1. INTRODUCTION

Since the hyperchaotic Lü system was generated in [1], the dynamical characteristics of it has attracted a lot of attention. And we know the fact that it is sometimes necessary to take account of time delays inherent in the phenomena in modeling in the biological and social sciences. So in this paper, we want to discuss Hopf bifurcation of the hyperchaotic Lü system with multiple time-delay and parameter controllers to derive the chaotic state to some periodic orbits. In the case of multiple delays, the dynamics of system could be more complicated and interesting. As far as we know, some simple two-dimensional networks with two delays were studied [3]. Usually it is difficult to analyze time-delay systems since the characteristic equation is transcendental. For example, Olien and Belair [6] pointed out that "finding all parameter values for all the roots of the characteristic equation to have negative real part is hopeless". It shows the difficulty in studying the distribution of the zeros of the characteristic equation.

In this paper, we plan to mainly discuss the four-dimensional hyperchaotic Lü system with four time-delay controllers. Although the two-dimensional system was analyzed, it is usually very difficult to achieve a simple relationship for the stability condition in terms of the system parameters by using the method of characteristic roots, the problem becomes more difficult as the dimension and the number of delays grow. Fortunately, when the first and third delays are zeros, the characteristic equation obviously has two negative real roots. Correspondingly, the transcendental equation becomes simple. So only this simple case is considered in the paper. Readers can study the general case, it maybe more interesting.

The paper is structured as follows. In Section 2, we briefly introduce the hyperchaotic Lü system, establish sufficient conditions when the controlled hyperchaotic Lü system occurs Hopf bifurcation. Section 3, Numerical Simulation indicates that choosing parameters in the determined region can control hyperchaotic Lü system well. Finally, we consider the differences between the analysis of the hyperchaotic Lorenz system, hyperchaotic Chen system and hyperchaotic Lü system.

2. HOPF BIFURCATION OF HYPERCHAOTIC LÜ SYSTEM WITH MULTIPLE TIME–DELAY CONTROLLERS

The 4-dimensional Lü system is described as follows [1]:

$$\begin{cases} \dot{x} = a(y-x) + u \\ \dot{y} = -xz + cy \\ \dot{z} = xy - bz \\ \dot{u} = xz + du \end{cases} \tag{1}$$

where a, b, c are constants of Lü system and d is the control parameter. In this paper, we always suppose $a = 36,\ b = 3,\ c = 20,\ d = 1.3$, calculating its Lyapunov exponents according to the Wolf algorithm [2] and [4], we know that the system (1) is a hyperchaotic system.

The system (1) with multiple time-delay controllers is given by:

$$\begin{cases} \dot{x} = 36(y-x) + u \\ \dot{y} = -xz + 20y + k_1[y(t) - y(t-\tau_1)] \\ \dot{z} = xy - 3z \\ \dot{u} = xz + 1.3u + k_2[u(t) - u(t-\tau_2)] \end{cases} \tag{2}$$

the characteristic equation is

$$P(\lambda) = (\lambda + 36)\,(\lambda + 3)\,[\lambda - (20 + k_1) + k_1 e^{-\lambda\tau_1}]\,[\lambda - (1.3 + k_2) + k_2 e^{-\lambda\tau_2}] = 0 \tag{3}$$

so the eigenvalues are

$$\lambda_1 = -36,\ \lambda_2 = -3,\ \lambda_3 = 20 + k_1 - k_1 e^{-\lambda_3\tau_1},\ \lambda_4 = 1.3 + k_2 - k_2 e^{-\lambda_4\tau_2}.$$

Let $\lambda_3 = \alpha + \mathrm{i}\,\omega,\ \lambda_4 = \alpha - \mathrm{i}\,\omega$, where ω is a positive constant, we have

$$\begin{cases} \alpha + \mathrm{i}\,\omega = 20 + k_1 - k_1 e^{-(\alpha+\mathrm{i}\,\omega)\tau_1} \\ \alpha - \mathrm{i}\,\omega = 1.3 + k_2 - k_2 e^{-(\alpha+\mathrm{i}\,\omega)\tau_2}. \end{cases} \tag{4}$$

