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COMPUTATIONAL TECHNIQUE FOR TREATING THE NONLINEAR BLACK–SCHOLES EQUATION WITH THE EFFECT OF TRANSACTION COSTS

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We deal with numerical computation of the nonlinear partial differential equations (PDEs) of Black–Scholes type which incorporate the effect of transaction costs. Our proposed technique surmounts the difficulty of infinite domains and unbounded values of the solutions. Numerical implementation shows the validity of our scheme.

Keywords: transaction costs, nonlinear partial differential equation, numerical computation

AMS Subject Classification: 91B28, 35K15

1. INTRODUCTION

Numerical evaluation is an important as well as an indispensable issue in the theory of option pricing. Exact pricing formula is not expected in general with the exception of happy cases such as the celebrated Black–Scholes formula for the plain vanilla European call options. As a consequence numerical schemes for computing the price have been a subject for researches and much progress has been made so far. Many studies pursue the numerical realization of stochastic processes. We refer for instance to Part Seven of [13].

Here we deal with the numerical computation of the nonlinear partial differential equations (PDEs) of Black–Scholes type in the presence of transaction costs. The principal feature of our investigation is twofold. One is that we directly discuss the partial differential equations. We do not attempt to simulate the Brownian motion in the computer. We hope that our methodology is easy to understand for large communities of scholars even without the knowledge of advanced probability theory.

The second point is that the effect of transaction costs is taken into account. It is conceded that the Black–Scholes analysis idealizes the situation so that the transaction costs associated with trading is excluded [1, 12]; this hypothesis, however, is invalid in practice; the influence of transaction costs is actually very important for practitioners. There exist already a plenty of studies which remedy the absence of transaction costs. We refer to [2, 5, 9, 11] for instance and the references cited therein. We follow the analysis performed by P. Wilmott and his coworkers [4, 13, 14]. The resulting PDE becomes nonlinear; the equation involves the square of asset price multiplied by the absolute value of option gamma. See the equations (1)(2) in the next section. We want to numerically solve these equations.

Drawbacks of our procedure now appear in numerically treating these nonlinear PDEs of Black–Scholes type; the domain is half interval and the solution generally grows infinitely large. Following [6] we overcome this difficulty of unboundedness by exploiting suitable transformations. We remark that from the practical viewpoint, in the numerical computation we had better argue the modified Black–Scholes equation by itself, although a change of variables makes the equation into a usual diffusion equation.

The plan of the paper is as follows. In § 2 we recall the model equation and our previous establishments. § 3 is devoted to explain our technique, which is followed by numerical implementation of § 4. We conclude with discussions in § 5.

2. MODEL EQUATION

In this paper we are concerned with the numerical computation of the next PDEs of Black–Scholes type:

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = \kappa F\Big(S^2\left|\frac{\partial^2 V}{\partial S^2}\right|\Big) \text{ in } (S,t) \in (0,\infty)\times(0,T) \\ &\qquad V(S,T) = V_0(S) \qquad \text{for } S \geq 0, \end{aligned} \tag{1}$$

where S is the price of the underlying asset and $V(S,t)$ denotes the option price written on S. The maturity data $V_0(S)$ (≥ 0) fulfills a suitable growth condition specified later. For technical reasons we principally assume $V_0(0) = V(0,t) = 0$ throughout the paper; that is, we consider a European call type options. The constants r and σ stand for as usual the risk-free interest rate and the asset volatility, respectively. The right hand side of (1) originates in the presence of transaction costs. The given smooth function $F(Q)$ is assumed to satisfy $|F'(Q)| \leq M$ for some constant $M > 0$. Transaction costs are assumed to be proportional to the value traded with a constant κ, which depends on the individual investor.

A typical example of (1) is given by $F(Q) = \sigma\sqrt{2/\pi\delta t}Q$; namely, the so-called Hoggard–Whalley–Wilmott equation [4], which is claimed as one of the first nonlinear PDEs in finance and is expressed as follows.

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = \kappa\sigma S^2\sqrt{\frac{2}{\pi\delta t}}\left|\frac{\partial^2 V}{\partial S^2}\right| \text{ in } (S,t) \in (0,\infty)\times(0,T) \\ &\qquad V(S,T) = V_0(S) \qquad \text{for } S \geq 0. \end{aligned} \tag{2}$$

In this model, the portfolio is considered to be revised every δt where δt is a non-infinitesimal fixed time-step not to be taken $\delta t \to 0$. The derivation of (2) and its variants like (1) is well known to specialists. We refer for instance to [10, 13, 14].

