KYBERNETIKA — VOLUME *43* (2007), NUMBER 6, PAGES 903 – 912

ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO AN AREA–PRESERVING MOTION BY CRYSTALLINE CURVATURE

Shigetoshi Yazaki

Asymptotic behavior of solutions of an area-preserving crystalline curvature flow equation is investigated. In this equation, the area enclosed by the solution polygon is preserved, while its total interfacial crystalline energy keeps on decreasing. In the case where the initial polygon is essentially admissible and convex, if the maximal existence time is finite, then vanishing edges are essentially admissible edges. This is a contrast to the case where the initial polygon is admissible and convex: a solution polygon converges to the boundary of the Wulff shape without vanishing edges as time tends to infinity.

Keywords: essentially admissible polygon, crystalline curvature, the Wulff shape, isoperimetric inequality

AMS Subject Classification: 34A26, 34A34, 34K25, 39A12, 53A04, 74N05, 82D25

1. INTRODUCTION

The present paper is an extension of [12, Part I] in which it has been proved that the solution admissible polygon of an area-preserving crystalline curvature flow equation converges to the prescribed Wulff shape. In the present paper, the asymptotic behavior of solutions starting from essentially admissible polygons will be investigated. This flow is a generalized version introduced by Yazaki [11] in which we discussed the gradient flow of the total length functional of convex polygon keeping the area enclosed by the polygon constant, and showed that any polygon which evolves by this gradient flow converges to the circumscribed polygon of a circle; This result is corresponding to a semi-discrete version introduced by Gage [3].

The so-called curvature flow equation is a general term which describes a motion of curves in the plane (or surfaces in space) which change its shape in time and depend on its bend, especially on its curvature. It has been investigated by many scientists and mathematicians since the 1950's. At the end of 1980's, J. E. Taylor, and S. Angenent and M. E. Gurtin focused on motion of polygonal curves by crystalline curvature in the plane, and since then crystalline curvature flow equation has been studied under various kinds of evolution law by several authors. We refer the reader to the pioneer works Taylor [8, 9] and Angenent and Gurtin [2], and the surveys by Taylor, Cahn and Handwerker [10] and the books by Gurtin [5] for a geometric and physical background. Also one can find essentially the same method of crystalline as a numerical scheme for curvature flow equation in Roberts [7]. See Almgren and Taylor [1] for detailed history. Besides this crystalline strategy, other strategies by subdifferential and level-set method have been extensively studied. See Giga [4] and references therein.

Polygons. Let $\mathcal{P}$ be an N-sided convex polygon in the plane $\mathbb{R}^2$, and label the position vector of vertices $\boldsymbol{p}_i$ $(i = 1, 2, \ldots, N)$ in an anticlockwise order: $\mathcal{P} = \bigcup_{i=1}^{N} \mathcal{S}_i$, where $\mathcal{S}_i = [\boldsymbol{p}_i, \boldsymbol{p}_{i+1}]$ is the ith edge $(\boldsymbol{p}_{N+1} = \boldsymbol{p}_1)$. The length of $\mathcal{S}_i$ is $d_i = |\boldsymbol{p}_{i+1} - \boldsymbol{p}_i|$, and then the ith unit tangent vector is $\boldsymbol{t}_i = (\boldsymbol{p}_{i+1} - \boldsymbol{p}_i)/d_i$ and the ith unit outward normal vector is $\boldsymbol{n}_i = -\boldsymbol{t}_i^{\perp}$, where $(a, b)^{\perp} = (-b, a)$. We define a set of normal vectors of $\mathcal{P}$ by $\mathcal{N} = \{\boldsymbol{n}_1, \boldsymbol{n}_2, \ldots, \boldsymbol{n}_N\}$. Let θ_i be the exterior normal angle of $\mathcal{S}_i$ such as $\boldsymbol{n}_i = \boldsymbol{n}(\theta_i)$ and $\boldsymbol{t}_i = \boldsymbol{t}(\theta_i)$, where $\boldsymbol{n}(\theta) = (\cos\theta, \sin\theta)$ and $\boldsymbol{t}(\theta) = (-\sin\theta, \cos\theta)$. We define the ith hight function $h_i = \boldsymbol{p}_i \cdot \boldsymbol{n}_i = \boldsymbol{p}_{i+1} \cdot \boldsymbol{n}_i$. By using N-tuple $h = (h_1, h_2, \ldots, h_N)$, d_i is described as follows:

$$d_i[h] = -(\cot\vartheta_i + \cot\vartheta_{i+1})h_i + h_{i-1}\operatorname{cosec}\vartheta_i + h_{i+1}\operatorname{cosec}\vartheta_{i+1}, \tag{1}$$

where $\vartheta_i = \theta_i - \theta_{i-1}$ for $i = 1, 2, \ldots, N$. Note that $0 < \vartheta_i < \pi$ holds for all i.

