KYBERNETIKA — VOLUME *43* (2007), NUMBER 4, PAGES 431 – 442

APPLICATIONS OF REGIME–SWITCHING MODELS BASED ON AGGREGATION OPERATORS

Jozef Komorník and Magda Komorníková

A synthesis of recent development of regime-switching models based on aggregation operators is presented. It comprises procedures for model specification and identification, parameter estimation and model adequacy testing. Constructions of models for real life data from hydrology and finance are presented.

Keywords: time series, regime-switching model, aggregation operator, model adequacy testing

AMS Subject Classification: 37M10, 68T37

1. INTRODUCTION

The aim of this paper is to sum up a series of studies of its authors and some of their collaborators on regime-switching models. Special attention will be paid to threshold variables, transition functions and their construction via aggregation operators. Some further problems related to testing adequacy will be indicated and discussed.

2. STRUCTURE OF REGIME–SWITCHING MODELS

A readable introduction to regime-switching models accompanied by procedures of model selection, testing and parameter estimation as well as applications to real life data is presented in [8].

2.1. Origins of regime-switching models

The idea of multi-regime forecasting models dates back to [2]. In [15] a *Threshold Autoregressive* (TAR) model has been proposed. It assumes that the regime that occurs at time t can be determined by an observable variable q_t relative to a *threshold value*, which is denoted as c. The resulting model is called a *Self-Exciting TAR* (SETAR) model. SETAR model is linear within a regime, but liable to move between regimes as the process crosses the threshold. For example, a two-regime model for a state variables series y_t with AR(p_1) and AR(p_2) has the form

$$\begin{aligned} y_t &= (\phi_{0,1} + \phi_{1,1} y_{t-1} + \cdots + \phi_{p_1,1} y_{t-p_1}) I[q_t \le c] \\ &\quad + (\phi_{0,2} + \phi_{1,2} y_{t-1} + \cdots + \phi_{p_2,2} y_{t-p_2})\, I[q_t > c] + \epsilon_t \end{aligned} \tag{1}$$

where $\phi_{i,j}$, $i = 0, 1, \ldots, p_{i_j}$, $j = 1, 2$ are autoregressive coefficients, $I[A]$ is an *indicator function* with $I[A] = 1$ if event A occurs and $I[A] = 0$ otherwise, and ϵ_t's are assumed to be a martingale difference sequence with respect to the history of the time series up to time $t - 1$.

A more gradual transition between different regimes can be obtained by replacing the indicator function $I[A]$ in (1) by a continuous function $G(q_t, \gamma, c)$, which changes smoothly from 0 to 1 as q_t increases. The resulting model is called a *Smooth Transition AR* (STAR) model and is given by (see, e. g. [14])

$$\begin{aligned} y_t = {} & (\phi_{0,1} + \phi_{1,1} y_{t-1} + \cdots + \phi_{p_1,1} y_{t-p_1})(1 - G(q_t, \gamma, c)) + \\ & + (\phi_{0,2} + \phi_{1,2} y_{t-1} + \cdots + \phi_{p_2,2} y_{t-p_2}) G(q_t, \gamma, c) + \epsilon_t \end{aligned} \tag{2}$$

Both models (1) and (2) allow multi-dimensional interpretation (when we interpret y_t as a state vector $\boldsymbol{y}_t = (y_{1,t}, \ldots, y_{s,t})'$ and coefficient $\Phi_{i,j}$ as matrices of the type $s \times s$.

A popular choice for the so-called *transition function* $G(q_t, \gamma, c)$ is the logistic function

$$G(q_t, \gamma, c) = \frac{1}{1 + e^{-\gamma(q_t - c)}} \tag{3}$$

and the resulting model is called a *Logistic STAR* (LSTAR) model. The parameter c in (3) can be interpreted as the threshold between the two regimes corresponding to $G(q_t, \gamma, c) = 0$ and $G(q_t, \gamma, c) = 1$, in the sense that the logistic function changes monotonously from 0 to 1 as q_t increases, while $G(c, \gamma, c) = 0.5$. The parameter γ determines the *smoothness* of the change in the value of the logistic function (the maximum of its derivative is $G'(c, \gamma, c) = \gamma/2$), and thus the transition from one regime to the other.

2.2. Models using aggregation operators

Let us recall that the most interesting types of aggregation operators (as mapping from $\mathbb{R}^n$ onto $\mathbb{R}$) are

- — arithmetic mean $\mathcal{M}(x_1, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^n x_i$;
- — weighted means $\mathcal{W}(x_1, \ldots, x_n) = \sum_{i=1}^n w_i x_i$, where the weights $w_i \in [0, 1]$, $\sum_{i=1}^n w_i = 1$;
- — OWA operators $\mathcal{W}'(x_1, \ldots, x_n) = \sum_{i=1}^n w_i x_i'$ with weights as in the case of weighted means, but with x_i' as non-decreasing permutation of x_i inputs, i. e., $x_1' \le \cdots \le x_n'$.

