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OVERLAPPING CONTROLLERS FOR UNCERTAIN DELAY CONTINUOUS–TIME SYSTEMS

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This papers extends the Inclusion Principle to a class of linear continuous-time uncertain systems with state as well as control delays. The derived expansion-contraction relations include norm bounded arbitrarily time-varying real uncertainties and a point delay. They are easily applicable also to polytopic uncertainties. These structural conditions are further specialized on closed-loop systems with arbitrarily time-varying parameters, a point delay, and guaranteed quadratic costs. A linear matrix inequality (LMI) delay independent procedure is used for control design in the expanded space. The results are specialized on the overlapping decentralized control design. A numerical illustrative example is supplied.

Keywords: decentralized control, large-scale complex systems, overlapping decompositions, continuous-time systems, uncertainty, delay, LMI

AMS Subject Classification: 93A14, 93A15, 93B51, 93B52, 93C41

1. INTRODUCTION

One of the principal objectives of robust decentralized control is to synthesize feedback controllers under a priori guarantees of the robust stability and performance under information structure constraints. These guarantees are achieved by means of well known sparse matrix forms of gain matrices. A block tridiagonal form (BTD) corresponds to the concept of overlapping decompositions. A general mathematical framework for this approach has been called the Inclusion Principle. This approach is useful mainly when designing decentralized controllers for subsystems with shared parts where the states of shared parts have no direct control input.

Robust control of linear systems with real parameter uncertainties has recently attracted a lot of attention. The focus has been concentrated mainly on systems with norm bounded arbitrarily fast time-varying parameter uncertainties and polytopic uncertainties for systems with time-invariant (or constant) parameters. A common quadratic Lyapunov function independent of uncertain parameters is used when considering quadratic stability of the system for arbitrarily fast time-varying admissible uncertainties. However, this concept leads to conservative results when considering this approach for constant uncertainties. In order to reduce the conservatism of quadratic stability, the notion of parameter-dependent Lyapunov functions was developed, i.e. quadratic Lyapunov functions which are dependent on uncertain parameters. Moreover, the guaranteed cost control approach ensuring besides the robust stability an upper bound on a given performance has been developed.

Time delays appear in many real world systems. The time-delays are a source of instabilities and bad performance. This fact underlines the importance of new control design methods for time-delay systems. For continuous-time systems, time delay problems can be treated by the infinite-dimensional system theory, which usually leads to solutions in terms of Riccati type partial differential equations. They are difficult to compute. Some attempts have been recently realized to derive simple solutions to control problems by using LMIs.

1.1. Relevant references

The Inclusion Principle has been introduced into the systems theory in [1, 2, 11, 12, 13, 20] and further extended to solve different problems such as for instance in [3, 4, 5].

A guaranteed cost control problem for a class of uncertain delayed systems has been solved using LMI for the state or output feedback controller in [6, 8, 15, 16, 17], and the references therein. Parameter dependent robust stabilization present for this class of systems using a modified Riccati equation [14], while [9] deal with parameter dependent stability and stabilization using LMIs. [7] derived sufficient conditions for the expansion-contractions relations for a class of uncertain state-delay discrete-time systems. These relations are based on complicated recurrence relations. This is in contrast to the considered uncertain delay continuous-time systems, where the solution of expansion-relations can be performed at a general level by using the Peano–Baker series. This paper deals with overlapping controllers for uncertain delay continuous-time systems.

1.2. Outline of the paper

The paper deals with the expansion-contraction structural relations within the framework of overlapping decompositions for a class of uncertain state and control-delayed continuous-time systems. Systems with arbitrarily fast time-varying parameters are considered. The derived expansion-contraction relations are easily extendable to polytopic systems with constant uncertainties and time-varying delays. These conditions are specialized on closed-loop systems with arbitrarily time-varying parameters, point delay and the performance with bounded costs. An LMI delay independent procedure is used as a control design tool for expanded systems. The results are specialized into the overlapping decentralized control setting. Its effectiveness illustrates a numerical example.

To the authors knowledge, the expansion-contraction relations as well as the overlapping controller design have not been extended up to now for the considered class of systems including their specialization on the guaranteed cost static output control design.

2. PROBLEM FORMULATION

2.1. Inclusion of systems

Consider a linear continuous-time uncertain system with state and control delay described by the state equation

$$\begin{aligned} \boldsymbol{S}: \quad & \dot{x}(t) = \bar{A}(t)x(t) + \bar{B}(t)\,u(t) + \bar{A}_1(t)x(t-d) + \bar{B}_1(t)\,u(t-d), \\ & y(t) = Cx(t), \\ & x(t) = \varphi(t), \quad -d \leqslant t \leqslant 0, \end{aligned} \tag{1}$$

where

$$\begin{aligned} \bar{A}(t) &= A + \Delta A(t), & \bar{B}(t) &= B + \Delta B(t), \\ \bar{A}_1(t) &= A_1 + \Delta A_1(t), & \bar{B}_1(t) &= B_1 + \Delta B_1(t). \end{aligned} \tag{2}$$

$x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^q$, $d > 0$, $\varphi(t)$ denote the state, the control, the output, a point delay, and a given vector corresponding to a continuous initial function, respectively. The set $\{x(t), u(s)\}$, $s \in [t-d, t]$, defines the *complete state* of the system (1). A, B, A_1, B_1, C are known constant matrices of appropriate dimensions. $\Delta A(t)$, $\Delta B(t)$, $\Delta A_1(t)$, $\Delta B_1(t)$ are real-valued matrices of uncertain parameters. Norm-bounded uncertainties have the form

$$[\Delta A(t)\ \Delta B(t)\ \Delta A_1(t)\ \Delta B_1(t)] = D\ F(t)\ [E_1\ E_2\ E_3\ E_4]\,, \tag{3}$$

where D, E_1, E_2, E_3, E_4 are known constant real matrices of appropriate dimensions and $F^{i\times j}(t)$ is an unknown matrix function with Lebesgue measurable elements satisfying

$$F^T(t)F(t) \leqslant I. \tag{4}$$

The unique solution of (1) for any complete initial state $\{x(0), u(s)\}$ is given by

$$\begin{aligned} x(t;\varphi(t),u(t)) = \Phi(t,0)x(0) &+ \int_0^t \Phi(t,s)\bar{A}_1(s)x(s-d)\,\mathrm{d}s \\ &+ \int_0^t \Phi(t,s)\left[\bar{B}(s)\,u(s) + \bar{B}_1(s)\,u(s-d)\right]\,\mathrm{d}s, \end{aligned} \tag{5}$$

where Φ is the transition matrix.