Interfacial energy. In the field of material sciences and crystallography, we need to explain the anisotropy: phenomenon of interface motion which depends on the normal direction $\boldsymbol{n}$. To explain the anisotropy, it is convenient to define an interfacial energy on the interface or the curve which has line density $\gamma(\boldsymbol{n}) > 0$. The function $\gamma(\boldsymbol{n})$ can be extended to the function $\boldsymbol{x} \in \mathbb{R}^2$ by putting $\gamma(\boldsymbol{x}) = |\boldsymbol{x}|\gamma(\boldsymbol{x}/|\boldsymbol{x}|)$ if $\boldsymbol{x} \neq \boldsymbol{0}$, otherwise $\gamma(\boldsymbol{0}) = 0$. This extension is called the extension of positively homogeneous of degree 1, since $\gamma(\lambda\boldsymbol{x}) = \lambda\gamma(\boldsymbol{x})$ holds for $\lambda \geq 0$ and $\boldsymbol{x} \in \mathbb{R}^2$. We will use the same notation γ for the extended function. To observe the characteristic of γ, the following *Frank diagram* is useful: $\mathcal{F}_\gamma = \{\boldsymbol{n}(\theta)/\gamma(\boldsymbol{n}(\theta));\ \theta \in S^1\} = \{\boldsymbol{x} \in \mathbb{R}^2;\ \gamma(\boldsymbol{x}) = 1\}$. If the Frank diagram $\mathcal{F}_\gamma$ is a convex polygon, γ is called *crystalline energy*. When $\mathcal{F}_\gamma$ is a J-sided convex polygon, there exists a set of angles $\{\phi_i \,|\, \phi_1 < \phi_2 < \cdots < \phi_J < \phi_1 + 2\pi\}$ such that the position vectors of vertices are labeled $\boldsymbol{n}(\phi_i)/\gamma(\boldsymbol{n}(\phi_i))$ in an anticlockwise order $(\phi_{J+1} = \phi_1)$:

$$\mathcal{F}_\gamma = \bigcup_{i=1}^{J} \left[\frac{\boldsymbol{\nu}_i}{\gamma(\boldsymbol{\nu}_i)}, \frac{\boldsymbol{\nu}_{i+1}}{\gamma(\boldsymbol{\nu}_{i+1})}\right].$$

Here and hereafter, we denote $\boldsymbol{\nu}_i = \boldsymbol{n}(\phi_i)$ for all i. In this case, the *Wulff shape* $\mathcal{W}_\gamma = \bigcap_{\theta \in S^1}\{\boldsymbol{x} \in \mathbb{R}^2;\ \boldsymbol{x} \cdot \boldsymbol{n}(\theta) \leq \gamma(\boldsymbol{n}(\theta))\}$ is also a J-sided convex polygon with the outward normal vector of the ith edge being $\boldsymbol{\nu}_i$:

$$\mathcal{W}_\gamma = \bigcap_{i=1}^{J} \{\boldsymbol{x} \in \mathbb{R}^2;\ \boldsymbol{x} \cdot \boldsymbol{\nu}_i \leq \gamma(\boldsymbol{\nu}_i)\}.$$

We define a set of normal vectors of $\mathcal{W}_\gamma$ by $\mathcal{N}_\gamma = \{\boldsymbol{\nu}_1, \boldsymbol{\nu}_2, \ldots, \boldsymbol{\nu}_J\}$.

Admissibility. Following [6], we call $\mathcal{P}$ *essentially admissible* if and only if the consecutive outward unit normal vectors $\boldsymbol{n}_i$, $\boldsymbol{n}_{i+1} \in \mathcal{N}$ ($\boldsymbol{n}_{N+1} = \boldsymbol{n}_1$) satisfy $\boldsymbol{\eta}/|\boldsymbol{\eta}| \notin \mathcal{N}_\gamma$, where $\boldsymbol{\eta} = (1-\lambda)\boldsymbol{n}_i + \lambda \boldsymbol{n}_{i+1}$ for $\lambda \in (0,1)$ and $i = 1, 2, \ldots, N$. Note that $\mathcal{P}$ is an essentially admissible convex polygon if and only if $\mathcal{N} \supseteq \mathcal{N}_\gamma$ holds. We call $\mathcal{P}$ *admissible* if and only if $\mathcal{P}$ is an essentially admissible polygon and $\mathcal{N} = \mathcal{N}_\gamma$ holds. In other words, $\mathcal{P}$ is an admissible convex polygon if and only if $\boldsymbol{n}_i = \boldsymbol{\nu}_i$ holds for all $i = 1, 2, \ldots, N = J$.

Gradient of the total interfacial energy. Let $\mathcal{P}$ be an essentially admissible N-sided convex polygon with the N-tuple of hight functions $h = (h_1, h_2, \ldots, h_N)$. Then the total interfacial (crystalline) energy on $\mathcal{P}$ is

$$\mathcal{E}_\gamma[h] = \sum_{i=1}^{N} \gamma(\boldsymbol{n}_i) d_i[h].$$

For two N-tuples $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_N), \psi = (\psi_1, \psi_2, \ldots, \psi_N) \in \mathbb{R}^N$, let us define the inner product on $\mathcal{P}$ as $(\varphi, \psi)_2 = \sum_{i=1}^{N} \varphi_i \psi_i d_i[h]$. Furthermore, we define the rate of variation of $\mathcal{E}_\gamma[h]$ in the direction φ and the first variation $\delta\mathcal{E}_\gamma[h]/\delta h$ as follows:

$$\frac{\delta \mathcal{E}_\gamma[h]}{\delta \varphi} = \frac{d}{d\varepsilon} \mathcal{E}_\gamma[h + \varphi] \bigg|_{\varepsilon=0} = \operatorname{grad} \mathcal{E}_\gamma[h] \cdot \varphi = \left(\frac{\delta \mathcal{E}_\gamma[h]}{\delta h}, \varphi \right)_2 .$$