In our previous paper [7, 8] we analytically prove the existence of solutions for (1). Precisely stated we have established the next theorem.

Theorem 1. (Imai et al [7], Ishimura [8]) Suppose $\kappa < 2^{-1}\sigma^2 M^{-1}$. Then for any maturity data $V_0(S)$ (≥ 0) with $V_0'(S) \simeq \alpha$ ($\alpha \geq 0$) exponentially as $S \to \infty$, there exists a solution $V(S,t)$ to the equation (1), whose behavior is given by $\partial V(S,t)/\partial S \simeq \alpha$ exponentially as $S \to \infty$.

We remark that the solution is not presumed a priori to be convex nor concave. We mention that although the nonlinear right hand side term is essential in (1), its treatment is rather cumbersome within the theory of PDE. The foregone literature thus customarily presupposes the convexity of V to remove the absolute value. In the real world, however, this restriction is not appropriate and there are portfolios which are not necessarily convex nor concave. On the other hand, in the numerical computation the absolute value does not cause so much trouble.

3. COMPUTATIONAL TECHNIQUE

Now we explain our scheme in order to numerically compute the PDE (1). To begin with we make the time inversion $t \mapsto T - t$ to consider the next initial boundary value problem.

$$\begin{gathered}\frac{\partial V_2}{\partial t} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_2}{\partial S^2} - \kappa F\left(S^2 \left|\frac{\partial^2 V_2}{\partial S^2}\right|\right) + rS\frac{\partial V_2}{\partial S} - rV_2 \text{ in } (S,t) \in (0,\infty)\times(0,T) \\ V_2(S,0) = V_0(S) \quad \text{for } S \geq 0, \qquad V_2(0,t) = 0 \quad \text{for } 0 \leq t < T,\end{gathered} \tag{3}$$

where $V_0(S) \geq 0$ with $V_0(0) = 0$, and $V_0'(S) \simeq \alpha$ exponentially as $S \to \infty$ with a nonnegative constant α. The original $V(S,t)$ is recovered by the formula $V(S,t) = V_2(S, T-t)$.

To handle the unboundedness of the problem we perform transformations on (3), which are divided into two steps.

Step 1. Transformations of V_2.

We put

$$\begin{aligned} V_3(S,t) &:= V_2(S,t) - \alpha S \\ V_4(S,t) &:= 1 - \frac{V_2(S,t)}{\alpha(\varepsilon + S)}, \qquad \text{where } \varepsilon > 0. \end{aligned}$$

It is easy to check that both V_3, V_4 are bounded as $S \to \infty$ and satisfy

$$\begin{gathered}\frac{\partial V_3}{\partial t} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_3}{\partial S^2} - \kappa F\left(S^2 \left|\frac{\partial^2 V_3}{\partial S^2}\right|\right) + rS\frac{\partial V_3}{\partial S} - rV_3 \text{ in } (S,t) \in (0,\infty)\times(0,T) \\ V_3(S,0) = V_0(S) - \alpha S \quad \text{for } S \geq 0, \qquad V_3(0,t) = 0 \quad \text{for } 0 \leq t < T\end{gathered}$$

$$\frac{\partial V_4}{\partial t} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_4}{\partial S^2} + \frac{\kappa}{\alpha(\varepsilon+S)} F\Big(\alpha S^2 \Big|(\varepsilon+S)\frac{\partial^2 V_4}{\partial S^2} + 2\frac{\partial V_4}{\partial S}\Big|\Big)$$
$$+ \Big\{\frac{\sigma^2 S^2}{\varepsilon+S} + rS\Big\}\frac{\partial V_4}{\partial S} + \frac{r\varepsilon(1-V_4)}{\varepsilon+S} \text{ in } (S,t)\in(0,\infty)\times(0,T)$$
$$V_4(S,0) = 1 - \frac{V_0(S)}{\alpha(\varepsilon+S)} \quad \text{for } S\ge 0, \qquad V_4(0,t)=1 \quad \text{for } 0\le t<T.$$

Step 2. Change of space variable.