Consider another system

$$\begin{aligned} \tilde{\boldsymbol{S}}: \quad & \dot{\tilde{x}}(t) = \bar{\tilde{A}}(t)\tilde{x}(t) + \bar{\tilde{B}}(t)\,u(t) + \bar{\tilde{A}}_1(t)\tilde{x}(t-d) + \bar{\tilde{B}}_1(t)\,u(t-d), \\ & \tilde{y}(t) = \tilde{C}\tilde{x}(t), \\ & \tilde{x}(t) = \tilde{\varphi}(t), \quad -d \leqslant t \leqslant 0, \end{aligned} \tag{6}$$

where

$$\begin{aligned} \bar{\tilde{A}}(t) &= \tilde{A} + \Delta\tilde{A}(t), & \bar{\tilde{B}}(t) &= \tilde{B} + \Delta\tilde{B}(t), \\ \bar{\tilde{A}}_1(t) &= \tilde{A}_1 + \Delta\tilde{A}_1(t), & \bar{\tilde{B}}_1(t) &= \tilde{B}_1 + \Delta\tilde{B}_1(t). \end{aligned} \tag{7}$$

$\tilde{x}(t) \in \mathbb{R}^{\tilde{n}}$, $u(t) \in \mathbb{R}^m$, $\tilde{y}(t) \in \mathbb{R}^{\tilde{q}}$, and $\tilde{\varphi}(t)$ denote the state, the control, the output, and a continuous initial function, respectively. Suppose that $n \leqslant \tilde{n}$, $q \leqslant \tilde{q}$. The set $\{\tilde{x}(t), u(s)\}$ with $s \in [t-d, t]$ defines a complete state for (6). Norm-bounded uncertainties have the form

$$\left[\Delta\tilde{A}(t)\ \Delta\tilde{B}(t)\ \Delta\tilde{A}_1(t)\ \Delta\tilde{B}_1(t)\right] = \tilde{D}\,\tilde{F}(t)\left[\tilde{E}_1\ \tilde{E}_2\ \tilde{E}_3\ \tilde{E}_4\right], \tag{8}$$

where $\tilde{D}$, $\tilde{E}_1$, $\tilde{E}_2$, $\tilde{E}_3$, $\tilde{E}_4$ are known constant real matrices of appropriate dimensions and $\tilde{F}^{\tilde{i}\times\tilde{j}}(t)$ is an unknown matrix function with Lebesgue measurable elements satisfying

$$\tilde{F}^T(t)\tilde{F}(t) \leqslant I. \tag{9}$$

The unique solution of (6) for any complete initial state $\{\tilde{x}(0), u(s)\}$ has the form

$$\begin{aligned}\tilde{x}(t;\tilde{\varphi}(t),u(t)) = \tilde{\Phi}(t,0)\tilde{x}(0) &+ \int_0^t \tilde{\Phi}(t,s)\tilde{\bar{A}}_1(s)\tilde{x}(s-d)\,\mathrm{d}s \\ &+ \int_0^t \tilde{\Phi}(t,s)\left[\tilde{\bar{B}}(s)\,u(s) + \tilde{\bar{B}}_1(s)\,u(s-d)\right]\mathrm{d}s,\end{aligned} \tag{10}$$

where $\tilde{\Phi}$ is the transition matrix.

Denote $x(t) = x(t;\varphi(t),u(t))$ and $\tilde{x}(t) = \tilde{x}(t;\tilde{\varphi}(t),u(t))$ the formal solutions of (1) and (6) for given inputs $u(t)$ and initial complete states $\{x(0), u(s)\}$ and $\{\tilde{x}(0), u(s)\}$, $s \in [-d, 0]$, respectively. Consider the standard relations between the states and the outputs within the Inclusion Principle. It means that the systems $\boldsymbol{S}$ and $\tilde{\boldsymbol{S}}$ are related by the following linear transformations

$$\tilde{x}(t) = Vx(t), \quad x(t) = U\tilde{x}(t), \quad \tilde{y}(t) = Ty(t), \quad y(t) = S\tilde{y}(t), \tag{11}$$

where V, $U = (V^TV)^{-1}V^T$, T and $S = (T^TT)^{-1}T^T$ are constant full rank matrices of appropriate dimensions [20]. Suppose given a quadruplet of matrices (U, V, S, T). Then the matrices $\tilde{A}$, $\Delta\tilde{A}(t)$, $\tilde{B}$, $\Delta\tilde{B}(t)$, $\tilde{A}_1$, $\Delta\tilde{A}_1(t)$, $\tilde{B}_1$, $\Delta\tilde{B}_1(t)$, and $\tilde{C}$ can be described as follows

$$\begin{gathered}\tilde{A} = VAU + M, \qquad \Delta\tilde{A}(t) = V\Delta A(t)\,U, \qquad \tilde{B} = VB + N, \\ \Delta\tilde{B}(t) = V\Delta B(t), \qquad \tilde{A}_1 = VA_1U + M_1, \qquad \Delta\tilde{A}_1(t) = V\Delta A_1(t)\,U, \\ \tilde{B}_1 = VB_1 + N_1, \qquad \Delta\tilde{B}_1(t) = V\Delta B_1(t),\end{gathered} \tag{12}$$

where M, N, M_1, N_1 and L are so called *complementary matrices.* Usually, the transformations (U, V, S, T) are selected a priori to define structural relations between the state variables in both systems $\boldsymbol{S}$ and $\tilde{\boldsymbol{S}}$. Given these transformations, the choice of the complementary matrices gives degrees of freedom to obtain different expanded spaces with desirable properties [1, 2].

Definition 1. A system $\boldsymbol{S}$ *includes* the system $\boldsymbol{S}$, denoted by $\boldsymbol{S} \supset \boldsymbol{S}$, if there exists a quadruplet of constant matrices (U, V, S, T) such that $UV = I_n$, $ST = I_q$ and for any initial function $\varphi(t)$ and any fixed input $u(t)$ of $\boldsymbol{S}$, $x(t) = U\tilde{x}(t)$ and $y[x(t)] = S\tilde{y}[\tilde{x}(t)]$ for all t. If $\tilde{\boldsymbol{S}} \supset \boldsymbol{S}$ holds then $\boldsymbol{S}$ is called and *expansion* of $\boldsymbol{S}$ and $\boldsymbol{S}$ is a *contraction* of $\boldsymbol{S}$.

Definition 2. A static output feedback controller $u(t) = \tilde{K}\tilde{y}(t)$ for $\tilde{\boldsymbol{S}}$ is *contractible* to $u(t) = Ky(t)$ for $\boldsymbol{S}$ if $Ky(t) = \tilde{K}\tilde{y}(t)$ for all t, any initial function $\varphi(t)$ and any fixed input $u(t)$.

2.2. Inclusion of quadratic costs

Consider the cost function associated with the system $\boldsymbol{S}$ by (1) in the form

$$J(x,u) = \int_0^{\infty} \left[x^T(t)\, Q^* x(t) + u^T(t) R^* u(t)\right] \mathrm{d}t, \tag{13}$$

where $Q^* \geqslant 0$ and $R^* > 0$.

Further, consider the cost function associated with the system $\boldsymbol{S}$ by (6) in the form

$$\tilde{J}(\tilde{x},u) = \int_0^{\infty} \left[\tilde{x}^T(t)\, \tilde{Q}^* \tilde{x}(t) + u^T(t) \tilde{R}^* u(t)\right] \mathrm{d}t, \tag{14}$$

where $\tilde{Q}^* \geqslant 0$ and $\tilde{R}^* > 0$.