Crystalline curvature. The first variation of $\mathcal{E}_\gamma[h]$ of $\mathcal{P}$ at $\mathcal{S}_i$ is

$$\frac{\delta \mathcal{E}_\gamma[h]}{\delta \varphi} = \sum_{i=1}^{N} \gamma_i d_i[\varphi] = \sum_{i=1}^{N} d_i[\gamma] \varphi_i = \sum_{i=1}^{N} \frac{d_i[\gamma]}{d_i[h]} \varphi_i d_i[h], \quad \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_N),$$

where $\gamma_i = \gamma(\boldsymbol{n}_i)$ for all i. Hence we have $(\delta\mathcal{E}_\gamma[h]/\delta h)_i = d_i[\gamma]/d_i[h]$ for all i in this metric $(\cdot,\cdot)_2$. This quantity is called *crystalline curvature* on the ith edge $\mathcal{S}_i$, and we denote it by $\Lambda_\gamma(\boldsymbol{n}_i) = d_i[\gamma]/d_i[h]$. The numerator $d_i[\gamma]$ is described as

$$d_i[\gamma] = l_\gamma(\boldsymbol{n}_i),$$

where $l_\gamma(\boldsymbol{n})$ is the length of the jth edge of $\mathcal{W}_\gamma$ if $\boldsymbol{n} = \boldsymbol{\nu}_j$ for some j, otherwise $l_\gamma(\boldsymbol{n}) = 0$. Therefore if $\mathcal{P} = \mathcal{W}_\gamma$, then the crystalline curvature is 1.

An area-preserving motion by crystalline curvature. The enclosed area $\mathcal{A}$ of $\mathcal{P}$ is given by

$$\mathcal{A}[h] = \frac{1}{2} \sum_{i=1}^{N} h_i d_i[h].$$

Then the rate of variation of $\mathcal{A}[h]$ in the direction φ is

$$\frac{\delta \mathcal{A}[h]}{\delta \varphi} = \frac{d}{d\varepsilon} \mathcal{A}[h + \varphi] \bigg|_{\varepsilon=0} = \sum_{i=1}^{N} \varphi_i d_i[h].$$

By taking $\varphi_i = -(\delta\mathcal{E}_\gamma[h]/\delta h)_i = -\Lambda_\gamma(\boldsymbol{n}_i)$, we have $\delta\mathcal{A}[h]/\delta\varphi = -\sum_{i=1}^N \Lambda_\gamma(\boldsymbol{n}_i)d_i[h]$. Hence by taking $\varphi_i = \overline{\Lambda}_\gamma - \Lambda_\gamma(\boldsymbol{n}_i)$, we have $\delta\mathcal{A}[h]/\delta\varphi = 0$. Here

$$\overline{\Lambda}_\gamma = \frac{\sum_{i=1}^N \Lambda_\gamma(\boldsymbol{n}_i)d_i[h]}{\sum_{k=1}^N d_k} = \frac{\sum_{i=1}^N l_\gamma(\boldsymbol{n}_i)}{\mathcal{L}}$$

is the average of the crystalline curvature, and $\mathcal{L}$ is the total length of $\mathcal{P}$, i. e., $\mathcal{L} = \sum_{i=1}^N d_i$. Thus we have the gradient flow of $\mathcal{E}_\gamma$ along $\mathcal{P}$ which encloses a fixed area:

$$V_i = \overline{\Lambda}_\gamma - \Lambda_\gamma(\boldsymbol{n}_i), \quad i = 1, 2, \dots, N, \tag{2}$$

where $V_i(t) = \dot{h}_i(t)$ is the normal velocity on $\mathcal{S}_i$ in the direction $\boldsymbol{n}_i$ at the time t. Here and hereafter, we denote $\dot{u}$ by du/dt. From (1), the time derivative of $d_i(t) = d_i[h]$ is given by

$$\dot{d}_i = -(\cot\vartheta_i + \cot\vartheta_{i+1})V_i + V_{i-1}\operatorname{cosec}\vartheta_i + V_{i+1}\operatorname{cosec}\vartheta_{i+1} \tag{3}$$

for $i = 1, 2, \dots, N$. Note that (2) and (3) are equivalent each other.

Problem 1. For a given essentially admissible closed curve $\mathcal{P}_0$, find a family of essentially admissible curves $\{\mathcal{P}(t)\}_{0\le t<T}$ satisfying (2) (or (3)) with $\mathcal{P}(0) = \mathcal{P}_0$. Since (3) are the system of ordinary differential equations, the maximal existence time is positive $T > 0$.

Main results. What might happen to $\mathcal{P}(t)$ as t tends to $T \le \infty$? For this question, we have the following three results. The first result is the case where motion is isotropic and polygon is admissible.

Proposition A. Let the interfacial energy be isotropic $\gamma \equiv 1$. Assume the initial polygon $\mathcal{P}_0$ is an N-sided admissible convex polygon. Then a solution admissible polygon $\mathcal{P}(t)$ of Problem 1 exists globally in time keeping the area enclosed by the polygon constant $\mathcal{A}$, and $\mathcal{P}(t)$ converges to the shape of the boundary of the Wulff shape $\partial\mathcal{W}_{\gamma_*}$ in the Hausdorff metric as t tends to infinity, where $\gamma_*(\boldsymbol{n}_i) \equiv \sqrt{2\mathcal{A}/\sum_{k=1}^N l_1(\boldsymbol{n}_k)}$ is constant. In particular, if $\mathcal{P}_0$ is centrally symmetric with respect to the origin, then we have an exponential rate of convergence.

