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KERMACK–McKENDRICK EPIDEMIC MODEL REVISITED

Josef Štěpán and Daniel Hlubinka

This paper proposes a stochastic diffusion model for the spread of a susceptible-infective-removed Kermack–McKendric epidemic (M1) in a population which size is a martingale N_t that solves the Engelbert–Schmidt stochastic differential equation (2). The model is given by the stochastic differential equation (M2) or equivalently by the ordinary differential equation (M3) whose coefficients depend on the size N_t. Theorems on a unique strong and weak existence of the solution to (M2) are proved and computer simulations performed.

Keywords: SIR epidemic models, stochastic differential equations, weak solution, simulation

Mathematics Subject Classification: 37N25, 60H10, 60H35, 92D25

1. INTRODUCTION

An epidemy of a highly infectious disease with a fast recovery (or fatality) in a homogeneous population is considered, the influenza being an example of such epidemics. This classical Kermack–McKendrick model [12] assumes a fixed sized population of n individuals, the population being divided into three subpopulations which change their respective sizes in the running time of the epidemic: *Susceptibles* (the individuals exposed to the infection), *infectives* (the infected individuals that are able to spread the disease) and *removals* (the individuals restored to health not able further to spread the infection or get themselves to be infected again) numbering by $x(t)$, $y(t)$ and $z(t)$ the individuals that are susceptible, infected and removed at some time $t \geq 0$, respectively. Hence, $x(t) + y(t) + z(t) = n$, $x(t)$ and $z(t)$ being generally a nonincreasing and nondecreasing function, respectively such that $z(0) = 0$.

The model assumes the dynamics given by the following three dimensional differential equation

$$\begin{aligned}
\dot{x}(t) &= -\beta x(t)y(t), & x(0) &= x_0 > 0, \\
\dot{y}(t) &= \beta x(t)y(t) - \gamma y(t), & y(0) &= y_0 = n - x_0 > 0, \\
\dot{z}(t) &= \gamma y(t), & z(0) &= 0,
\end{aligned} \tag{M1}$$

where the intensity $\beta > 0$ is higher for more infectious diseases and the parameter $\gamma^{-1} > 0$ is proportional to the average duration of the "being infected" status, i. e. to the average time for which an individual is infected.

In general there is no explicit solution to (M1), the approximation $e^{-u} \sim 1 - u + \frac{1}{2}u^2$ is known, (see [7]) to deliver a unique solution z as

$$z(t) \sim \frac{\rho^2}{x_0}\left(\frac{x_0}{\rho} - 1\right) + \frac{\alpha\rho^2}{x_0}\tanh\left(\frac{1}{2}\gamma\alpha t - \varphi\right) \tag{S1}$$

with

$$\begin{aligned} \alpha &= \left[\frac{2x_0}{\rho^2}(n - x_0) + \left(\frac{x_0}{\rho} - 1\right)^2\right]^{1/2} \\ \varphi &= \tanh^{-1}\left[\frac{1}{\alpha}\left(\frac{x_0}{\rho} - 1\right)\right] \end{aligned} \tag{1}$$

where $\rho = \gamma/\beta$ denotes the relative removal rate of the disease. In some cases we may get a precise solution assuming a non constant intensity $\beta = \beta(x, y, z)$ or adopting a more simple and still realistic choice $\beta = \beta(z)$. Assuming, for example, that $\beta(z) \geq 0$ is a decreasing function we in fact propose a model in which the population grows to be more cautious and hence the epidemy will slow down its spread. See Theorem 1 and the examples that follow.

Our aim is to propose and justify a diffusion version of (M1) model that allows both more general intensities β and to model the spread of epidemic in a population that changes its size N_t due to a diffusion type emigration and immigration processes defined for example by the Engelbert–Schmidt stochastic differential equation

$$\mathrm{d}N_t = N_t\sigma(N_t)\,\mathrm{d}W_t, \quad N_0 = n_0 := x_0 + y_0. \tag{2}$$