We introduce $S = x/(1-x^2)$ (see Imai [6]). Then it is easy to see that $\{S\ge 0\}$ corresponds to $\{0\le x<1\}$ and $x = 2S/(1+\sqrt{1+4S^2})$. We further define

$$u_3(x,t) := V_3(x/(1-x^2),t)$$
$$u_4(x,t) := V_4(x/(1-x^2),t).$$

After a little tedious calculation we find that

$$\frac{\partial u_3}{\partial t} = \frac{1}{2}\sigma^2\frac{x^2(1-x^2)^2}{(1+x^2)^2}\frac{\partial^2 u_3}{\partial x^2}$$
$$-\kappa F\Big(\frac{x^2(1-x^2)}{(1+x^2)^3}\Big|(1-x^2)(1+x^2)\frac{\partial^2 u_3}{\partial x^2} - 2x(3+x^2)\frac{\partial u_3}{\partial x}\Big|\Big)$$
$$+\frac{x(1-x^2)}{(1+x^2)^3}\{r(1+x^2)^2 - \sigma^2 x^2(3+x^2)\}\frac{\partial u_3}{\partial x} - ru_3 \text{ in } (x,t)\in(0,1)\times(0,T) \tag{4}$$
$$u_3(x,0) = V_0(x/(1-x^2)) - \frac{\alpha x}{1-x^2} \quad \text{for } 0<x<1,$$
$$u_3(0,t) = u_3(1,t) = 0 \quad \text{for } 0\le t<T,$$

$$\frac{\partial u_4}{\partial t} = \frac{1}{2}\sigma^2\frac{x^2(1-x^2)^2}{(1+x^2)^2}\frac{\partial^2 u_4}{\partial x^2} + \frac{x(1-x^2)}{1+x^2}\Big\{r - \sigma^2\frac{x^2(3+x^2)}{(1+x^2)^2} + \frac{\sigma^2 x}{\varepsilon(1-x^2)+x}\Big\}\frac{\partial u_4}{\partial x}$$
$$+\frac{\kappa(1-x^2)}{\alpha(\varepsilon(1-x^2)+x)}F\Big(\frac{\alpha x^2(\varepsilon(1-x^2)+x)}{(1+x^2)^3}\Big|(1-x^2)(1+x^2)\frac{\partial^2 u_4}{\partial x^2}$$
$$+2\Big\{\frac{(1+x^2)^2}{\varepsilon(1-x^2)+x} - x(3+x^2)\Big\}\frac{\partial u_4}{\partial x}\Big|\Big) \tag{5}$$
$$+\frac{r\varepsilon(1-x^2)(1-u_4)}{\varepsilon(1-x^2)+x} \quad \text{in } (x,t)\in(0,1)\times(0,T)$$
$$u_4(x,0) = 1 - \frac{1-x^2}{\alpha\varepsilon(1-x^2)+x}V_0(x/(1-x^2)) \quad \text{for } 0<x<1,$$
$$u_4(0,t)=\varepsilon, \quad u_4(1,t) = 0 \quad \text{for } 0\le t<T.$$

This is our procedure. We note that the proof of Theorem 1 does not rely on this transformation.

Now our numerical experiments are manipulated on these PDEs, which are defined on a bounded interval and whose solution take bounded values. We employ the explicit Euler method with second order finite difference method in space; the computation is carried out by the double precision. To be more specific, our algorithm is schematically described as follows.

$$\frac{u^{i,j+1}-u^{i,j}}{\Delta t} = R[u^{i-1,j}, u^{i,j}, u^{i+1,j}],$$

where we define $u^{i,j} := u(i\Delta x, j\Delta t)$ $(0 \le i \le 1/\Delta x, 0 \le j \le T/\Delta t)$ with $u^{(1/\Delta x)+1,j} = u^{1/\Delta x,j} = 0 = u^{-1,j} = u^{0,j}$ by the Dirichlet condition, and $R[\cdot]$ means the right hand side of (4) and/or (5). As is already mentioned, the absolute value and the nonlinear terms do not cause serious troubles in the numerical implementation so long as we use the explicit Euler method in time.

It is to be noted that a singularity $x = 1$ appeared in the transformation $S = x/(1-x^2)$ is just apparent. Indeed, as to the initial condition, we simply set $u(i\Delta, 0) = V_0(S_i)$ with $S_i := i\Delta x/(1-(i\Delta x)^2)$ for $0 \le i \le (1/\Delta x) - 1$, and for $i = 1/\Delta x$ it suffices to set $u(1,0) = 0$ by the Dirichlet condition. Moreover this singularity is not involved in the PDEs (4)(5).