This proposition is proved by Yazaki [11] by using the isoperimetric inequality and the theory of dynamical systems. We note that $\partial\mathcal{W}_{\gamma_*}$ is the circumscribed polygon of a circle with radius γ_*, and then this result is a semi-discrete version introduced by Gage [3].

The second result is the case where motion is anisotropic and polygon is admissible.

Proposition B. Let the crystalline energy be $\gamma > 0$. Assume the initial polygon $\mathcal{P}_0$ is an N-sided admissible convex polygon. Then a solution admissible polygon $\mathcal{P}(t)$ of Problem 1 exists globally in time keeping the area enclosed by the polygon constant $\mathcal{A}$, and $\mathcal{P}(t)$ converges to the shape of the boundary of the Wulff shape $\partial\mathcal{W}_{\gamma_*}$ in the Hausdorff metric as t tends to infinity, where $\gamma_*(\boldsymbol{n}_i) = \gamma(\boldsymbol{n}_i)/W$, $W = \sqrt{|\mathcal{W}_\gamma|/\mathcal{A}}$ for all $i = 1, 2, \ldots, N$ and $|\mathcal{W}_\gamma| = \sum_{k=1}^{N} \gamma(\boldsymbol{n}_k) l_\gamma(\boldsymbol{n}_k)/2$ is enclosed area of $\mathcal{W}_\gamma$.

This proposition is proved in Yazaki [12, Part I] by using the anisoperimetric inequality or Brünn and Minkowski's inequality and the theory of dynamical systems which is a similar technique as in Yazaki [11].

The last result is the case where motion is anisotropic and polygon is essentially admissible.

Theorem C. Let the crystalline energy be $\gamma > 0$. Assume the initial polygon $\mathcal{P}_0$ is an N-sided essentially admissible convex polygon. If the maximal existence time of a solution essentially admissible polygon $\mathcal{P}(t)$ of Problem 1 is finite $T < \infty$, then there exists the ith edge $\mathcal{S}_i$ such that $\lim_{t\to T} d_i(t) = 0$ and $l_\gamma(\boldsymbol{n}_i) = 0$ hold. That is, the normal vector of vanishing edge does not belong to $\mathcal{N}_\gamma$, and $\inf_{0<t<T} d_k(t) > 0$ holds for all $\boldsymbol{n}_k \in \mathcal{N}_\gamma$.

For any essentially admissible convex polygon $\mathcal{P}_0$, is T a finite value? This is still open. If the answer of this question is yes, then we have the finite time sequence $T_1 < T_2 < \cdots < T_M$ such that $\mathcal{P}(T_i)$ is essentially admissible for $i = 1, 2, \ldots, M-1$ and $\mathcal{P}(T_M)$ is admissible. In the general case where $V_i = g(\boldsymbol{n}_i, \Lambda_\gamma(\boldsymbol{n}_i))$ for all i under certain conditions of g, the answer of the above question is yes. See Yazaki [13]. However, g does not include $\overline{\Lambda}_\gamma$.

In the next section, we will prove this theorem.

2. PROOF OF THEOREM C

Suppose the assumption of Theorem C. From the general theory of ordinary differential equations, a solution of (3) exists uniquely and locally in time. Let $T > 0$ be the maximal existence time.

Lemma 1. Assume that the maximal existence time is finite $T < \infty$. Then there exists $i_0 \in \{1, 2, \ldots, N\}$ such that $\liminf_{t\to T} d_{i_0}(t) = 0$ holds.

P r o o f. By the CBS inequality, the time derivative of the total interfacial energy $\mathcal{E}_\gamma(t) = \mathcal{E}_\gamma[h]$ is decreasing in time:

$$\dot{\mathcal{E}}_\gamma = \left.\frac{\delta\mathcal{E}_\gamma[h]}{\delta\varphi}\right|_{\varphi=\dot{h}} = \sum_{i=1}^{N} \Lambda_\gamma(\boldsymbol{n}_i)\dot{h}_i d_i = \sum_{i=1}^{N} \Lambda_\gamma(\boldsymbol{n}_i) V_i d_i \le 0.$$

Hence we have a finite upper bound of $d_i(t)$, since

$$\mathcal{E}_\gamma(0) \geq \mathcal{E}_\gamma(t) = \sum_{j=1}^{N} \gamma(\boldsymbol{n}_j) d_j(t) \geq \min_{1\leq k\leq N} \gamma(\boldsymbol{n}_k)\mathcal{L}(t) \geq \min_{1\leq k\leq N} \gamma(\boldsymbol{n}_k) d_i(t) \tag{4}$$

holds for all $i = 1, 2, \ldots, N$ and $t \in [0, T]$. Therefore diameter of $\mathcal{P}(t)$ is finite for $t \in [0, T]$. To prove the lemma, we assume that the lower bound of $d_i(t)$ is positive, i. e., $\inf_{0<t<T} d_i(t) > 0$ holds for all i. Then from (3), we have $\sup_{0<t<T} |\dot{d}_i(t)| < \infty$ for all i. Therefore a solution polygon $\mathcal{P}(t)$ exists up to $t = T$, and $\mathcal{P}(T)$ is an N-sided essentially admissible convex polygon. Hence from the general theory of ordinary differential equations, a solution polygon exists after T. This is a contradiction. Therefore there exists at least one edge $\mathcal{S}_{i_0}$ such that $\liminf_{t\to T} d_{i_0}(t) = 0$ holds. □

Lemma 2. Assume the same assumption as in Lemma 1. Then $\lim_{t\to T} d_{i_0}(t) = 0$ holds.