In the class of OWA operators we can find also $\mathcal{MIN}$ and $\mathcal{MAX}$ operators, corresponding to the extreme cases $w_1 = 1$ and $w_i = 0$ otherwise (alternatively $w_n = 1$

and $w_i = 0$ otherwise) and all order statistics. Similarly projection to k-ary coordinate with $w_k = 1$ and $w_i = 0$ otherwise is a special weighted mean. A convenient way of producing a decreasing sequence $(w_1, \ldots, w_n)$ of weight coefficients is based on utilization of increments of a generating increasing convex bijection φ of $[0,1]$, if we put $w_k = \varphi\left(\frac{n-k+1}{n}\right) - \varphi\left(\frac{n-k}{n}\right)$, $k = 1, \ldots, n$. More about aggregation operators can be found, for example, in [6].

Outputs of aggregation operators have been used in [12] in the role of threshold variables. For a fixed number $k > 0$ of observations of a suitable auxiliary variable z_t and an aggregation operator $\mathcal{A}$ the values of output threshold variable

$$q_t = \mathcal{A}(z_{t-1}, \ldots, z_{t-k}) \tag{4}$$

will indicate (by crossing a threshold level c) switching between 2 individual regimes described by the models (1) or (2).

3. MODEL IDENTIFICATION AND PARAMETER ESTIMATION

3.1. Model identification

Identification of the appropriate threshold value c, lag order d and orders p_1, p_2 for AR in two regimes can be chosen from model that will minimize an information criterion. Liew and Chong in [13] presented AIC (Akaike's Information Criterion) and SIC (Schwarz's information criterion) for a 2-regime STAR model as:

$$AIC(p_1, p_2) = \ln \widehat{\sigma}^2 + \frac{2(p_1 + p_2)}{n} \tag{5}$$

$$SIC(p_1, p_2) = \ln \widehat{\sigma}^2 + \frac{(p_1 + p_2)\ln n}{n} \tag{6}$$

where $\widehat{\sigma}^2$ is the estimate of variance of the residuals (equal to the average of the sum of their squares).

3.2. Estimation of SETAR models

The parameters of interest in the 2-regime SETAR model (1) can be conveniently estimated by sequential conditional least squares.

We can rewrite (1) as

$$y_t = \boldsymbol{\phi}_1 \boldsymbol{x}_{t,1} I[q_t \le c] + \boldsymbol{\phi}_2 \boldsymbol{x}_{t,2} I[q_t > c] \tag{7}$$

where $\boldsymbol{\phi}_j = \left(\phi_{0,j}, \phi_{1,j}, \ldots, \phi_{p_j,j}\right)'$, $\boldsymbol{x}_{t,j} = \left(1, y_{t-1}, y_{t-2}, \ldots, y_{t-p_j}\right)'$, $j = 1, 2$.

Estimates of $\boldsymbol{\phi} = (\boldsymbol{\phi}_1, \boldsymbol{\phi}_2)'$ can be obtained by Ordinary Least Squares (OLS) as

$$\widehat{\boldsymbol{\phi}}(c) = \left(\sum_{t=1}^{n} \boldsymbol{x}_t(c) \cdot \boldsymbol{x}_t(c)'\right)^{-1} \left(\sum_{t=1}^{n} \boldsymbol{x}_t(c) y_t\right) \tag{8}$$

where $\boldsymbol{x}_t(c) = (\boldsymbol{x}_{t,1}{}' I[q_t \le c], \boldsymbol{x}_{t,2}{}' I[q_t > c])'$. The corresponding residuals are denoted as $\hat{\epsilon}_t(c) = y_t - \hat{\boldsymbol{\phi}}(c)' \boldsymbol{x}_t(c)$ with the estimate of variance $\hat{\sigma}^2(c) = \frac{1}{n}\sum_{t=1}^{n} \hat{\epsilon}_t(c)^2$.

The least squares estimate of c can be obtained by minimizing this residual variance $\hat{c} = \text{argmin}_{c \in C} \hat{\sigma}^2(c)$, where C denotes the set of all allowable threshold values.

3.3. Estimation of STAR models

Estimation of the parameters $\hat{\boldsymbol{\theta}} = (\boldsymbol{\phi}_1,\ \boldsymbol{\phi}_2,\ \gamma, c)'$ in the STAR model (2) is a relatively straight-forward application of nonlinear least squares

$$\hat{\boldsymbol{\theta}} = \text{argmin}_{\boldsymbol{\theta}} \sum_{t=1}^{n} \left(y_t - F(x_t; \boldsymbol{\theta})\right)^2,$$

where $F(x_t; \boldsymbol{\theta}) = \boldsymbol{\phi}_1 \boldsymbol{x}_{t,1}(1 - G[q_t, \gamma, c]) + \boldsymbol{\phi}_2 \boldsymbol{x}_{t,2} G[q_t, \gamma, c]$.