4. NUMERICAL IMPLEMENTATION

Here we undertake the numerical computation. First we treat the Hoggard–Whalley–Wilmott equation (2). This is partly due to an intention to clarify the utility of our method.

To start with we ascertain the validity of our scheme; we check the error of computation. Since $V_0 \equiv S = x/(1-x^2)$ $(\alpha = 1)$ solves (4) with $u_3(x,t) \equiv 0$ and (5) $(\varepsilon = 1)$ with $u_4(x,t) \equiv (1-x^2)/(1-x^2+x)$, respectively, we estimate the $err(T) := \max_{0\le i\le 1/\Delta x} |u_3^{\text{num}}(i\Delta x, T)|$ and $\max_{0\le i\le 1/\Delta x} |u_4^{\text{num}}(i\Delta x, T) - u_4(i\Delta x, T)|$, respectively, where $u_j^{\text{num}}(x,t)$ $(j = 3,4)$ denote the numerical solution and we set $T = 50$. The computation is performed under fixed $\Delta t = 10^{-5}$. The results are depicted in Figure 1, which shows $err(T) = O((\Delta x)^2)$. Although the nonlinear effect must be negligible in this case of $V_0 = S$, our numerical scheme is judged to be well constructed.

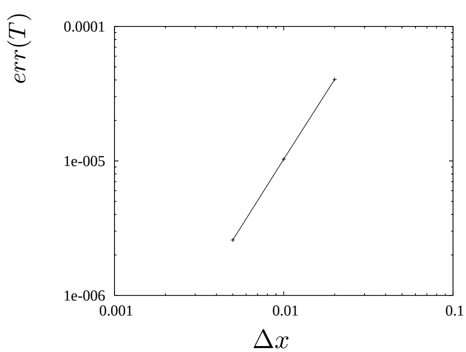

Fig. 1. Numerical error of (5) (fixed $\Delta t = 10^{-5}$).

Now we set $\Delta t = 10^{-5}$, $\Delta x = 1/200$, and various parameters as follows.

$$\begin{aligned} &\sigma = 1, \quad \kappa = \frac{1}{4}, \quad \delta t = \frac{2}{\pi} \\ &T = 10, \quad r = 0.1, \quad \text{and } \varepsilon = 1 \text{ in (5)}. \end{aligned} \tag{6}$$

As to the initial conditions we choose

$$\begin{aligned} &V_0^1(S) = S \tanh \frac{S}{2} \quad (\alpha = 1) \\ &V_0^2(S) = \max\{S-1, 0\} - 2\max\{S-2, 0\} + \max\{S-3, 0\} \quad (\alpha = 0). \end{aligned} \tag{7}$$