Proof. If $\lim_{t\to T} d_{i_0}(t) = 0$ does not hold, then there exists a positive constant $D > 0$ such that $\limsup_{t\to T} d_{i_0}(t) = D$ holds. Hence there exist time sequences $\{t_m\}$ and $\{s_m\}$ converging to T as $m \to \infty$ such that $\lim_{m\to\infty} d_{i_0}(t_m) = D$ and $\lim_{m\to\infty} d_{i_0}(s_m) = 0$ hold. Without loss of generality, we can assume that $s_m < t_m < s_{m+1}$ and that $d_{i_0}(s_m) < D/2 < d_{i_0}(t_m)$. Put $r_m = \sup\{t < t_m;\ d_{i_0}(t) < D/2\}$. Then $r_m \in (s_m, t_m)$ and $d_{i_0} \geq D/2$ holds for $t \in [r_m, t_m]$. By mean value theorem, there exists $\mu_m \in (r_m, t_m)$ such that $d_{i_0}(\mu_m) \geq D/2$ and $\dot{d}_{i_0}(\mu_m) = (d_{i_0}(t_m) - d_{i_0}(r_m))/(t_m - r_m)$ hold. This yields $\lim_{m\to\infty} \dot{d}_{i_0}(\mu_m) = \infty$.

We will use repeatedly the following isoperimetric inequality:

$$\frac{\mathcal{L}^2}{4c\mathcal{A}} \geq 1, \quad c = \sum_{j=1}^{N} \tan\frac{\vartheta_j}{2}. \tag{5}$$

See e. g., Yazaki [11, Lemma 2.4] for the proof. From (3), we have

$$\dot{d}_{i_0}(\mu_m) \leq \frac{a l_\gamma(\boldsymbol{n}_{i_0})}{d_{i_0}(\mu_m)} + \frac{b\sum_{k=1}^{N} l_\gamma(\boldsymbol{n}_k)}{\mathcal{L}(\mu_m)} \leq \frac{2a l_\gamma(\boldsymbol{n}_{i_0})}{D} + \frac{b\sum_{k=1}^{N} l_\gamma(\boldsymbol{n}_k)}{2\sqrt{c\mathcal{A}}},$$

where $a = 1/\sin\vartheta_{i_0} + 1/\sin\vartheta_{i_0+1}$ and $b = \tan(\vartheta_{i_0}/2) + \tan(\vartheta_{i_0+1}/2)$. This contradicts to $\lim_{m\to\infty} \dot{d}_{i_0}(\mu_m) = \infty$. □

Lemma 3. Assume the same assumption as in Lemma 2. Put

$$\mathcal{Q} = \left\{\boldsymbol{n}_i \in \mathcal{N};\ \lim_{t\to T} d_i(t) = 0\right\}.$$

Then $\mathcal{Q} \subseteq \mathcal{N}\backslash\mathcal{N}_\gamma$ holds.

Proof. By the isoperimetric inequality (5), it holds that $\max_{1\leq i\leq N} d_i(t) \geq 2\sqrt{c\mathcal{A}}/N$, and then there exists at least one k such that $\inf_{0<t<T} d_k(t) > 0$. Therefore $\mathcal{Q} \neq \mathcal{N}$ holds.

One can represent $\mathcal{Q}$ as a disjoint sum of $\mathcal{Q}_k$; namely $\mathcal{Q} = \bigoplus_k \mathcal{Q}_k$, where $\mathcal{Q}_k$'s are maximal subsets having m_k consecutive elements $\boldsymbol{n}_j$ of the form

$$\mathcal{Q}_k = \{\boldsymbol{n}_i \in \mathcal{Q};\ i = j_k, j_k+1, \dots, j_k+m_k-1\},$$

with the boundary of $\mathcal{Q}_k$: $\partial\mathcal{Q}_k = \{\boldsymbol{n}_i;\ i = j_k-1, j_k+m_k\}$. By the definition, $m_k \geq 1$ holds for each k. Since $\mathcal{Q} \neq \mathcal{N}$, we have $\partial\mathcal{Q}_k \subseteq \mathcal{N}\backslash\mathcal{Q}$, i. e., $\inf_{0<t<T} d_i(t) > 0$ holds for $\boldsymbol{n}_i \in \bigoplus_k \partial\mathcal{Q}_k$.

Let $L_i(t)$ be the straight line extending the ith edge $\mathcal{S}_i$ of $\mathcal{P}(t)$ for $\boldsymbol{n}_i \in \mathcal{N}$, and let $\boldsymbol{p}_i(t)$ be the intersection point of $L_i(t)$ and $L_{i-1}(t)$, i. e., $\boldsymbol{p}_i(t)$ is the ith vertex of $\mathcal{P}(t)$ and is described as follows:

$$\boldsymbol{p}_i = h_i\boldsymbol{n}_i + \frac{h_{i-1} - (\boldsymbol{n}_{i-1}\cdot\boldsymbol{n}_i)h_i}{\boldsymbol{n}_{i-1}\cdot\boldsymbol{t}_i}\boldsymbol{t}_i = h_i\boldsymbol{n}_i + \frac{h_i\cos\vartheta_i - h_{i-1}}{\sin\vartheta_i}\boldsymbol{t}_i. \tag{6}$$

We denote $p = j_k - 1$ and $q = j_k + m_k$ for simplicity.