Separating the real and imaginary parts, the following equalities are fulfilled:

$$\begin{cases} \omega = k_1 e^{-\alpha\tau_1} \sin\omega\tau_1 \\ \alpha = 20 + k_1 - k_1 e^{-\alpha\tau_1} \cos\omega\tau_1 \end{cases} \qquad \begin{cases} \omega = k_2 e^{-\alpha\tau_2} \sin\omega\tau_2 \\ \alpha = 1.3 + k_2 - k_2 e^{-\alpha\tau_2} \cos\omega\tau_2. \end{cases} \tag{5}$$

If the system (2) occurs Hopf bifurcation at some $k_1,\ k_2$, the parameters α and ω must satisfy conditions: $\alpha = 0, \omega > 0,\ \alpha' \neq 0$. At which equation (3) has a pair of purely imaginary roots, $\pm\,\mathrm{i}\,\omega$, equations (5) become as follows:

$$\begin{cases} \omega = k_1 \sin\omega\tau_1 \\ 0 = 20 + k_1 - k_1\cos\omega\tau_1 \end{cases} \qquad \begin{cases} \omega = k_2 \sin\omega\tau_2 \\ 0 = 1.3 + k_2 - k_2\cos\omega\tau_2 \end{cases} \tag{6}$$

which leads to

$$\begin{cases} \omega^2 + (20 + k_1)^2 = k_1^2 \\ \omega^2 + (1.3 + k_2)^2 = k_2^2 \\ \tan\omega\tau_1 = \frac{\omega}{20+k_1} \\ \tan\omega\tau_2 = \frac{\omega}{1.3+k_2}. \end{cases} \tag{7}$$

The next step is to test the transversality conditions, that is, to compute the terms $\mathrm{d}\alpha/\mathrm{d}\tau_1$, $\mathrm{d}\omega/\mathrm{d}\tau_1$.

From equation(5), obtaining

$$\begin{aligned} &(1 - k_1\tau_1 e^{-\alpha\tau_1}\cos\omega\tau_1)\frac{\mathrm{d}\alpha}{\mathrm{d}\tau_1} \\ &\quad = k_1\alpha e^{-\alpha\tau_1}\cos\omega\tau_1 + k_1\omega e^{-\alpha\tau_1}\sin\omega\tau_1 + k_1\tau_1 e^{-\alpha\tau_1}\sin\omega\tau_1\frac{\mathrm{d}\omega}{\mathrm{d}\tau_1} \end{aligned}$$

$$\begin{aligned} &(1 - k_1\tau_1 e^{-\alpha\tau_1}\cos\omega\tau_1)\frac{\mathrm{d}\omega}{\mathrm{d}\tau_1} \\ &\quad = -k_1\alpha e^{-\alpha\tau_1}\sin\omega\tau_1 + k_1\omega e^{-\alpha\tau_1}\cos\omega\tau_1 - k_1\tau_1 e^{-\alpha\tau_1}\sin\omega\tau_1\frac{\mathrm{d}\alpha}{\mathrm{d}\tau_1} \end{aligned}$$

so we have

$$\begin{aligned} \left.\frac{\mathrm{d}\alpha}{\mathrm{d}\tau_1}\right|_{\alpha=0} &= \left.\frac{k_1\omega\sin\omega\tau_1}{(1 - k_1\tau_1 e^{-\alpha\tau_1}\cos\omega\tau_1)^2 + (k_1\tau_1 e^{-\alpha\tau_1}\sin\omega\tau_1)^2}\right|_{\alpha=0} \\ &= \left.\frac{\omega^2}{(1 - k_1\tau_1 e^{-\alpha\tau_1}\cos\omega\tau_1)^2 + (k_1\tau_1 e^{-\alpha\tau_1}\sin\omega\tau_1)^2}\right|_{\alpha=0} \\ &> 0. \end{aligned}$$