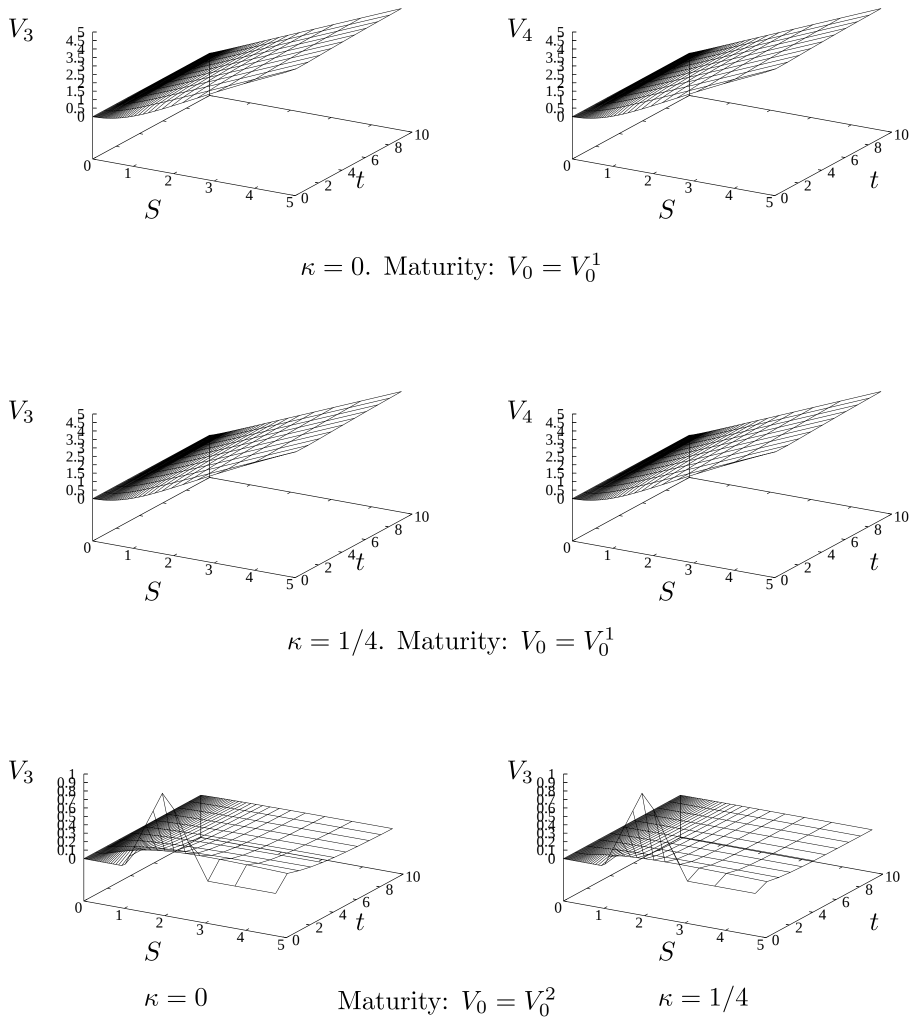

Fig. 2. Numerical results of (2) with different κ and maturity condition.

It is easy to see that $V_0^1(S)$ is not convex nor concave and $V_0^2(S)$ gives rise to the linear combination of plain vanilla European call options. Since V_0^2 is bounded the transformation V_4, u_4 should be vacuous; only V_3, u_3 will be enough. Figure 2 summarizes the results. We note that in the case $\kappa = 0$, the equation (2) of course reduces to the ordinary Black–Scholes equation, and with the initial condition $V_0^2(S)$ the solution is given by the combination of famous Black–Scholes formulas of respective maturities.

Table 1. Values of V_3 and for $V_0 = V_0^2$, $t = 10$.

S	Exact sol.	$\kappa = 0$	$\kappa = 1/4$
		V^3	V^3
1.007	0.00838983	0.00840841	0.00115789
1.992	0.01121360	0.01124995	0.00155121
2.955	0.01298491	0.01303859	0.00180054
4.083	0.01447570	0.01454901	0.00201198
5.001	0.01541521	0.01550385	0.00214596

Next we consider a special case of the so-called extended Leland model [3].

$$F(Q) = \frac{\kappa_0}{\delta t} + \sigma\sqrt{\frac{2}{\pi \delta t}}Q. \tag{8}$$

The parameters are set to be the same as (6) and $\kappa_0 = 2/\pi$. We similarly transform the equation and implement the computation. The initial condition is V_0^2. The result is shown in Figure 3. Since the constant term in (8) makes the option value negative, the pricing of this model resulted in an unrealistic situation.

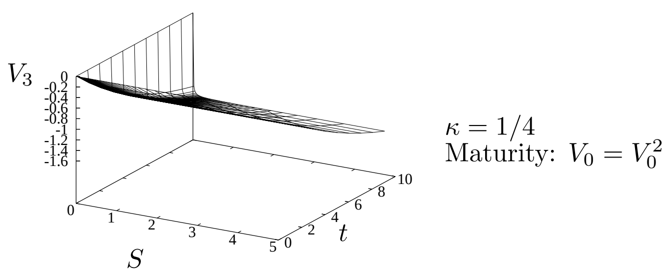

Fig. 3. Numerical results of (1) (8).

5. DISCUSSIONS

We have proposed a numerical scheme to effectively compute the nonlinear partial differential equations (PDEs) of Black–Scholes type which incorporates the effects of transaction costs. The influence of transaction costs makes the PDE nonlinear and we directly argue the PDE rather than simulate the underlying stochastic processes. We hope that our strategy is accessible to large part of researchers and at the same time it is easy to understand.

Since the domain of these modified PDEs of Black–Scholes type is half line and the solution generally grows infinitely large, we are forced to be confronted with unboundedness. We overcome this difficulty by exploiting suitable transformations. We eventually discuss the PDE on a bounded interval whose solution stays bounded.

Numerical implementation shows that our scheme is robust enough and well works. It can be seen that transaction costs certainly make the value lower; certain model forces the option value even negative. However they round off the behavior of solutions in a sense.

Our method may be applicable to other PDEs obtained in finances; for example the PDEs derived from the Bellman principle in the stochastic control setting would be worth investigating. This will be our next topics for researches.

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