By the isoperimetric inequality (5), we have

$$V_i = \frac{\sum_{k=1}^N l_\gamma(\boldsymbol{n}_k)}{\mathcal{L}} - \frac{l_\gamma(\boldsymbol{n}_i)}{d_i} \leq \frac{\sum_{k=1}^N l_\gamma(\boldsymbol{n}_k)}{2\sqrt{c\mathcal{A}}} - \frac{l_\gamma(\boldsymbol{n}_i)}{d_i}, \quad p < i < q.$$

Therefore there exists a constant $\delta \in (0, T)$ such that $\sup_{T_\delta<t<T} V_i(t) < 0$ for $p < i < q$ and $T_\delta = T - \delta$. Hence by the definition of $\mathcal{Q}_k$, vertices $\boldsymbol{p}_{p+1}(t), \dots, \boldsymbol{p}_q(t)$ converge to a point $\boldsymbol{p}_*$ as $t \to T$:

$$\boldsymbol{p}_* \in \bigcap_{T_\delta<t<T} \bigcap_{p<i<q} \left\{\boldsymbol{x} \in \mathbb{R}^2;\ (\boldsymbol{p}_i(t) - \boldsymbol{x})\cdot\boldsymbol{n}_i \geq 0\right\}.$$

Note that the intersection is taken over $p < i < q$ since the sign of V_p and V_q is unknown. We denote $|\mathcal{Q}_k| = |\theta_p - \theta_q|$.

Claim: $|\mathcal{Q}_k| \leq \pi$ holds.

Suppose $|\mathcal{Q}_k| > \pi$. Without loss of generality, we may assume that $\pi < \theta_q - \theta_p < 2\pi$. Then we have

$$(\boldsymbol{p}_q - \boldsymbol{p}_{p+1})\cdot\boldsymbol{n}_q = (\boldsymbol{p}_q - \boldsymbol{p}_p)\cdot\boldsymbol{n}_q - d_p(\boldsymbol{t}_p\cdot\boldsymbol{n}_q) \geq -\inf_{T_\delta<t<T} d_p(t)S > 0,$$

since $S = \sin(\theta_q - \theta_p) < 0$. Therefore $\inf_{T_\delta<t<T}(\boldsymbol{p}_q(t) - \boldsymbol{p}_{p+1}(t))\cdot\boldsymbol{n}_q > 0$ holds, which contradicts $\lim_{t\to T}\boldsymbol{p}_i(t) = \boldsymbol{p}_*$ for $i = p+1, q$. Hence assertion holds.

Claim: $|\mathcal{Q}_k| < \pi$ holds.

Suppose $|\mathcal{Q}_k| = \pi$. By a geometric inspection, there exist exactly two sets $\mathcal{Q}_1, \mathcal{Q}_2$ such that $\mathcal{Q} = \bigoplus_{k=1}^2 \mathcal{Q}_k$ and $\mathcal{N}\backslash\mathcal{Q} = \{\theta_p, \theta_q\}$ hold. Then $\lim_{t\to T} d_i(t) = 0$ holds for all $i \neq p, q$, and $\inf_{0<t<T} d_i(t) > 0$ holds for $i = p, q$. For any choice of $\boldsymbol{n}_k \in \mathcal{Q}_1$ and $\boldsymbol{n}_l \in \mathcal{Q}_2$, one can construct a trapezoid surrounded by four lines $L_p(t)$, $L_k(t)$, $L_q(t)$ and $L_l(t)$. Here we have assumed that $\theta_p < \theta_k < \theta_q = \theta_p + \pi < \theta_l < \theta_p + 2\pi$, without loss of generality. Then this trapezoid includes the enclosed region of $\mathcal{P}(t)$ at each t. Let $w(t)$ be the width between $L_p(t)$ and $L_q(t)$. Then the area of this trapezoid is greater than or equal to $\mathcal{A}$, i. e., $(D_p(t)+D_q(t))w(t)/2 \geq \mathcal{A}$, where $D_i(t)$ is the length of the edge of this trapezoid on the line $L_i(t)$. Let $\boldsymbol{y}_{ij}(t)$ be the intersection point of $L_i(t)$ and $L_j(t)$:

$$\boldsymbol{y}_{ij}(t) = \boldsymbol{p}_{i+1}(t) + a_{ij}(t)\boldsymbol{t}_i, \quad a_{ij}(t) = \frac{(\boldsymbol{p}_j(t) - \boldsymbol{p}_{i+1}(t)) \cdot (\boldsymbol{t}_i - \mu_{ij}\boldsymbol{t}_j)}{1-{\mu_{ij}}^2},$$

where $\mu_{ij} = \boldsymbol{t}_i \cdot \boldsymbol{t}_j = \cos(\theta_i - \theta_j)$. Then we have

$$D_p = (\boldsymbol{y}_{pk} - \boldsymbol{y}_{lp}) \cdot \boldsymbol{t}_p = (\boldsymbol{p}_{p+1} - \boldsymbol{p}_{l+1}) \cdot \boldsymbol{t}_p + a_{pk} - a_{lp}\mu_{lp}.$$