For the same reason, we can get

$$\begin{aligned} \left.\frac{\mathrm{d}\alpha}{\mathrm{d}\tau_2}\right|_{\alpha=0} &= \left.\frac{k_2\omega\sin\omega\tau_2}{(1 - k_2\tau_2 e^{-\alpha\tau_2}\cos\omega\tau_2)^2 + (k_2\tau_2 e^{-\alpha\tau_2}\sin\omega\tau_2)^2}\right|_{\alpha=0} \\ &= \left.\frac{\omega^2}{(1 - k_2\tau_2 e^{-\alpha\tau_2}\cos\omega\tau_2)^2 + (k_2\tau_2 e^{-\alpha\tau_2}\sin\omega\tau_2)^2}\right|_{\alpha=0} \\ &> 0 \end{aligned}$$

that is, the transversality conditions are fulfilled.

Then we get the following result:

Theorem 1. Let $\tau_1 > 0$, $\tau_2 > 0$, k_1 and k_2 are parameters, ω is the imaginary part of the solution, the hyperchaotic Lü system with delays will have Hopf bifurcation if the following conditions are satisfied:

1. $k_1 < -10, \quad k_2 = \frac{400+40k_1-1.3^2}{2.6}$;
2. $\tau_{1k} = \left\{ \frac{1}{\omega}[\arctan(\frac{40\omega}{400-\omega^2}) + k\pi], k = 0, 1, \ldots \right\}$,

   $\tau_{2k} = \left\{ \frac{1}{\omega}[\arctan(\frac{2.6\omega}{1.3^2-\omega^2}) + k\pi], k = 0, 1, \ldots \right\}$.

From this theorem, we know that τ_{1k}, τ_{k2} are bifurcation values, the relationship between k_1 and τ_{1k}, k_2 and τ_{2k} can be illustrated by Figure 1.

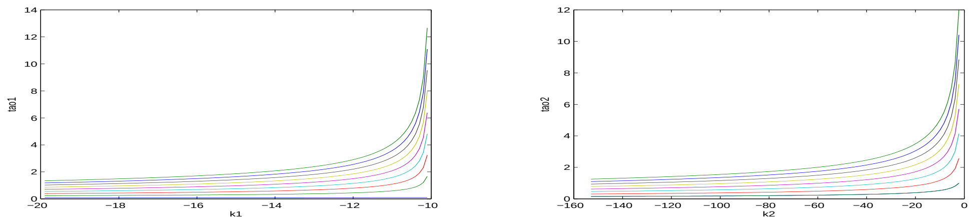

Fig. 1. When $k_1 \in [-19.9, -10.1]$, $k = 0, 1, \ldots, 8$, the graphs show the relationship between the parameters k_1 and τ_1, k_2 and τ_2 respectively.

It is well-known that the stability of the zero equilibrium solution depends on the roots of its characteristic equation, in order to discuss the stability of the equilibrium O, we shall use the following lemma [5].

Lemma 1. Let $f(\lambda, \tau) = \lambda^2 + a\lambda + b\lambda e^{-\tau\lambda} + c + de^{-\tau\lambda}$, where a, b, c, d, τ are real numbers and $\tau \geq 0$. Then, as τ varies, the sum of the multiplicities of zeros of f in the open right half-plane can change only if a zero appears on or crosses the imaginary axis.

For a delay equation, the supermum of the real parts of the roots of the characteristic equation varies continuously with τ [9]. By the previous Lemma, if there is a transition from stability to instability, vice verse, as τ varies, it must correspond to a purely imaginary root $\mathrm{i}\omega$, and the root must be simple. Since $\frac{\mathrm{d}\alpha}{\mathrm{d}\tau_1}|_{\alpha=0} > 0$ and $\frac{\mathrm{d}\alpha}{\mathrm{d}\tau_2}|_{\alpha=0} > 0$, the roots cross the imaginary axis from left to right as τ_1 and τ_2 increase. That is,there are k switches from stability to instability.

3. NUMERICAL SIMULATION

In this section, numerical simulations are given to explain that our theory is reasonable.