Suppose the relation between matrices in (13) and (14) in the form

$$\tilde{Q}^* = U^T Q^* U + M_{Q^*}, \qquad \tilde{R}^* = R^* + N_{R^*}, \tag{15}$$

where M_{Q^*} and N_{R^*} are complementary matrices.

Definition 3. A pair $(\boldsymbol{S}, \tilde{J})$ *includes* the pair $(\boldsymbol{S}, J)$, denoted by $(\tilde{\boldsymbol{S}}, \tilde{J}\,) \supset (\boldsymbol{S}, J)$, if $\tilde{\boldsymbol{S}} \supset \boldsymbol{S}$ and $J(x,u) = \tilde{J}(\tilde{x},u)$. If $(\boldsymbol{S}, \tilde{J}\,) \supset (\boldsymbol{S}, J)$ holds then $(\tilde{\boldsymbol{S}}, \tilde{J})$ is called and expansion of $(\tilde{\boldsymbol{S}}, J)$ and $(\boldsymbol{S}, J)$ is a contraction of $(\tilde{\boldsymbol{S}}, \tilde{J})$.

Definition 4. A control law $u(t) = Ky(t)$ is said to be a *quadratic guaranteed cost control* with associated cost matrix $P > 0$ for the delay system (1) and cost function (13) if the corresponding closed-loop system is quadratically stable and the cost function satisfies the bound $J \leqslant J_0$ for all admissible uncertainties, that is

$$\frac{\mathrm{d}}{\mathrm{d}t}\, x^T(t) P x(t) + x^T(t) \left[Q^* + C^T K^T R^* K C\right] x(t) < 0 \tag{16}$$

for all nonzero $x \in \mathbb{R}^n$.

There are available different approaches to compute quadratic guaranteed cost control laws. A delay independent linear matrix inequality approach is selected to design a linear output feedback controller guaranteeing that the system is quadratically stable with a desired upper bound on the quadratic cost function. The following proposition gives sufficient conditions to get a guaranteed cost control law [17]. To simplify, the result is presented only for the problem (1), (13), but it evidently holds also for the expanded problem (6), (14).

Theorem 1. Consider the problem (1), (13). A static output feedback controller $u(t) = Ky(t)$ is a guaranteed cost controller if there exist a constant parameters $\mu > 0$, $\epsilon > 0$, a symmetric positive-definite matrices P, S, $Z \in \mathbb{R}^{n\times n}$ and a matrix $K \in \mathbb{R}^{m\times q}$ such that the following matrix inequality

$$\begin{bmatrix} \Psi & PB_1KC & [E_1+E_2KC]^T & I_n & [KC]^T & PA_1 & 0 & I_n \\ [PB_1KC]^T & -PZP & [E_4KC]^T & 0 & 0 & 0 & 0 & 0 \\ E_1+E_2KC & E_4KC & -\mu I_j & 0 & 0 & 0 & 0 & 0 \\ I_n & 0 & 0 & -[Q^*]^{-1} & 0 & 0 & 0 & 0 \\ KC & 0 & 0 & 0 & -[R^*]^{-1} & 0 & 0 & 0 \\ A_1^TP & 0 & 0 & 0 & 0 & -S & E_3^T & 0 \\ 0 & 0 & 0 & 0 & 0 & E_3 & -\epsilon I_j & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 & 0 & -S^{-1} \end{bmatrix} < 0 \tag{17}$$

is feasible, where $\Psi := [A + BKC]^T P + P[A + BKC] + PZP + (\mu + \epsilon)PDD^TP$. Moreover, the cost function given in (13) satisfies

$$J \leqslant \varphi^T(0)P\varphi(0) + \int_{-d}^{0} \varphi^T(s)\ \left[S^{-1}+PZP\right] \varphi(s)\,\mathrm{d}s = J_0. \tag{18}$$

Proof. The proof can be derived by using conveniently the results presented in [17]. □

2.3. The problem

Suppose given a linear continuous-time uncertain delayed system $\boldsymbol{S}$ given by (1)–(4) with an associated cost function J by (13). Consider an expanded system $\tilde{\boldsymbol{S}}$ represented in (6)–(9) with an associated cost function $\tilde{J}$ by (14). Denote $\boldsymbol{S}_C$ the closed-loop system composed of the system $\boldsymbol{S}$ by (1) and a static output controller K. Suppose $\tilde{\boldsymbol{S}}_C$ denotes a closed-loop system for the expanded system $\tilde{\boldsymbol{S}}$ by (6) with a static output controller $\tilde{K}$. Then, the specific goals are as follows:

- Derive conditions under which $\tilde{\boldsymbol{S}} \supset \boldsymbol{S}$ and $\tilde{\boldsymbol{S}}_C \supset \boldsymbol{S}_C$. Present these conditions in terms of complementary matrices.
- Derive conditions under which $(\tilde{\boldsymbol{S}}_C, \tilde{J}_0) \supset (\boldsymbol{S}, J_0)$. Use the concept of quadratic guaranteed cost control within the delay independent LMI approach.
- Specialize the global system results into decentralized control setting.
- Supply these results with a numerical example.

3. MAIN CONTRIBUTION

3.1. Closed-loop systems

Definition 1 can be rewritten in terms of complementary matrices. The derivation of these relations lead first to the relations based on the transition matrices. They are presented by the following theorem.

Theorem 2. Consider the systems (1)–(4) and (6)–(9). A system $\tilde{\boldsymbol{S}}$ includes the system $\boldsymbol{S}$ if and only if

$$\begin{aligned} &U\tilde{\Phi}(t,0)V = \Phi(t,0), \quad U\tilde{\Phi}(t,s)M_1V = 0, \quad U\tilde{\Phi}(t,s)N = 0, \quad U\tilde{\Phi}(t,s)N_1 = 0, \\ &SL\tilde{\Phi}(t,0)V = 0, \quad SL\tilde{\Phi}(t,s)M_1V = 0, \quad SL\tilde{\Phi}(t,s)N = 0, \quad SL\tilde{\Phi}(t,s)N_1 = 0, \end{aligned} \tag{19}$$

hold for all t and s.

Proof. Impose $x(t) = U\tilde{x}(t)$ for all t by Definition 1. Substitute (5) and (10) into this relation and compare both sides. We obtain the equalities: $U\tilde{\Phi}(t,0)V = \Phi(t,0)$; $U\tilde{\Phi}(t,s)\bar{\tilde{A}}_1(s)V = \Phi(t,s)\bar{A}_1(s)$; $U\tilde{\Phi}(t,s)\bar{\tilde{B}}(s) = \Phi(t,s)\bar{B}(s)$ and $U\tilde{\Phi}(t,s)\bar{\tilde{B}}_1(s) = \Phi(t,s)\bar{B}_1(s)$. However, we get from (2) and (12) that $U\tilde{\Phi}(t,s)\bar{\tilde{A}}_1(s)V = \Phi(t,s)\bar{A}_1(s)$ is equivalent to $U\tilde{\Phi}(t,s)M_1V = 0$; $U\tilde{\Phi}(t,s)\bar{\tilde{B}}(s) = \Phi(t,s)\bar{B}(s)$ is equivalent to $U\tilde{\Phi}(t,s)N = 0$ and $U\tilde{\Phi}(t,s)\bar{\tilde{B}}_1(s) = \Phi(t,s)\bar{B}_1(s)$ is equivalent to $U\tilde{\Phi}(t,s)N_1 = 0$, for all t, s.