Note that $|\mu_{pk}| < 1$ and $|\mu_{lp}| < 1$ hold. Since $\boldsymbol{p}_j = \boldsymbol{p}_i + \sum_{k=i}^{j-1} d_k \boldsymbol{t}_k$, we have $|\boldsymbol{p}_j - \boldsymbol{p}_i| \leq \mathcal{L}$ for any $j > i$ (mod N). Hence $D_p \leq C\mathcal{L}$ and in the same way $D_q \leq C\mathcal{L}$ hold with a positive constant C depending only on $\mathcal{N}$. By (4), we obtain $D_i \leq C$ for $i = p, q$ with a positive constant C depending only on $\mathcal{N}$, $\mathcal{P}_0$ and γ. However, since $\lim_{t\to T} w(t) = 0$ and $\mathcal{A}$ is constant, $\lim_{t\to T} D_i(t) = \infty$ holds for $i = p, q$. This is a contradiction. Hence assertion holds.

Let $\boldsymbol{y}(t)$ be the intersection point of $L_p(t)$ and $L_q(t)$: $\boldsymbol{y}(t) = \boldsymbol{y}_{pq}(t)$. Note that $|\mu_{pq}| < 1$ holds since $0 < |\theta_p - \theta_q| < \pi$, and that $\boldsymbol{y}(t)$ converges to $\boldsymbol{p}_*$ as $t \to T$.

By (6), the time derivative of the ith vertex is

$$\dot{\boldsymbol{p}}_i = V_{i-1}\boldsymbol{n}_{i-1} + \frac{V_i - V_{i-1}\cos\vartheta_i}{\sin\vartheta_i}\boldsymbol{t}_{i-1}, \tag{7}$$

$$= V_i\boldsymbol{n}_i + \frac{V_i\cos\vartheta_i - V_{i-1}}{\sin\vartheta_i}\boldsymbol{t}_i, \tag{8}$$

for $i = 1, 2, \ldots, N$. Note that (2), (3) and (7), (8) are equivalent each other. By using $\dot{\boldsymbol{p}}_{p+1}$ with (7) and $\dot{\boldsymbol{p}}_q$ with (8), we have

$$\dot{\boldsymbol{y}} = V_p\boldsymbol{n}_p + \frac{\cos(\theta_q - \theta_p)V_p - V_q}{\sin(\theta_q - \theta_p)}\boldsymbol{t}_p.$$

Claim: Either $\boldsymbol{n}_p \in \mathcal{N}_\gamma$ or $\boldsymbol{n}_q \in \mathcal{N}_\gamma$ hold.

Suppose that $\boldsymbol{n}_i \in \mathcal{N}\backslash\mathcal{N}_\gamma$ holds for $i = p, q$. Then we have $V_i = \overline{\Lambda}_\gamma$ for $i = p, q$ and

$$\dot{\boldsymbol{y}} = \overline{\Lambda}_\gamma\boldsymbol{n}_p - C\boldsymbol{t}_p, \quad C = \overline{\Lambda}_\gamma\frac{1-\cos(\theta_q - \theta_p)}{\sin(\theta_q - \theta_p)}.$$

Hence $\dot{\boldsymbol{y}} \cdot \boldsymbol{n}_p = \overline{\Lambda}_\gamma > 0$ and $\dot{\boldsymbol{y}} \cdot \boldsymbol{t}_p = -C < 0$ hold, and $\boldsymbol{p}_* = \boldsymbol{y}(T)$ is in the region $\{\boldsymbol{x} \in \mathbb{R}^2;\ \boldsymbol{z} \cdot \boldsymbol{n}_p > 0,\ \boldsymbol{z} \cdot \boldsymbol{t}_p < 0,\ \boldsymbol{z} = \boldsymbol{x} - \boldsymbol{y}(T_\delta)\}$. On the other hand, $\dot{\boldsymbol{y}} \cdot \boldsymbol{n}_q = \overline{\Lambda}_\gamma > 0$ and $\dot{\boldsymbol{y}} \cdot \boldsymbol{t}_q = C > 0$ hold, and $\boldsymbol{p}_* = \boldsymbol{y}(T)$ is also in the region $\{\boldsymbol{x} \in \mathbb{R}^2;\ \boldsymbol{z} \cdot \boldsymbol{n}_q > 0,\ \boldsymbol{z} \cdot \boldsymbol{t}_q > 0,\ \boldsymbol{z} = \boldsymbol{x} - \boldsymbol{y}(T_\delta)\}$. This is a contradiction. Hence assertion holds.

Since $\boldsymbol{n}_p, \boldsymbol{n}_q \notin \mathcal{Q}$, $\inf_{T_\delta<t<T} d_i(t) > 0$ holds for $i = p, q$, and by the isoperimetric inequality (5), $\sup_{T_\delta<t<T} |V_i(t)| < \infty$ holds for $i = p, q$. Then there is a constant C_* such that $\sup_{T_\delta<t<T} |\dot{\boldsymbol{y}}(t)| \leq C_*$ holds.