Let $\tau = (\tau_1, \tau_2)^{\mathrm{T}}, \tau_0 = (\tau_{10}, \tau_{20})^{\mathrm{T}}$, where $\tau_{10} > 0, \tau_{20} > 0$, and $\tau_{10} = \min\{\frac{1}{\omega}[\arctan(\frac{40\omega}{400-\omega^2}) + k\pi], k = 0, 1, \ldots\}$, $\tau_{20} = \min\{\frac{1}{\omega}[\arctan(\frac{2.6\omega}{1.3^2-\omega^2}) + k\pi], k = 0, 1, \ldots\}$. Let $k_1 = -10.1$, by theorem 1, $k_2 = -2.2, \tau_{10} = 0.09, \tau_{20} = 0.99$, the controlled system is:

$$\begin{cases} \dot{x} = 36(y - x) + u \\ \dot{y} = -xz + 20y - 10.1[y(t) - y(t - 0.09)] \\ \dot{z} = xy - 3z \\ \dot{u} = xz + 1.3u - 2.2[u(t) - u(t - 0.99)] \end{cases} \tag{8}$$

the numerical simulations of the controlled hyperchaotic Lü system are showed as the Figure 2 and 3. We can see that the impact is obvious, i. e. the system has a periodic solution.

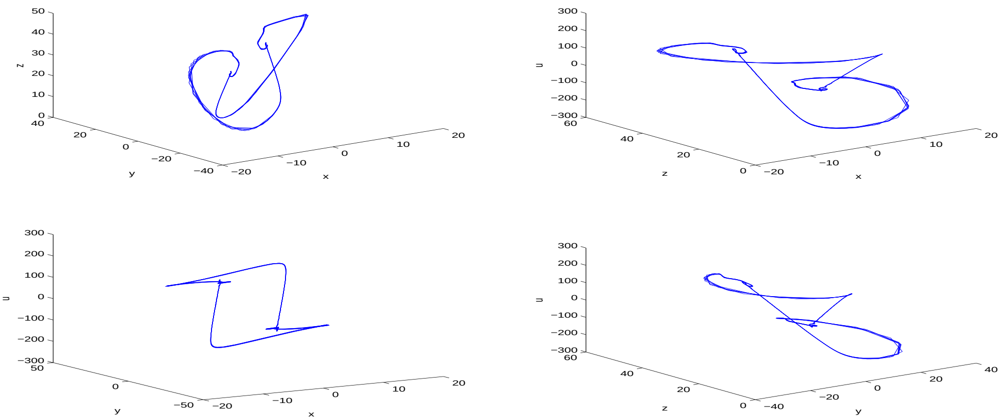

Fig. 2. When $k_1 = -10.1$, $k_2 = -2.2$, $\tau_{10} = 0.09$, $\tau_{20} = 0.99$, the state trajectories of the controlled hyperchaotic Lü system.

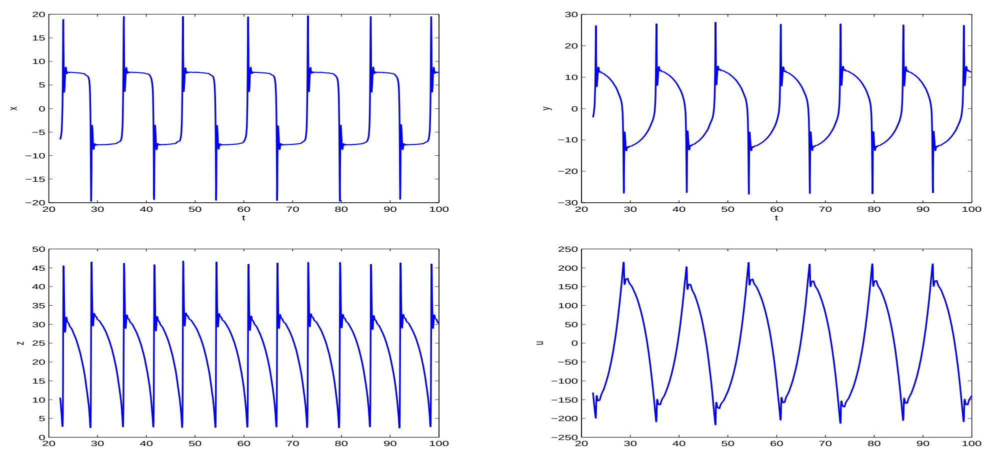

Fig. 3. When $k_1 = -10.1$, $k_2 = -2.2$, $\tau_{10} = 0.09$, $\tau_{20} = 0.99$, the time response of the controlled hyperchaotic Lü system.