Now, impose $y[x(t)] = S\tilde{y}[\tilde{x}(t)]$. We get: $S\tilde{C}\tilde{\Phi}(t,0)V = C\Phi(t,0)$; $S\tilde{C}\tilde{\Phi}(t,s)\bar{\tilde{A}}_1(s)V = C\Phi(t,s)\bar{A}_1(s)$; $S\tilde{C}\tilde{\Phi}(t,s)\bar{\tilde{B}}(s) = C\Phi(t,s)\bar{B}(s)$ and $S\tilde{C}\tilde{\Phi}(t,s)\bar{\tilde{B}}_1(s) = C\Phi(t,s)\bar{B}_1(s)$. The relations (2) and (12) yield that $S\tilde{C}\tilde{\Phi}(t,0)V = C\Phi(t,0)$ is equivalent to $SL\tilde{\Phi}(t,0)V = 0$; $S\tilde{C}\tilde{\Phi}(t,s)\bar{\tilde{A}}_1(s)V = C\Phi(t,s)\bar{A}_1(s)$ is equivalent to the relation $SL\tilde{\Phi}(t,s)M_1V = 0$; $S\tilde{C}\tilde{\Phi}(t,s)\bar{\tilde{B}}(s) = C\Phi(t,s)\bar{B}(s)$ to $SL\tilde{\Phi}(t,s)N = 0$ and finally $S\tilde{C}\tilde{\Phi}(t,s)\bar{\tilde{B}}_1(s) = C\Phi(t,s)\bar{B}_1(s)$ is equivalent to $SL\tilde{\Phi}(t,s)N_1 = 0$, for all t, s. □

Remark 1. It is well known that to obtain a general solution for time-varying systems is almost impossible. The problem is solved using approximations of transition matrices. However, even to compute such approximation via Peano–Baker series is complicated task when excluding trivial cases [4], [19]. Theorem 2 can be rewritten without a precise knowledge of transition matrices only in terms of complementary matrices.

Theorem 3. Consider the systems (1)–(4) and (6)–(9). A system $\tilde{\boldsymbol{S}}$ includes the system $\boldsymbol{S}$ if and only if

$$\begin{aligned} &UM^iV = 0, \quad UM^{i-1}M_1V = 0, \quad UM^{i-1}N = 0, \quad UM^{i-1}N_1 = 0, \\ &SLM^{i-1}V = 0, \quad SLM^{i-1}M_1V = 0, \quad SLM^{i-1}N = 0, \quad SLM^{i-1}N_1 = 0 \end{aligned} \tag{20}$$

hold for all $i = 1, 2, \ldots, \tilde{n}$.

P r o o f. Consider the transition matrix $\tilde{\Phi}(t,s)$ of the expanded system $\tilde{\boldsymbol{S}}$ as a function of two variables defined by the Peano–Baker series [18]

$$\begin{aligned}\tilde{\Phi}(t,s) = I + \int_s^t \bar{\tilde{A}}(\sigma_1)\,\mathrm{d}\sigma_1 + \int_s^t \bar{\tilde{A}}(\sigma_1) \int_s^{\sigma_1} \bar{\tilde{A}}(\sigma_2)\,\mathrm{d}\sigma_2\mathrm{d}\sigma_1 \\ + \int_s^t \bar{\tilde{A}}(\sigma_1) \int_s^{\sigma_1} \bar{\tilde{A}}(\sigma_2) \int_s^{\sigma_2} \bar{\tilde{A}}(\sigma_3)\,\mathrm{d}\sigma_3 d\sigma_2 \mathrm{d}\sigma_1 + \cdots,\end{aligned} \tag{21}$$

where according to (2) and (12), $\bar{\tilde{A}}(\sigma_i) = \tilde{A} + \Delta\tilde{A}(\sigma_i) = VAU + M + V\Delta A(\sigma_i)\,U$ for all $i = 1, 2, \ldots$ Pre and post-multiplying both sides of $\tilde{\Phi}(t,0)$ by U and V and after a tedious manipulations results in $U\tilde{\Phi}(t,0)V = \Phi(t,0)$ is equivalent to $UM^iV = 0$ for all $i = 1, 2, \ldots, \tilde{n}$. A similar process, when applied on each condition given in (19) leads to the equivalence relation for the corresponding condition in (20). □

Remark 2. The expansion-contraction relations are expressed in terms of the transition matrices for continuous-time delayed systems. This is an essential difference from the case of their discrete-time counterpart, where the general terms using transition matrices lead to complicated recurrence relations. Such relations can be used to derive general expansion-contraction relations. Therefore, only sufficient conditions can be obtained for the discrete-time case [7].

We have presented necessary and sufficient conditions for the inclusion relation $\tilde{\boldsymbol{S}} \supset \boldsymbol{S}$. Unfortunately, the requirements (20) are very difficult to verify due to the powers of M and matrix products. It motivates the derivation of only more simple sufficient conditions which are more convenient for verification and computations. These conditions presents the next proposition.

Proposition 1. Consider the problems (1), (13) and (6), (14). A system $\tilde{\boldsymbol{S}}$ includes the system $\boldsymbol{S}$ if

$$\begin{aligned}&\text{a)}\quad MV = 0, \quad M_1V = 0, \quad N = 0, \quad N_1 = 0, \quad LV = 0 \qquad \text{or} \\ &\text{b)}\quad UM = 0, \quad UM_1 = 0, \quad UN = 0, \quad UN_1 = 0, \quad SL = 0\end{aligned} \tag{22}$$

hold.

P r o o f. The proof is a direct consequence of Theorem 3. □

Remark 3. The cases a) and b) when $M_1 = 0$, $N_1 = 0$ in (22), i. e. $\boldsymbol{S}$ is a system without any delays, correspond with particular cases of the Inclusion Principle known as *restrictions* and *aggregations*, respectively, [20].

Though Definition 2 presents the contractibility condition, it does not guarantee that the closed-loop system $\tilde{\boldsymbol{S}}_C$ includes the closed-loop system $\boldsymbol{S}_C$ in the sense of the Inclusion Principle, i. e. $\boldsymbol{S}_C \supset \boldsymbol{S}_C$. Consider such conditions now.