For fixed values of the parameters γ and c, the STAR model is linear in the autoregressive parameters $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$. Thus, conditional upon γ and c, estimates of $\boldsymbol{\phi} = (\boldsymbol{\phi}_1, \boldsymbol{\phi}_2)'$ can be obtained by OLS as

$$\hat{\boldsymbol{\phi}}(\gamma, c) = \left(\sum_{t=1}^{n} \boldsymbol{x}_t(\gamma, c) \cdot \boldsymbol{x}_t(\gamma, c)'\right)^{-1} \left(\sum_{t=1}^{n} \boldsymbol{x}_t(\gamma, c) y_t\right),$$

where $\boldsymbol{x}_t(\gamma, c) = (\boldsymbol{x}_{t,1}{}'(1 - G[q_t, \gamma, c],\ \boldsymbol{x}_{t,2}{}' G[q_t, \gamma, c])'$.

The corresponding residuals are denoted as $\hat{\epsilon}_t(\gamma, c) = y_t - \hat{\boldsymbol{\phi}}(\gamma, c)' \boldsymbol{x}_t(\gamma, c)$ with the estimate of variance $\hat{\sigma}^2(\gamma, c) = \frac{1}{n}\sum_{t=1}^{n} \hat{\epsilon}_t(\gamma, c)^2$.

4. TESTING MODEL ADEQUACY

4.1. Tests for univariate STAR model

For testing the linearity of (2) (or H_0: $\gamma = 0$) we can proceed similarly as in [8] and utilizing the difference

$$G^*(q_t) = G(q_t, \gamma, c) - 1/2 = h(\gamma(q_t - c)),$$

where $h(u) = \frac{1}{1+e^{-u}} - \frac{1}{2}$.

In the reparametrized model equation the linearity can be tested by means of a Lagrange Multiplier [LM] statistic with a standard asymptotic χ^2-distribution under the null hypothesis. We denote $p = \max(p_1, p_2)$, $x_t = (1, \tilde{x}_t')'$ with $\tilde{x}_t = (y_{t-1}, \dots, y_{t-p})'$ and $\boldsymbol{\phi}_i = (\phi_{i,0}, \phi_{i,1}, \dots, \phi_{i,p})'$, $i = 1, 2$. Then we rewrite the model (2) as

$$y_t = \boldsymbol{\phi}_1' \boldsymbol{x}_t [1/2 - G^*(q_t)] + \boldsymbol{\phi}_2' \boldsymbol{x}_t [1/2 + G^*(q_t)] + \varepsilon_t$$

or

$$\begin{aligned} y_t &= 1/2(\boldsymbol{\phi}_1 + \boldsymbol{\phi}_2)'\boldsymbol{x}_t + (\boldsymbol{\phi}_2 - \boldsymbol{\phi}_1)'\boldsymbol{x}_t G^*(q_t) + \varepsilon_t \\ &= 1/2(\boldsymbol{\phi}_1 + \boldsymbol{\phi}_2)'\boldsymbol{x}_t + (\boldsymbol{\phi}_2 - \boldsymbol{\phi}_1)'\boldsymbol{x}_t h(\gamma(q_t - c)) + \varepsilon_t \,. \end{aligned}$$

In order to derive a linearity test against (2), we approximate the function $G^*(q_t)$ with a third-order Taylor approximation around $\gamma = 0$. This results in the auxiliary regression

$$y_t = \alpha_0' + \boldsymbol{\beta}_0'\boldsymbol{x}_t + \boldsymbol{\beta}_1'\boldsymbol{x}_t q_t + \boldsymbol{\beta}_2'\boldsymbol{x}_t q_t^2 + \boldsymbol{\beta}_3'\boldsymbol{x}_t q_t^3 + e_t \tag{9}$$

where $q_t = \mathcal{A}(y_{t-1}, \ldots, y_{t-d})$, α_0 and $\boldsymbol{\beta}_i = (\beta_{i,0}, \beta_{i,1}, \ldots, \beta_{i,p})'$, $i = 0, 1, 2, 3$ are functions of the parameters $\boldsymbol{\phi}_1$, $\boldsymbol{\phi}_2$, γ, c and $e_t = \varepsilon_t + (\boldsymbol{\phi}_2 - \boldsymbol{\phi}_1)'\boldsymbol{x}_t R_3(q_t)$ with $R_3(q_t)$ the remainder term from the Taylor approximation. Under the null hypothesis, $R_3(q_t) \equiv 0$ and $e_t = \varepsilon_t$ (see e. g. [8]). Consequently, this remainder term does not affect the properties of the errors under the null hypothesis. Inspection of the exact relationships shows that the null hypothesis H_0': $\gamma = 0$ corresponds to H_0'' : $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 = \boldsymbol{\beta}_3 = \mathbf{0}$, which can be tested by a standard LM-type test. Under the null hypothesis of linearity, the test statistic, to be denoted as LM_3, has an asymptotic χ^2 distribution with $3(p+1)$ degrees of freedom.