4. SUPPLEMENTARY REMARKS

In this section, we will make similar analysis on the hyperchaotic Chen system and hyperchaotic Lorenz system and consider the differences between the above two systems and hyperchaotic Lü system. Although Lorenz system, Chen system and Lü system have different backgrounds, the corresponding hyperchaotic systems are all constructed by introducing a state feedback controller to the original ones. According to the different requirements, the hyperchaotic Lorenz system and hyperchaotic Chen system can be described in different ways, such as [8, 10]:

$$\begin{cases} \dot{x} = 10(y-x) + u \\ \dot{y} = -xz - y + 28x \\ \dot{z} = xy - \frac{3}{8}z \\ \dot{u} = -xz + d_1 u \end{cases} \tag{9}$$

$$\begin{cases} \dot{x} = 35(y-x) + u \\ \dot{y} = -xz + 12y + 7x \\ \dot{z} = xy - 3z \\ \dot{u} = yz + d_2 u \end{cases} \tag{10}$$

where $d_1 \in (0.85, 1.3]$, $d_2 \in [0.085, 0.798]$. Obviously they all have the equilibrium point $O(0,0,0,0)$, the Jacobi matrices at this equilibrium point are:

$$A_1 = \begin{pmatrix} -10 & 10 & 0 & 1 \\ 28 & -1 & 0 & 0 \\ 0 & 0 & -\frac{3}{8} & 0 \\ 0 & 0 & 0 & d_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -35 & 35 & 0 & 1 \\ 7 & 12 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & d_2 \end{pmatrix}.$$

The Jacobi matrices show that the forms of the hyperchaotic Lorenz system and hyperchaotic Chen system are the same, so we only discuss the Hopf bifurcation analysis of hyperchaotc Lorenz system with the same kind controllers. In order to compare it with hyperchaotic Lü system, we also add the delay terms to the second and the fourth equations, and let $d_1 = 1.3$, the controlled hyperchaotic Lorenz system is given by:

$$\begin{cases} \dot{x} = 10(y-x) + u \\ \dot{y} = -xz - y + 28x + k_1[y(t) - y(t-\tau_1)] \\ \dot{z} = xy - \frac{8}{3}z \\ \dot{u} = -xz + 1.3u + k_2[u(t) - u(t-\tau_2)] \end{cases} \tag{11}$$

where τ_i are time delays($\tau_i > 0$), k_i are parameters,($i = 1, 2$). The characteristic equation of (11) is:

$$\left[(\lambda + 1 - k_1 + k_1 e^{-\lambda\tau_1})(\lambda + 10) - 280\right]\left[\lambda - (1.3 + k_2) + k_2 e^{-\lambda\tau_2}\right]\left(\lambda + \frac{8}{3}\right) = 0. \tag{12}$$

Obviously, this equation is more complex than equation (3), we will discuss the following equations respectively:

$$\lambda - (1.3 + k_2) + k_2 e^{-\lambda\tau_2} = 0 \tag{13}$$

$$(\lambda + 1 - k_1 + k_1 e^{-\lambda\tau_1})(\lambda + 10) - 280 = 0. \tag{14}$$

Because the complex roots appear in pairs, equation (13) only has real root. When equation (12) has complex eigenvalues, we can only find the complex eigenvalues in equation (14). In that case, we need to introduce the following lemma [7] to find out the eigenvalues' distribution of the system (11).

Lemma 2. All roots of the equation $(z + a)\, e^z + b = 0$, where a and b are real, have negative real parts if and only if

1. $a > -1$
2. $a + b > 0$
3. $b < \xi \sin \xi - a \cos \xi$

where ξ is the roots of $\xi = -a \tan \xi, 0 < \xi < \pi$, if $a \neq 0$ and $\xi = \frac{\pi}{2}$ if $a = 0$.