Theorem 4. Consider the systems (1)–(4) and (6)–(9) satisfying the relation $\tilde{\boldsymbol{S}} \supset \boldsymbol{S}$. Suppose that $u(t) = \tilde{K}\tilde{y}(t)$ is a contractible control law designed in $\tilde{\boldsymbol{S}}$. If $MV = 0$, $M_1V = 0$, $N = 0$, $N_1 = 0$, and $LV = 0$ hold, then $\tilde{\boldsymbol{S}}_C \supset \boldsymbol{S}_C$.

Proof. Suppose that $\tilde{\boldsymbol{S}} \supset \boldsymbol{S}$ and $u(t) = \tilde{K}\tilde{y}(t)$ is a contractible control law designed in $\tilde{\boldsymbol{S}}$. The corresponding closed-loop expanded system $\tilde{\boldsymbol{S}}_C$ has the form

$$\begin{aligned}\tilde{\boldsymbol{S}}_C : \dot{\tilde{x}}(t) &= \left[\tilde{A}+\Delta\tilde{A}(t)+[\tilde{B}+\Delta\tilde{B}(t)]\,\tilde{K}\tilde{C}\right]\tilde{x}(t)\\ &+\left[\tilde{A}_1+\Delta\tilde{A}_1(t)+[\tilde{B}_1+\Delta\tilde{B}_1(t)]\,\tilde{K}\tilde{C}\right]\tilde{x}(t-d) = \tilde{A}_p(t)\tilde{x}(t)+\tilde{A}_q(t)\tilde{x}(t-d).\end{aligned} \tag{23}$$

An analogous expression holds for the closed-loop system $\tilde{\boldsymbol{S}}_C$. Consider the relation between the state matrices $\tilde{A}_p(t)$ and $A_p(t)$ and between $\tilde{A}_q(t)$ and $A_q(t)$ of the closed-loop systems $\tilde{\boldsymbol{S}}_C$ and $\boldsymbol{S}_C$, respectively. The relation $\tilde{A}_p(t) = VA_p(t)\,U + M_p$ implies $M_p = M + [VB + V\Delta B(t) + N]\,\tilde{K}L + N\tilde{K}TCU$ and $M_q = M_1 + [VB_1 + V\Delta B_1(t) + N_1]\,\tilde{K}L + N_1\tilde{K}TCU$. M_p and M_q are complementary matrices to be determined. Since $\tilde{\boldsymbol{S}}_C \supset \boldsymbol{S}_C$ is desired, the conditions (20) must be satisfied. Imposing these requirements, we obtain that the relations $MV = 0$, $M_1V = 0$, $N = 0$, $N_1 = 0$ and $LV = 0$ are sufficient conditions satisfying (20). □

The expansion-contraction relations are structural relations. They are derived for arbitrarily time-varying real parameter uncertainties. However, they include the case of uncertain time-invariant (constant) real parameters when considering $F(t)$ in (4) and $\tilde{F}(t)$ in (9) as unknown constants. Then, the controller design can be performed in the expanded space for instance by [14] to reduce the conservatism using the parameter-dependent Lyapunov functions. The presented expansion-contraction relations are easily adaptable also on the case when considering polytopic uncertainties. For instance, suppose that uncertain terms in (1) and (6) are unknown constants and the delay is unknown time-varying with a given bound and bounded rate of variations. Denote $\Omega = [\bar{A}\ \ \bar{A}_1]$ and suppose that $\Omega \in Co\,\{\Omega_j, j = 1,\dots,N\}$ with $\Omega = \sum_{j=1}^{N} f_j\Omega_j$ for any $0 \leqslant f_j \leqslant 1, \sum_{j=1}^{N} f_j = 1$, where the N vertices of the polytope are described by $\Omega_j = [A^{(j)}\ \ A_1^{(j)}]$. Now, when applying the derived expansion-contraction relations with $\bar{A} = A^{(j)}, \bar{A}_1 = A_1^{(j)}$ for all j, we get these relations for a polytopic system. Then, we can use for instance the results in [9] to design the state controller in the expanded space. Thus reducing the conservatism using the parameter-dependent Lyapunov functions in this case.

The solution of the most general case of systems with arbitrarily time-varying real parameter uncertainties using a single Lyapunov function presents the following theorem.

3.2. Guaranteed cost control

Suppose that the relations $\tilde{\boldsymbol{S}} \supset \boldsymbol{S}$ given by Theorems 2, 3, and Proposition 1, as well as the relations $\tilde{\boldsymbol{S}}_C \supset \boldsymbol{S}_C$ given by Theorem 4 hold. Consider the expansion-inclusion relations within the standard quadratic performance (13) and (14) for the corresponding systems $\boldsymbol{S}$ and $\tilde{\boldsymbol{S}}$, respectively.

Theorem 5. Consider the systems (1) – (4) and (6) – (9). A pair $(\tilde{\boldsymbol{S}}, \tilde{J})$ includes the pair $(\tilde{\boldsymbol{S}}, \tilde{J})$ if and only if the conditions (19) and $V^T M_{Q^*} V = 0$, $N_{R^*} = 0$ hold for all t and s.

Proof. This theorem is a direct consequence of Theorem 2. The conditions $V^T M_{Q^*} V = 0$ and $N_{R^*} = 0$ follows from $J(x_0, u) = \tilde{J}(V x_0, u)$. □

An analogous extension holds for Theorem 3 and Proposition 1.

Proposition 2. Consider the problems (1), (13) and (6), (14). A pair $(\tilde{\boldsymbol{S}}, \tilde{J})$ includes the pair $(\tilde{\boldsymbol{S}}, \tilde{J})$ if $V^T M_{Q^*} V = 0$, $N_{R^*} = 0$ and the conditions (22) hold.

Proof. The proof follows from Proposition 1 and the equality $J(x_0, u) = \tilde{J}(V x_0, u)$. □

Theorem 6. Consider the systems (1) – (4) and (6) – (9) satisfying the relation $\tilde{\boldsymbol{S}} \supset \boldsymbol{S}$. Suppose that $u(t) = \tilde{K}\tilde{y}(t)$ is a contractible control law designed in $\tilde{\boldsymbol{S}}$. If $MV = 0$, $M_1 V = 0$, $N = 0$, $N_1 = 0$, $LV = 0$, $V^T M_{Q^*} V = 0$, and $N_{R^*} = 0$ hold, then $\tilde{\boldsymbol{S}}_C \supset \boldsymbol{S}_C$.

Proof. The proof is a direct consequence of Theorem 4 and the equality $J(x_0, u) = \tilde{J}(V x_0, u)$. □

Remark 4. The assumptions in Theorem 6 coincide with the requirements a) in Proposition 2. Consider only this case in the remaining part of the paper.

The objective is to implement a guaranteed cost contracted controller $u(t) = Ky(t)$ into the delay system (1) obtained from a guaranteed cost controller $u(t) = \tilde{K}\tilde{y}(t)$ designed for the problem (6), (14). It remains to show that also the contracted controller is a quadratic guaranteed cost controller and the cost bounds of both systems are identical. This condition presents the following theorem.