Suppose that $\mathcal{Q} \subseteq \mathcal{N}\backslash\mathcal{N}_\gamma$ does not hold. Then we may choose a k such that $\mathcal{Q}_k \cap \mathcal{N}_\gamma \neq \emptyset$. Hence there exists at least one normal vector, say $\boldsymbol{n}_r \in \mathcal{Q}_k \cap \mathcal{N}_\gamma$, such that $p < r < q$ holds, and $\sup_{T_\delta < t < T} V_r(t) < 0$ and $\lim_{t\to T} V_r(t) = -\infty$ hold. We define

$$a(t) = (\boldsymbol{y}(t) - \boldsymbol{p}_*) \cdot \boldsymbol{n}_r, \quad b(t) = \mathrm{dist}(\boldsymbol{p}_*, L_r(t)) = (\boldsymbol{p}_r(t) - \boldsymbol{p}_*) \cdot \boldsymbol{n}_r.$$

Then $a(t) \geq b(t)$ holds for $t \in (T_\delta, T)$ and $\lim_{t\to T} a(t) = \lim_{t\to T} b(t) = 0$ holds.

Therefore by $\dot{a}(t) = -\dot{\boldsymbol{y}}(t) \cdot \boldsymbol{n}_r$, $|\dot{a}(t)| \leq C_*$ and $\dot{b} = V_r < 0$, for $t \in (T_\delta, T)$ there exists $\eta \in (t, T)$ such that

$$0 < -\int_t^T V_r(\tau)\,\mathrm{d}\tau = -\int_t^T \dot{b}(\tau)\,\mathrm{d}\tau = b(t) \leq a(t) = -\dot{a}(\eta)(T-t) \leq C_*(T-t).$$

This contradicts the fact $V_r \to -\infty$ as $t \to T$.

Hence $\mathcal{Q}_k \cap \mathcal{N}_\gamma = \emptyset$ for all k, i.e., $\mathcal{Q} \subseteq \mathcal{N}\backslash\mathcal{N}_\gamma$ holds. □

Proof of Theorem C. By Lemma 2, if the maximal existence time if finite $T < \infty$, then there exists at least one edge $\mathcal{S}_i$ such that $\lim_{t\to T} d_i(t) = 0$ holds. Furthermore, Lemma 3 follows that if the ith edge $\mathcal{S}_i$ disappears at T, then $\boldsymbol{n}_i \in \mathcal{N}\backslash\mathcal{N}_\gamma$ holds, and $\inf_{0<t<T} d_k(t) > 0$ holds for all $\boldsymbol{n}_k \in \mathcal{N}_\gamma$. □

ACKNOWLEDGEMENT

The author would like to thank the organizers of Czech–Japanese Seminar in Applied Mathematics 2006 which was held while the author was visiting Czech Republic within the period from August 18 to September 19, 2006; this visit was sponsored by Czech Technical University in Prague, Faculty of Nuclear Sciences and Physical Engineering within the Jindřich Nečas Center for Mathematical Modeling (Project of the Czech Ministry of Education, Youth and Sports, project No. LC 06052). Furthermore, the author is grateful to the anonymous referee for her or his comments. The present research is/was supported by Grant-in-Aid for Encouragement of Young Scientists No. 17740063.

(Received November 30, 2006.)

REFERENCES

[1] F. Almgren and J. E. Taylor: Flat flow is motion by crystalline curvature for curves with crystalline energies. J. Differential Geom. *42* (1995), 1–22.

[2] S. Angenent and M. E. Gurtin: Multiphase thermomechanics with interfacial structure, 2. Evolution of an isothermal interface. Arch. Rational Mech. Anal. *108* (1989), 323–391.

[3] M. Gage: On an area-preserving evolution equations for plane curves. Contemp. Math. *51* (1986), 51–62.

[4] Y. Giga: Anisotropic curvature effects in interface dynamics. Sūgaku *52* (2000), 113–127; English transl., Sūgaku Expositions *16* (2003), 135–152.

[5] M. E. Gurtin: Thermomechanics of Evolving Phase Boundaries in the Plane. Clarendon Press, Oxford 1993.

[6] H. Hontani, M.-H. Giga, Y. Giga, and K. Deguchi: Expanding selfsimilar solutions of a crystalline flow with applications to contour figure analysis. Discrete Appl. Math. *147* (2005), 265–285.

[7] S. Roberts: A line element algorithm for curve flow problems in the plane. CMA Research Report *58* (1989); J. Austral. Math. Soc. Ser. B *35* (1993), 244–261.
[8] J. E. Taylor: Constructions and conjectures in crystalline nondifferential geometry. In: Proc. Conference on Differential Geometry, Rio de Janeiro, Pitman Monographs Surveys Pure Appl. Math. *52* (1991), 321–336, Pitman London.
[9] J. E. Taylor: Motion of curves by crystalline curvature, including triple junctions and boundary points. Diff. Geom.: partial diff. eqs. on manifolds (Los Angeles, CA, 1990), Proc. Sympos. Pure Math., *54* (1993), Part I, 417–438, AMS, Providence.
[10] J. E. Taylor, J. Cahn, and C. Handwerker: Geometric models of crystal growth. Acta Metall. *40* (1992), 1443–1474.
[11] S. Yazaki: On an area-preserving crystalline motion. Calc. Var. *14* (2002), 85–105.
[12] S. Yazaki: On an anisotropic area-preserving crystalline motion and motion of non-admissible polygons by crystalline curvature. Sūrikaisekikenkyūsho Kōkyūroku *1356* (2004), 44–58.
[13] S. Yazaki: Motion of nonadmissible convex polygons by crystalline curvature. Publ. Res. Inst. Math. Sci. *43* (2007), 155–170.

Shigetoshi Yazaki, Faculty of Engineering, University of Miyazaki, 1-1 Gakuen Kibanadai Nishi, Miyazaki 889-2192. Japan.
e-mail: yazaki@cc.miyazaki-u.ac.jp