The LM_3 statistic based on (9) can be computed as follows:

1. Estimate the model under the null hypothesis of linearity by regressing y_t on $\boldsymbol{x}_t$. Compute the residuals $\hat{\varepsilon}_t$ and the sum of squared residuals $SSR_0 = \sum_{t=1}^n \hat{\varepsilon}_t^2$.
2. Estimate the auxiliary regression of y_t on $\boldsymbol{x}_t$ and $\boldsymbol{x}_t q_t^i$, $i = 1, 2, 3$. Compute the residuals $\hat{e}_t$ and the sum of squared residuals $SSR_1 = \sum_{t=1}^n \hat{e}_t^2$.
3. The LM_3 statistic can be computed as

$$LM_3 = \frac{n(SSR_0 - SSR_1)}{SSR_0} \,. \tag{10}$$

4.2. Tests for multivariate TAR model

Testing linearity of the 2-regimes model alternative with a threshold variable q_t satisfying (4) is based on the method of arranged regression described in [17]. Let

$$\boldsymbol{y}_t' = \boldsymbol{X}_t'\boldsymbol{\Phi} + \boldsymbol{\varepsilon}_t \,, t = h+1, \ldots, n \tag{11}$$

be the vector version of (2), where $h = \max(p, k)$, $\boldsymbol{X}_t' = \left(1, \boldsymbol{y}_{t-1}', \ldots, \boldsymbol{y}_{t-p}'\right)$ is a $(2p+1)$ dimensional regressor and $\boldsymbol{\Phi}$ denotes the parameter matrix. Let us denote $q_{(i)}$ the ith smallest value of the threshold variable q_t for $i = 1, \ldots, n-h$. Let $t(i)$ be the index of $z_{(i)}$, i. e., $z_{t(i)} = z_{(i)}$. When we rewrite (11) in the form

$$\boldsymbol{x}_{t(i)}' = \hat{\boldsymbol{X}}_{t(i)}'\boldsymbol{\Phi} + \boldsymbol{\varepsilon}_{t(i)} \,, i = 1, \ldots, n-h \tag{12}$$

the dynamic of $\boldsymbol{y}_t$ will not change. Only the order in which data enter the regression will change. The arranged regression transforms a threshold model into a change-point problem.

Predictive residuals $\hat{\boldsymbol{\varepsilon}}_{t(i+1)}$ in the arranged regression can be used for detecting a change point. Namely, they are uncorrelated with the regressor $\boldsymbol{X}_{t(i+1)}$ in case that $\boldsymbol{y}_t$ is linear. Let $\hat{\boldsymbol{\Phi}}_m$ be a least squares estimate of $\boldsymbol{\Phi}$ in the arranged regression (12) using data points associated with m smallest values of q_t. Let

$$\hat{\boldsymbol{e}}_{t(m+1)} = \boldsymbol{y}_{t(m+1)} - \hat{\boldsymbol{\Phi}'}_m \boldsymbol{X}_{t(m+1)} \tag{13}$$

be the residual of the $1 - step$ ahead prediction in the arranged regression. Let the standardized version of $\hat{\boldsymbol{e}}_{t(m+1)}$ be

$$\hat{\boldsymbol{\eta}}_{t(m+1)} = \frac{\hat{\boldsymbol{e}}_{t(m+1)}}{\left[1 + \boldsymbol{X}'_{t(m+1)} \boldsymbol{V}_m \boldsymbol{X}_{t(m+1)}\right]^{1/2}} \tag{14}$$

where $\boldsymbol{V}_m = \left[\sum_{i=1}^m \boldsymbol{X}_{t(i)} \boldsymbol{X}'_{t(i)}\right]^{-1}$.