Theorem 7. Consider the problems (1), (13) and (6), (14). Suppose that the relations $MV = 0$, $M_1 V = 0$, $N = 0$, $N_1 = 0$, $LV = 0$, $V^T M_{Q^*} V = 0$, and $N_{R^*} = 0$ are satisfied. Suppose that $u(t) = \tilde{K}\tilde{y}(t)$ is a quadratic guaranteed cost controller designed in the system $\tilde{\boldsymbol{S}}$ with a cost matrix $\tilde{P} > 0$. Then $u(t) = Ky(t) = \tilde{K}Ty(t)$ is the quadratic guaranteed cost controller with a cost matrix $P = V^T \tilde{P} V > 0$ for $\boldsymbol{S}$. Moreover, $J_0 = \tilde{J}_0$.

Proof. Suppose $u(t) = \tilde{K}\tilde{y}(t)$ a contractible quadratic guaranteed cost controller for the system $\tilde{\boldsymbol{S}}$. Then the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\tilde{x}^T(t)\tilde{P}\,\tilde{x}(t)+\tilde{x}^T(t)\left[\tilde{Q}^*+\tilde{C}^T\tilde{K}^T\tilde{R}^*\tilde{K}\tilde{C}\right]\tilde{x}(t)<0 \tag{24}$$

is satisfied by Definition 4. Suppose $\tilde{x}(t) = Vx(t)$, $V^T M_{Q^*} V = 0$, $N_{R^*} = 0$, $P = V^T\tilde{P}V$, $K = \tilde{K}T$ and apply (12). Then the inequality (24) leads directly to the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\,x^T(t)P\,x(t)+x^T(t)\left[Q^*+C^TK^TR^*KC\right]x(t)<0. \tag{25}$$

Moreover, the cost bounds satisfy

$$\begin{aligned}\tilde{J}_0 &= \tilde{\varphi}^T(0)\tilde{P}\tilde{\varphi}(0)+\int_{-d}^{0}\tilde{\varphi}^T(s)\left[\tilde{S}^{-1}+\tilde{P}\tilde{Z}\tilde{P}\right]\tilde{\varphi}(s)\,\mathrm{d}s\\ &= \varphi^T(0)V^T\tilde{P}V\varphi(0)+\int_{-d}^{0}\varphi^T(s)\left[V^T\tilde{S}^{-1}V+V^T\tilde{P}(VZV^T)\tilde{P}V\right]\varphi(s)\,\mathrm{d}s\\ &= \varphi^T(0)P\varphi(0)+\int_{-d}^{0}\varphi^T(s)\left[S^{-1}+PZP\right]\varphi(s)\,\mathrm{d}s=J_0.\end{aligned} \tag{26}$$

The bounds J_0 and $\tilde{J}_0$ are identical. If $u(t) = \tilde{K}\tilde{y}(t)$ is a quadratic guaranteed cost controller for $\tilde{\boldsymbol{S}}$, then the contracted controller $u(t) = Ky(t) = \tilde{K}Ty(t)$ of $\boldsymbol{S}$ keeps the same property. □

Remark 5. The equality $J_0 = \tilde{J}_0$ in Theorem 7 presupposes that three matrix assumptions $P = V^T\tilde{P}V$, $S^{-1} = V^T\tilde{S}^{-1}V$ and $\tilde{Z} = VZV^T$ are simultaneously satisfied. However, the condition $\tilde{Z} = VZV^T$ does not offer the freedom required when designing controller in the expanded space using the LMI by (17). $\tilde{Z} = VZV^T$ implies $Z = U\tilde{Z}U^T$, but it does not hold inversely. Therefore, Z can be used in the initial system together with the matrices $P = V^T\tilde{P}V$ and $S^{-1} = V^T\tilde{S}^{-1}V$.

3.3. Overlapping control design using LMI

The inequality (17) is not an LMI. It must be conveniently modified. The following proposition gives a sufficient condition for the existence of a guaranteed cost controller by Theorem 1 in terms of computational LMI. The result is presented only for the problem (1), (13). It evidently holds also for the expanded problem (6), (14). However, the result is in fact used only for the controller design in the expanded space.

Proposition 3. Consider the problem (1), (13). Suppose the existence of a positive definite matrices P, S, $Z \in \mathbb{R}^{n\times n}$, a matrix $K \in \mathbb{R}^{m\times q}$ and constant parameters $\mu > 0$, $\epsilon > 0$ satisfying Theorem 1. The inequality (17) holds if and only if there exist a symmetric positive-definite matrices X, Y, $Z \in \mathbb{R}^{n\times n}$ a matrix $W \in \mathbb{R}^{m\times n}$ and constant parameters $\mu > 0$, $\epsilon > 0$ such that the following linear matrix inequality

$$\begin{bmatrix} \Psi_1 & B_1W & [E_1X+E_2W]^T & X & W^T & A_1Y & 0 & X \\ [B_1W]^T & -Z & [E_4W]^T & 0 & 0 & 0 & 0 & 0 \\ [E_1X+E_2W] & E_4W & -\mu I_j & 0 & 0 & 0 & 0 & 0 \\ X & 0 & 0 & -[Q^*]^{-1} & 0 & 0 & 0 & 0 \\ W & 0 & 0 & 0 & -[R^*]^{-1} & 0 & 0 & 0 \\ YA_1^T & 0 & 0 & 0 & 0 & -Y & YE_3^T & 0 \\ 0 & 0 & 0 & 0 & 0 & E_3Y & -\epsilon I_j & 0 \\ X & 0 & 0 & 0 & 0 & 0 & 0 & -Y \end{bmatrix} < 0 \tag{27}$$

is feasible, where $\Psi_1 := [AX+BW]+[AX+BW]^T+Z+(\mu+\epsilon)DD^T$.

Proof. Let us introduce the matrices $X = P^{-1}$, $W = KCP^{-1}$ and $Y = S^{-1}$. Pre and post-multiplication of both sides of (17) by the non-singular matrix block $\operatorname{diag}\{P^{-1}, P^{-1}, I_j, I_n, I_m, S^{-1}, I_j, I_n\}$ leads directly to the inequality (27). □

Remark 6. We get $W = KCX$ from Proposition 3. However, the goal is to obtain the gain matrix K. Such problem has been solved in [22]. We use this result in the form of an algorithm.

Algorithm.

Step 1. Select a full rank matrix Q of $n\times(n-q)$ dimension such that $CQ = 0$.

Step 2. Solve the LMI given in (27) with

$$X = QX_qQ^T + C^T\left[CC^T\right]^{-1}C + C^TX_cC, \qquad W = W_cC, \tag{28}$$

where X_q and X_c are unknown symmetric matrices of dimensions $(n–q)\times(n–q)$ and $q \times q$, respectively, and W_c is an unknown $m \times q$ dimensional matrix.

Step 3. By supposing feasible the LMI (27), to compute the gain matrix K as

$$K = W_c\left[I - CX_0^{-1}C^TX_c\left[I + CX_0^{-1}C^TX_c\right]^{-1}\right], \tag{29}$$

where $X_0 = QX_qQ^T + C^T\left[CC^T\right]^{-1}C$. The procedure guarantees $WX^{-1} = KC$.