Firstly, we transfer equation (13) into:

$$[\lambda - (1.3 + k_2)]e^{\lambda\tau_2} + k_2 = 0. \tag{15}$$

Let $\lambda\tau_2 = z$, equation (15) turns into:

$$[z - (1.3 + k_2)\tau_2]e^z + k_2\tau_2 = 0. \tag{16}$$

Suppose $a = -(1.3 + k_2)\tau_2$, $b = k_2\tau_2$, using Lemma 2, we have:

$$a + b = -(1.3 + k_2)\tau_2 + k_2\tau_2 = -1.3\tau_2 < 0. \tag{17}$$

So equation (13) has no solution of negative real part.

The analysis above proves that equation(13) only has one positive real root.

If the system (11) occurs Hopf bifurcation at some k_1, k_2, equation (12) must have a pair of pure imaginary roots and the remaining eigenvalues have strictly negative real parts.

Therefore, the system (11) cannot occur Hopf bifurcation at any parameters k_1, k_2.

We have already studied the case that adding the feedback delay terms to the second and the fourth equations, however, there are still another five different methods to add the delay terms to: (1) the first and the second equations; (2) the first and the third equations; (3) the first and the fourth equations; (4) the second and the third equations; (5) the third and the fourth equations. Then we discuss the eigenvalues' distribution of the controlled system under the above five conditions correspondingly:

1. $[(\lambda + 10 - k_1 + k_1 e^{-\lambda\tau_1})\,(\lambda + 1 - k_2 + k_2 e^{-\lambda\tau_2}) - 280](\lambda - d_1)\,(\lambda + \frac{3}{8}) = 0$, so $\lambda = d_1 > 0$;
2. $[(\lambda + 10 - k_1 + k_1 e^{-\lambda\tau_1})\,(\lambda + 1) - 280](\lambda - d_1)\,(\lambda + \frac{3}{8} - k_2 + k_2 e^{-\lambda\tau_2}) = 0$, so $\lambda = d_1 > 0$;
3. $[(\lambda + 10 - k_1 + k_1 e^{-\lambda\tau_1})\,(\lambda + 1) - 280](\lambda + \frac{3}{8})\,(\lambda - d_1 - k_2 + k_2 e^{-\lambda\tau_2}) = 0$, so $\lambda - d_1 - k_2 + k_2 e^{-\lambda\tau_2} = 0$, but, in consequence of Lemma 2, the last equation has a positive root;
4. $[(\lambda + 10)\,(\lambda + 1 - k_1 + k_1 e^{-\lambda\tau_1}) - 280]\,(\lambda - d_1)\left(\lambda + \frac{3}{8} - k_2 + k_2 e^{-\lambda\tau_2}\right) = 0$, so $\lambda = d_1 > 0$;
5. $[(\lambda + 10)(\lambda + 1) - 280]\left(\lambda + \frac{3}{8} - k_1 + k_1 e^{-\lambda\tau_1}\right)\,(\lambda - d_1 - k_2 + k_2 e^{-\lambda\tau_2}) = 0$, so $(\lambda - d_1 - k_2 + k_2 e^{-\lambda\tau_2}) = 0$, but, in consequence of Lemma 2, the last equation has a positive root.

In conclusion, the controlled hyperchaotic Lorenz system cannot occur Hopf bifurcation at any parameters k_1, k_2. The analysis and results of hyperchaotic Lorenz system are very different from those of hyperchaotic Lü system.

5. CONCLUSION

In this paper, we determine the relationship between the parameters of the hyperchaotic Lü system with multiple time-delay controllers when it has a Hopf bifurcation, that is, the system has a periodic solution. And we find that the Hopf bifurcation analysis of hyperchaotic Lorenz system and hyperchaotic Chen system are different, the controlled hyperchaotic Lorenz system and the controlled hyperchaotic Chen system cannot occur Hopf bifurcation at any parameters. The numerical results support the correctness of our calculated values for the bifurcation parameter. It shows that our theory is reasonable.

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