Under the hypothesis H_0 of linearity of $\boldsymbol{y}_t$,$\hat{\boldsymbol{\eta}}_{(j)}$ and $\boldsymbol{X}_{t(j)}$ should be uncorrelated for all $j = 1, \ldots, n - h$. We test this hypothesis using the regression

$$\hat{\boldsymbol{\eta}}'_{t(j)} = \boldsymbol{X}'_{t(j)} \boldsymbol{\Psi} + \boldsymbol{w}'_{t(l)}, \qquad j = m_0 + 1, \ldots n - h \tag{15}$$

where m_0 is the starting point of recursive regression ($m_0 \approx 3\sqrt{n}$). The Tsay's test statistic applied in [17] is

$$C = [n - h - m_0 - p - 1)] \times [\ln(\det S_0) - \ln(\det S_1)], \tag{16}$$

where

$$S_0 = \frac{1}{n - h - m_0} \sum_{j=m_0+1}^{n-h} \hat{\boldsymbol{\eta}}_{t(j)} \hat{\boldsymbol{\eta}}'_{t(j)}, \qquad S_1 = \frac{1}{n - h - m_0} \sum_{j=m_0+1}^{n-h} \hat{\boldsymbol{w}}_{t(j)} \hat{\boldsymbol{w}}'_{t(j)},$$

Under the null that $\boldsymbol{y}_t$ is linear, C is asymptotically a χ^2 random variable with $(p + 1)$ degrees of freedom.

Let for $i = 1, 2$, $\boldsymbol{\Sigma}_i$ be the covariance matrix of residuals $\boldsymbol{\epsilon}^{(i)}$ in regime i. Let $\sum_t^{(i)}$ denotes summing over observations in regime i. Then under certain regularity conditions (cf. [17]) the least square estimates

$$\hat{\boldsymbol{\Phi}}'_i = \left(\sum_t^{(i)} \boldsymbol{X}_t \boldsymbol{X}'_t\right)^{-1} \left(\sum_t^{(i)} \boldsymbol{X}_t \boldsymbol{x} y'_t\right), \tag{17}$$

$$\hat{\boldsymbol{\Sigma}}_i = \frac{\sum_t^{(i)} \left(\boldsymbol{y}_t - \boldsymbol{X}'_t \hat{\boldsymbol{\Phi}}_i\right) \left(\boldsymbol{y}_t - \boldsymbol{X}'_t \hat{\boldsymbol{\Phi}}_i\right)'}{n_i - s} \tag{18}$$

are strongly consistent estimators of parameters $\boldsymbol{\Phi}^{(i)}$ and $\boldsymbol{\Sigma}_i$. Moreover, $\sqrt{n_i}\left(\hat{\boldsymbol{\Phi}}_i - \boldsymbol{\Phi}_i\right)$ is asymptotically normal with zero mean and covariance matrix equal to the Kronecker product $\boldsymbol{\Gamma}_i \otimes \boldsymbol{\Sigma}_i$, where $\boldsymbol{\Sigma}_i$ can be estimated by $\hat{\boldsymbol{\Sigma}}_i$ and $\boldsymbol{\Gamma}_i$ by $\sum_t^{(i)} \boldsymbol{X}_t \boldsymbol{X}'_t / n$.

For selection between alternative 2-regimes models we use the Akaike information criterion for multivariate TAR models

$$AIC = \sum_{j=1}^{2} n_j \ln\left(\det\left(\hat{\mathbf{\Sigma}}_j\right)\right) + 2s(sp+1). \tag{19}$$

5. PRACTICAL APPLICATIONS

5.1. Applications in Hydrology (univariate model)

In first example discussed in this article we analyze 12 univariate time series of monthly average stream flows of the Slovak rivers. The data used for testing regime-switching nonlinearity with aggregation operators are residuals obtained after removing periodic components. We followed the approach of [8] estimating appropriate orders p of linear models AR(p) for considered time series by minimizing the above mentioned AIC and BIC information criteria. The number of delayed state variables that enter any aggregation operator (for individual time series models) is $h = k - 1$, where k is the first value of delay for which the values of the autocorrelation function are not significantly different from 0. In the role of aggregation operators we used Arithmetic Mean ($\mathcal{M}$), Weighted average with the generating functions x^2 ($\mathcal{W}_2$) and x^3 ($\mathcal{W}_3$) and OWA operators $\mathcal{MIN}$ and $\mathcal{MAX}$. Usual LSTAR models with threshold variables y_{t-d} can be considered as products of a special trivial type of aggregation operators (that map the sequence $(y_{t-1}, \ldots, y_{t-h})$ on its single components).

The following Table 1 shows the results of testing of adequacy of 2-regime models for 12 univariate time series of monthly average stream flows [m^3/s] from observation stations at the Slovak rivers in the period November 1930 – October 2003. For each river, items in the last 7 columns of its row represent p-values for LM_3 tests given by (10). The first 2 model classes are standard LSTAR models with threshold variables y_{t-d}, $d = 1, 2$ (without an explicit aggregation operator). The next 5 model classes correspond to LSTAR models with the threshold variable q_t as the output from the aggregation operators $\mathcal{MAX}$, $\mathcal{MIN}$, $\mathcal{M}$, $\mathcal{W}_2$ and $\mathcal{W}_3$.

From results in Table 1 we can derive several basic conclusions:

- All p-values in the row of the Kysuca river greatly exceed the standard threshold of weak significance 0.1, which indicates that 1-regime models outperform 2-regime alternatives.
- In case of the operator $\mathcal{MIN}$ only for one river (Dobšiná) the result of testing is significant. Other operators provide models with significant non-linearity for much larger numbers of rivers.