Theorem 8. Suppose given the problem (1), (13). Suppose that there exist symmetric matrices X_q, X_c, a matrix W_c and positive constant parameters $\mu > 0$, $\epsilon > 0$ such that the linear matrix inequality (27) is feasible with X and W given in (28). Then the static output feedback controller $u(t) = Ky(t)$ is a guaranteed cost controller for $\boldsymbol{S}$, where K given by (29). Moreover, the cost function J satisfies the upper bound

$$J \leqslant \varphi^T(0)P\varphi(0) + \int_{-d}^{0}\varphi^T(s)\,\left[S^{-1}+PZP\right]\varphi(s)\,\mathrm{d}s = J_0. \tag{30}$$

P r o o f. Theorem 8 is a direct consequence of Proposition 3. □

Let us specialize the above results into the decentralized setting. Overlapping structures correspond with the BTD form of sparse matrices. The basic structures with two overlapping subsystems for the matrices A, $\Delta A(t)$, A_1, $\Delta A_1(t)$, B_1, $\Delta B_1(t)$, and C are well known [6, 20, 21]. We suppose this basic structure for Type II case [21]. Suppose a standard particular selection of the matrices V and T

$$V = \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & I_{n_2} & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & I_{n_3} \end{bmatrix}, \qquad T = \begin{bmatrix} I_{q_1} & 0 & 0 \\ 0 & I_{q2} & 0 \\ 0 & I_{q2} & 0 \\ 0 & 0 & I_{q3} \end{bmatrix}. \tag{31}$$

This choice leads in a natural manner to an expanded system. The overlapped components x_2, q_2 have dimensions n_2, q_2, respectively. These components appear repeated in $\tilde{x}^T$ and $\tilde{y}^T$, where $x^T = (x_1^T, x_2^T, x_3^T)$ and $y^T = (y_1^T, y_2^T, y_3^T)$. The dimensions m_1 and m_2 are dimensions of vectors u_1 and u_2, where $u^T = (u_1^T, u_2^T)^T$. It means that the decentralized controller designed in the expanded state space is a block diagonal matrix with two subblocks of dimensions $m_1 \times (n_1 + n_2)$ and $m_2 \times (n_2 + n_3)$ of the gain matrix. It has the form

$$\tilde{K}_D = \left[\begin{array}{cc:cc} \tilde{K}_{11} & \tilde{K}_{12} & 0 & 0 \\ \hdashline 0 & 0 & \tilde{K}_{23} & \tilde{K}_{24} \end{array}\right]. \tag{32}$$

Denote $(*)_{TD}$ the BTD form of a matrix $(*)$. The contracted gain matrix has the form

$$K_{TD} = \left[\begin{array}{c:cc} \tilde{K}_{11} & \tilde{K}_{12} & 0 \\ \hdashline 0 & \tilde{K}_{23} & \tilde{K}_{24} \end{array}\right]. \tag{33}$$

Let us introduce the corresponding concept for this case.

Definition 5. Suppose given the problem (1), (13). A static output feedback controller $u_{TD}(t) = K_{TD}\, y(t)$ is said to be a *td-quadratic guaranteed cost controller* P_{TD} if it is a quadratic guaranteed cost controller with $K = K_{TD}$ and $P = P_{TD} = V^T \tilde{P}_D V > 0$, where P_D is the solution provided by the corresponding LMI.

Theorem 9. Suppose given the problems (1), (13) and (6), (14). Suppose that $MV = 0$, $M_1 V = 0$, $N = 0$, $N_1 = 0$, $LV = 0$, $V^T M_{Q^*} V = 0$, and $N_{R^*} = 0$ hold. If $u_D(t) = \tilde{K}_D \tilde{y}(t)$ is a contractible quadratic guaranteed cost controller with a cost matrix $\tilde{P}_D > 0$ for the system $\tilde{\boldsymbol{S}}$, then the contracted controller $u_{TD}(t) = K_{TD}\, y(t) = \tilde{K}_D T y(t)$ is a td-quadratic guaranteed cost controller with a cost matrix $P_{TD} = V^T \tilde{P}_D V > 0$ for $\boldsymbol{S}$ and $J_0 = \tilde{J}_0$.

P r o o f. Theorem 9 is a particular case of Theorem 7. □

4. EXAMPLE

4.1. Problem statement

Consider the problem (1), (13) with the initial data

$$A = \begin{bmatrix} -2 & 0 & -1 & 1 \\ -1 & 0 & 2 & 0 \\ 0 & -2 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}, \quad B = B_1 = D = \begin{bmatrix} 0.5 & 0 \\ 0.3 & 0.4 \\ 0 & 0.4 \\ 0 & 0.1 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -0.2 & 0 & 0 & 0 \\ 0 & 0.2 & 0.1 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix}, \qquad C = \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0.1 & -0.1 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix}, \tag{34}$$

$$E_1 = E_3 = \begin{bmatrix} 0.1 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix}, \qquad E_2 = E_4 = \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0.1 \end{bmatrix},$$
$$Q^* = \mathrm{diag}\{1, 2, 2, 1\}, \quad R^* = I_2, \quad \varphi(t) = [0.2,\ t,\ 0,\ 0]^T, \quad d = 1.$$

The overlapped subsystems are $A_{22} = \left[\begin{smallmatrix} 0 & 2 \\ -2 & -1 \end{smallmatrix}\right]$ and $C_{22} = [0.1 \quad -0.1]$ in the matrices A and C, respectively. The remaining overlapped subsystems corresponding to the matrices $\Delta A(t)$, A_1 and $\Delta A_1(t)$ are also 2×2 dimensional blocks. Find the guaranteed cost controller with the BTD structure of gain matrix for the above system using the delay independent LMI.

4.2. Results

Decentralized controller. Consider the expansion of the system $\boldsymbol{S}$ with the transformations V and T given in (31) in the form

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{35}$$

Select $N = 0$, $M_1 = 0$, $N_1 = 0$, $N_{R^*} = 0$. The remaining complementary matrices

M, L and M_{Q^*} have the standard form by [20]. It results in the forms

$$M = \begin{bmatrix} 0 & 0 & -0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & -0.5 & 1 & 0.5 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0.5 & -1 & -0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$M_{Q^*} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & -0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & -0.5 & 0 \\ 0 & -0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & -0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{36}$$

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.05 & -0.05 & -0.05 & 0.05 & 0 \\ 0 & -0.05 & 0.05 & 0.05 & -0.05 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

These matrices satisfy the required relations

$$MV = 0, \quad LV = 0 \quad \text{and} \quad V^T M_{Q^*} V = 0.$$

The resulting expanded weighting matrices are

$$\tilde{Q}^* = I_6, \quad \tilde{R}^* = I_2$$

and the matrix $\tilde{Q}$ is selected as

$$\tilde{Q} = \begin{bmatrix} 0 & 0.1 & 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0.1 & 0 \end{bmatrix}.$$

It satisfies $\tilde{C}\tilde{Q} = 0$. These choices satisfy the results presented in Subsection 3.3. Find the decentralized output guaranteed cost controller for the above system by using a delay independent LMI approach. Compare the results with the centralized output control design as a reference.