In the subsequent analysis, we restrict our attention to these combinations of rows and columns of the Table 1, where the corresponding p-value are smaller then the threshold of strong significance 0.01 (this way we eliminate the row corresponding to Kysuca river and the column corresponding to the $\mathcal{MIN}$ operator).

Table 1. The results of testing of adequacy of 2-regime models.

River	k	p	p-value for LM_3 test						
			for standard LSTAR with		for LSTAR with aggregation operator				
			$d = 1$	$d = 2$	$\mathcal{MAX}$	$\mathcal{MIN}$	$\mathcal{M}$	$\mathcal{W}_2$	$\mathcal{W}_3$
Vlkyňa	4	2	0.0554	0.0018	0.0001	0.0119	0.0005	0.0004	0.0013
Štítnik	4	1	0.0002	×	0.0033	0.4339	0.0452	0.0024	0.0007
Boca	5	2	0.0060	0.1246	0.0039	0.2048	0.0535	0.0046	0.0019
Ipel'	3	1	0.0073	×	0.0239	0.4751	0.0494	0.0133	0.0071
Kysuca1	2	2	0.4746	0.9972	0.7756	0.4144	0.8558	0.5313	0.5168
Kysuca2	2	1	0.1413	×	0.1613	0.6591	0.3093	0.1729	0.1532
Litava	15	2	0.2190	0.0005	0.1303	0.0261	0.0007	0.0004	0.0067
Bebrava	4	2	0.0006	0.9173	0.0065	0.4222	0.0642	0.0366	0.0330
Dobšiná	15	2	0.0013	0.2494	0.0056	0.0059	0.0004	0.0001	0.00001
Krupinica	15	2	0.0555	9.4981	0.0004	0.6796	0.0015	0.0051	0.0098
Hron1	5	2	0.0011	0.9913	0.0019	0.2586	0.0011	0.0002	0.0001
Hron2	6	1	0.0014	×	0.0079	0.6757	0.4191	0.0346	0.0102

The next Table 2 shows an overview of the models characteristics for 10 observation stations, where the adequacy of regime-switching models has been demonstrated by the previous tests.For each p-value smaller than 0.05 we estimates parameters of the model with $p_1, p_2 \leq 3$ of the considered river flow in the corresponding class of models. The best model of the class is determined by information criteria AIC and BIC modified for regime-switching models (see [13]). The criterion $RMSE = \sqrt{\frac{1}{m}\sum_{i=1}^{m}(y_i - F_i)^2}$ (where m is the number of predictions, y_i is an observed value and F_i a prediction, $i = 1, \ldots, m$) is measure of out-of-sample fitting (for the last year data left out from the model building sample and used for a subsequent testing, see [8]).

The above results demonstrate that regime-switching models based on aggregation operators provide clearly better fit in 8 of 10 cases and comparable fit in 2 cases.

5.2. Applications in Finance (multivariate model)

Czech and Slovak Crowns separated soon after the velvet divorce of Czechoslovakia on January 1, 1993. Thanks to continuing intensive economic ties between both countries one can expect that the development of exchange rates of their currencies to EURO has common features that can be described by a 2-dimensional time series model. Furthermore, the Polish economy is the largest one in the so-called Visegrad group of new EU countries (that also includes the Czech Republic, Slovakia and Hungary). It is believed by many local bank specialists that London banks dealers treat Visegrad currencies as one group lead by the Polish Zloty. Consequently shocks in the Polish political and economic life that cause drops in the value of its national

Table 2. The measures of out-of-sample fitting for 12 predictions.

River	Standard LSTAR		Aggregation operator							
	(p_1, p_2, d)	$RMSE$	Type	$RMSE$	Type	$RMSE$	Type	$RMSE$	Type	$RMSE$
Vlkyňa	(2,2,2)	5.068	$\mathcal{MAX}$	**4.835**	$\mathcal{M}$	5.326	$\mathcal{W}_2$	5.223	$\mathcal{W}_3$	5.209
Štítnik	(1,1,1)	0.695	$\mathcal{MAX}$	**0.593**	$\mathcal{W}_2$	0.663	$\mathcal{W}_3$	0.654	×	×
Ipel'	(2,1,1)	**1.089**	$\mathcal{W}_3$	1.093	×	×	×	×	×	×
Boca	(2,2,1)	0.960	$\mathcal{MAX}$	1.127	$\mathcal{W}_2$	1.210	$\mathcal{W}_3$	**0.954**	×	×
Hron1	(1,2,1)	18.501	$\mathcal{MAX}$	23.761	$\mathcal{M}$	20.496	$\mathcal{W}_2$	19.830	$\mathcal{W}_3$	**18.184**
Dobšiná	(2,2,1)	0.273	$\mathcal{MAX}$	**0.167**	$\mathcal{M}$	0,182	$\mathcal{W}_2$	0.179	$\mathcal{W}_3$	0.185
Litava	(2,1,2)	0.473	$\mathcal{M}$	0.426	$\mathcal{W}_2$	**0.419**	$\mathcal{W}_2$	0.423	×	×
Bebrava	(2,1,1)	**0.405**	$\mathcal{MAX}$	0.541	×	×	×	×	×	×
Krupinica	(2,2,1)	0.685	$\mathcal{MAX}$	**0.590**	$\mathcal{M}$	0.924	$\mathcal{W}_2$	0.995	$\mathcal{W}_3$	0.944
Hron2	(1,2,1)	24.08	$\mathcal{MAX}$	**23.67**	×	×	×	×	×	×

currency Zloty are also supposed to influence negatively the values of other currencies of the Visegrad group countries. On the other hand, strong appreciations of the Polish Zloty are believed to causes appreciations of other currencies of the Visegrad group. Therefore, we have chosen the exchange rate of EURO to Polish Zloty as an auxiliary variable. The results of the modelling procedure are as follows.

We denoted by $y_{1,t}$, $y_{2,t}$ and z_t daily values of EURO in Slovak Crowns, Czech Crowns and Polish Zlotys for working days in the period January 4, 1999 – January 13, 2006 ($\bar{y}_1 = 41.8,\ s_1^2 = 3.84,\ \bar{y}_2 = 32.9,\ s_2^2 = 6.3,\ \bar{z} = 4.11,\ s_z^2 = 0.1$).

First we found a 2-dimensional Autoregressive model for the vector $\boldsymbol{y}$. Its optimal form (minimizing the AIC and BIC criterion) has the order of autoregression equal to 8.

The following Table 3 contains the results of tests of linearity for different values of the delay parameter d and standard 2-regimes alternative MTAR models. The resulting p-values correspond to test statistics given by (16). We see that the minimum of p-values was attained for the delay $d = 6$.

Table 3. Results of tests of linearity
for standard MTAR ($p = 8,\ df = 34$).

Delay d	Test statistic $C(d)$	p-value
3	39.15	0.3452
4	42.85	0.1421
5	49.32	0.0433
6	52.00	**0.0247**
7	51.34	0.0285
8	49.59	0.0410

In the role of aggregation operators we used Arithmetic Mean ($\mathcal{M}$), Weighted average with the normalized Fibonacci triangle ($\mathcal{W}_1$) (see, e. g. [6]), operators with the generating functions x^2 ($\mathcal{W}_2$), x^3 ($\mathcal{W}_3$) and OWA operators $\mathcal{MIN}$ and $\mathcal{MAX}$.

Analogical results for linearity tests against optimal 2-regimes alternatives based on threshold variables in the form (4) for aggregation operators in different classes (described above) are in Table 4. For each class of selected operators we find the minimal p-value and corresponding number of inputs.

Table 4. Results of tests of linearity
for standard MTAR with aggregation operators ($p = 6$).

Type	Number of inputs	C	p-value
$\mathcal{M}$	15	39.63	0.0423
$\mathcal{W}_1$	20	34.62	0.0417
$\mathcal{W}_2$	15	39.02	0.0386
$\mathcal{W}_3$	20	40.26	0.0384
$\mathcal{MAX}$	10	35.69	**0.0382**
$\mathcal{MIN}$	20	43.94	**0.0153**

We see the minimal levels of p-values for best models corresponding to individual classes of aggregation operators (that are derived from values of the statistics (16)). We can conclude that the extreme value operators $\mathcal{MAX}$ (with number of inputs equal to 10) and $\mathcal{MIN}$ (with number of inputs equal to 20) attain lower minimal p-values than the remaining classes of operators.

We continue by selection of the best model for each investigated class of aggregation operators applying minimalization of the AIC criterion given by (19) over all model classes.

Table 5. Results of AIC model selection.

Type MTAR	c (Euro/PlZ)	AIC
Standard ($d = 6$)	4.044	−15 328.1
with $\mathcal{M}$	4.075	−15 325.5
with $\mathcal{W}_1$	4.069	−15 296.1
with $\mathcal{W}_2$	4.058	−15 323.2
with $\mathcal{W}_3$	4.058	−15 325.9
with $\mathcal{MAX}$	3.970	**−15 354.3**
with $\mathcal{MIN}$	4.041	−15 343.3

We see that the values of the AIC criterion for individual operator classes differ only slightly. Nevertheless, the standard methodology of model selection procedures leads to the choice of the operator $\mathcal{MAX}$ for which the minimum of the AIC criterion is attained.