Impose necessary structural constraints on the matrices $\tilde{X}_c$, $\tilde{X}_q$ and $\tilde{W}_c$ to get the block diagonal form of the matrix $\tilde{K}_D$

$$\tilde{X}_c = \begin{bmatrix} x_{c11} & x_{c12} & 0 & 0 \\ x_{c12} & x_{c22} & 0 & 0 \\ 0 & 0 & x_{c33} & x_{c34} \\ 0 & 0 & x_{c34} & x_{c44} \end{bmatrix}, \quad \tilde{X}_q = \begin{bmatrix} x_{q11} & 0 \\ 0 & x_{q22} \end{bmatrix},$$

$$\tilde{W}_c = \begin{bmatrix} w_{11} & w_{12} & 0 & 0 \\ 0 & 0 & w_{23} & w_{24} \end{bmatrix}. \tag{37}$$

The LMI computation by (27) for the system $\tilde{\boldsymbol{S}}$ results in the gain matrix

$$\tilde{K}_D = \begin{bmatrix} -0.7667 & -0.6133 & 0 & 0 \\ 0 & 0 & 0.1278 & 0.5530 \end{bmatrix}. \qquad (38)$$

The corresponding contracted gain matrix has the following form

$$K_D = \begin{bmatrix} -0.7667 & -0.6133 & 0 \\ 0 & 0.1278 & 0.5530 \end{bmatrix}. \qquad (39)$$

The associated bound on the cost is $J_0 = 1.83$.

Centralized controller. The standard computation on the original system results in the controller

$$K = \begin{bmatrix} -5.8917 & -2.3209 & -1.4840 \\ 5.1895 & 0.9130 & -1.1156 \end{bmatrix}. \qquad (40)$$

The cost bound in this case is $J_0 = 1.42$.

The centralized control design case serves only as a reference to compare the cost bounds in both cases. The upper bound is little greater in the decentralized case because of given information structure constraints. All computations have been performed using Matlab LMI Control Toolbox [10].

5. CONCLUSION

A new set of the expansion-contraction relations extending the Inclusion Principle is proved for a class of linear continuous-time uncertain systems with state and control delays. Norm bounded arbitrarily time-varying uncertainties and a given point delay are considered. The resulting structural relations are easily extendable to polytopic systems with constant uncertainties. The presented inclusion relations are applied on the quadratic guaranteed cost control design. Conditions preserving the expansion-contraction relations for closed-loop systems including the equality of cost bounds have been proved. The guaranteed cost control design is performed using the LMI delay independent procedure in the expanded space and subsequently contracted into the original system. The results are specialized on the overlapping static output feedback design. A numerical illustrative example is supplied.

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REFERENCES

[1] L. Bakule, J. Rodellar, and J. Rossell: Generalized selection of complementary matrices in the inclusion principle. IEEE Trans. Automat. Control *45* (2000), 6, 1237–1243.

[2] L. Bakule, J. Rodellar, and J. Rossell: Structure of expansion-contraction matrices in the inclusion principle for dynamic systems. SIAM J. Matrix Analysis and Appl. *21* (2000), 4, 1136–1155.

[3] L. Bakule, J. Rodellar, and J. Rossell: Controllability-observability of expanded composite systems. Linear Algebra Appl. *332–334* (2001), 381–400.

[4] L. Bakule, J. Rodellar, and J. Rossell: Overlapping quadratic optimal control of linear time-varying commutative systems. SIAM J. Control Optim. *40* (2002), 5, 1611–1627.

[5] L. Bakule, J. Rodellar, and J. Rossell: Overlapping resilient H_∞ control for uncertain time-delayed systems. In: Proc. 44th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC'05), Sevilla 2005, pp. 2290–2295.

[6] L. Bakule, J. Rodellar, and J. Rossell: Overlapping guaranteed cost control for uncertain continuous-time delayed systems. In: Proc. 16th IFAC World Congress, Prague 2005.

[7] L. Bakule, J. Rodellar, and J. Rossell: Robust overlapping guaranteed cost control of uncertain state-delay discrete-time systems. IEEE Trans. Automat. Control *51* (2006), 12, 1943–1950.

[8] S. Esfahani and I. Petersen: An LMI approach to output-feedback-guaranteed cost control for uncertain time-delay systems. Internat. J. Robust and Nonlinear Control *10* (2000), 157–174.

[9] E. Fridman and U. Shaked: Parameter dependent stability and stabilization of uncertain time-delay systems. IEEE Trans. Automat. Control *48* (2003), 861–866.

[10] P. Gahinet, A. Nemirovski, A. Laub, and M. Chilali: The LMI Control Toolbox. The MathWorks, Inc., 1995.

[11] M. Ikeda and D. Šiljak: Overlapping decompositions, expansions and contractions of dynamic systems. Large Scale Systems *1* (1980), 1, 29–38.

[12] M. Ikeda and D. Šiljak: Overlapping decentralized control with input, state, and output inclusion. Control Theory and Advanced Technology *2* (1986), 2, 155–172.

[13] M. Ikeda, D. Šiljak, and D. White: Decentralized control with overlapping information sets. J. Optim. Theory Appl. *34* (1981), 2, 279–310.

[14] V. Kapila and W. Haddad: Robust stabilization with parametric uncertainty and time delay. J. Franklin Institute *336* (1999), 473–480.

[15] V. Kapila, W. Haddad, R. Erwin, and D. Bernstein: Robust controller synthesis via shifted parameter-dependent quadratic cost bounds. IEEE Trans. Automat. Control *43* (1998), 1003–1007.

[16] O. Moheimani and I. Petersen: Optimal quadratic guaranteed cost control of a class of uncertain time-delay systems. IEE Proc. Control Theory Appl. *44* (1997), 2, 183–188.

[17] H. Mukaidani: An LMI approach to guaranteed cost control for uncertain delay systems. IEEE Trans. Circuits and Systems – I: Fundamental Theory Appl. *50* (2003), 6, 795–800.

[18] W. Rugh: Linear System Theory. Second edition. Prentice Hall, Upper Saddle River, N. J. 1996.

[19] S. Stanković and D. Šiljak: Inclusion principle for linear time-varying systems. SIAM J. Control Optim. *42* (2003), 1, 321–341.

[20] D. Šiljak: Decentralized Control of Complex Systems. Academie Press, New York 1991.

[21] D. Šiljak and A. Zečević: Control of large-scale systems: Beyond decentralized feedback. Annual Reviews of Control *29* (2005), 169–179.

[22] A. Zečević and D. Šiljak: Design of robust static output feedback for large-scale systems. IEEE Trans. Automat. Control *49* (2004), 11, 2040–2044.

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