Next we calculate estimates of model parameters for this type of model using (16). We present estimates of covariance matrices of residuals for individual regimes.

The diagonal elements of these matrices are lower by more than 2 orders than the variances of the original components of y_t.

Table 6. Covariance matrices $\hat{\Sigma}_1$ and $\hat{\Sigma}_2$.

$\hat{\Sigma}_1$			$\hat{\Sigma}_2$	
0.0126	0.0054		0.0137	0.0039
0.0054	0.0145		0.0039	0.0152

From the previous results we can conclude that extremes of exchange rate of Polish Zloty in the period of 2 weeks (10 working days) exhibit expected influence on the pair of exchange rate of EURO to Slovak and Czech Crowns.
This is the first example of aggregation operators based construction of threshold variables when extreme value operators $\mathcal{MIN}$ and $\mathcal{MAX}$ have demonstrated superior applicability. It seems to confirm that currency markets are exceptionally sensitive to shocks.

6. CONCLUSION

The above practical modeling results indicate that the regime-switching models based on aggregation operators provide promising methods for analysis of complex real life data from various fields. Concerning the theoretical methodology, it looks tempting to elaborate testing procedures of part 4.1 to multivariate STAR models (replacing test statistic (10) by an analogy of (16)). Similarly, the testing methods based on arranged regression from the part 4.2 seem to be extendable to multivariate STAR models. Except for a careful checking of main theoretical fundamentals (or their suitable modification), appropriate simulation studies (similarly to those in [1],[17]), as well as practical modeling experiments would be desirable.

ACKNOWLEDGEMENT

The research summarized in this paper was partly supported by the Grants APVT-20-003204 and VEGA 1/3006/06.

(Received April 5, 2006.)

REFERENCES

[1] T. Bacigál: Multivariate threshold autoregressive models in Geodesy. J. Electrical Engineering *55* (2004), 12/s, 91–94.
[2] D. W. Bacon and D. G. Watts: Estimating the transition between two intersecting lines. Biometrica *58* (1971), 525–534.
[3] T. Bognár: Time Series Analysis Applied in Geodesy and Geodynamics. Ph.D. Thesis, Bratislava 2005.
[4] T. Bognár, J. Komorník, and M. Komorníková: New STAR models of time series and their application in finance. In: Proc. 16th Symposium COMPSTAT 2004, Prague 2004 (J. Antoch, ed.), Physica-Verlag, Heidelberg 2004, pp. 713–720.

[5] B. Y. Brown and R. S. Mariano: Predictors in dynamic nonlinear models: large sample behaviour. Econometric Theory *5* (1989), 430–452.

[6] T. Calvo, R. Mesiar, and G. Mayor (eds.): Aggregation Operators (Studies in Fuzziness and Soft Computing 31). Physica–Verlag, Heidelberg 2002.

[7] F. X. Diebold and R. S. Mariano: Comparing predictive accuracy. J. Business and Economic Statistics *13* (1995), 253–263.

[8] P. H. Franses and D. Dijk: Non-linear Time Series Models in Empirical Finance. Cambridge University Press, Cambridge 2000.

[9] B. E. Hansen: Inference in TAR models. Studies in Nonlinear Dynamic and Econometrics *2* (1997), 1–14.

[10] B. E. Hansen: Sample splitting and threshold estimation. Econometrica *68* (2000), 3, 575–603.

[11] M. Komorníková and J. Komorník: Application of aggregation operators in regime-switching models for exchange rates. In: Proc. EUSFLAT-LFA 2005, Barcelona 2005, pp. 1297–1300.

[12] J. Komorník, M. Komorníková, and D. Szökeová: Testing the adequacy of regime-switching time series models nased on aggregation pperators. Submitted.

[13] V. K. Liew and T. T. Chong: Effect of STAR and TAR types nonlinearities on order selection criteria. http://econwpa.wustl.edu:80/eps/em/papers/0307/0307005.pdf (2003)

[14] T. Teräsvirta: Specification, estimation, and evaluation of smooth transition models. J. Amer. Statist. Assoc. *89* (1994), 208–218.

[15] H. Tong: On a threshold model. In: Pattern Recognition and Signal Processing (C. H. Chen, ed.), Amsterdam 1978, pp. 101–141.

[16] H. Tong: Non-linear Time Series: A Dynamical Systems Approach. Oxford University Press, Oxford 1990.

[17] R. S. Tsay: Testing and modeling multivariate threshold models. J. Amer. Statist. Assoc. *93* (1998), 1188–1202.

Jozef Komorník, Faculty of Management, Comenius University, 820 05 Bratislava. Slovak Republic.
e-mail: Jozef.Komornik@fm.uniba.sk

Magda Komorníková, Faculty of Civil Engineering, Slovak University of Technology, Radlinského 11, 813 68 Bratislava. Slovak Republic.
e-mail: magda@math.sk