If this is the case we assume

$$\sigma \geq 0 \quad \text{bounded such that} \quad \operatorname{supp}(\sigma) \subset [a, b], \quad \text{where} \quad 0 \leq a \leq n_0 \leq b < \infty, \tag{3}$$

which assumption keeps the size of population N_t in the interval $[a, b]$. Further we assume the ratios $x(t)$, $y(t)$ and $z(t)$ of susceptibles, infectives and removals, respectively, to derive their dynamics from a generalized (M1) model that is given as the ordinary differential equation with random time dependent coefficients

$$\begin{aligned} \dot{x}(t) &= -\alpha(x(t), y(t), z(t), N_t) \cdot x(t)y(t), & x(0) &= \frac{x_0}{n_0} \\ \dot{y}(t) &= \alpha(x(t), y(t), z(t), N_t) \cdot x(t)y(t) - \gamma \cdot y(t), & y(0) &= \frac{y_0}{n_0}, \\ \dot{z}(t) &= \gamma y(t), & z(0) &= 0, \end{aligned} \tag{M3}$$

(note that $x(t) + y(t) + z(t) = 1$), where $\alpha(x, y, z, n)$ is a function that is Lipschitz continuous on

$$\left\{(x, y, z) \in [0, 1]^3, x + y + z = 1\right\} \quad \text{uniformly for} \quad n \in [a, b].$$

If this is the case we are able to represent uniquely (Theorem 5 and Corollary 1) the size $X_t = x(t) \cdot N_t$, $Y_t = y(t) \cdot N_t$ and $Z_t = z(t) \cdot N_t$ of suspectibles, infectives and removals, respectively, as the solution to a three dimensional SDE

$$\begin{aligned} \mathrm{d}X_t &= -\beta(X_t, Y_t, Z_t)X_tY_t\,\mathrm{d}t + X_t\sigma(N_t)\,\mathrm{d}W_t, & X_0 &= x_0 > 0 \\ \mathrm{d}Y_t &= \beta(X_t, Y_t, Z_t)X_tY_t\,\mathrm{d}t - \gamma Y_t\,\mathrm{d}t + Y_t\sigma(N_t)\,\mathrm{d}W_t, & Y_0 &= y_0 > 0 \\ \mathrm{d}Z_t &= \gamma Y_t\,\mathrm{d}t + Z(t)\sigma(N_t)\,\mathrm{d}W_t, & Z_0 &= 0, \end{aligned} \tag{M2}$$

where the intensities $\alpha(x, y, z, n)$ and $\beta(x, y, z)$ rescale each other as $\alpha(x, y, z, n) = n \cdot \beta(nx, ny, nz)$. Note that the size of population $N_t = X_t + Y_t + Z_t$ where (X, Y, Z) solves (M2) is a solution to (2). Theorems 2, 5 and Corollary 2 offer sufficient conditions for (M2) to have a unique strong and weak solution, respectively. Section 1, namely Theorem 1, summarizes and extends well-known properties of the solution to the deterministic model (M1) and also possible choices of the intensity $\beta(z)$ are listed. The article is closed by Example 5 that delivers a visible computer illustration of the above results.

We refer the reader to [4, 7], and to more recent [8] and [3] for the history and present state of art of stochastic modelling of epidemics.

The martingale and diffusion models probably first appeared in [6] where (in our setting and notation) the stochastic process $M_t := Y_t - \int_0^t (\beta X_u Y_u - \gamma Y_u)\,\mathrm{d}u$ is assumed to be a martingale which assumption makes it possible to estimate the constant intensity β.

References [1, 2, 13, 14] propose a multidimensional diffusion model built up on the top of the deterministic infection in a population that consists only of suspectibles and infectives: Having interpreted γ as the disease death rate and β as its transmission rate, denoting by $N_t = X_t + Y_t$ the size of the population we may simplify this model to the dimension one as

$$\dot{X}_t = -\frac{\beta}{N_t}X_tY_t, \quad \dot{Y}_t = \frac{\beta}{N_t}X_tY_t - \gamma Y_t, \quad \text{hence} \quad \dot{N}_t = -\gamma Y_t.$$

The authors propose its diffusion version given by the non-linear stochastic differential equation

$$\begin{aligned} \mathrm{d}X_t &= -\beta\frac{X_tY_t}{N_t}\,\mathrm{d}t + b_{11}(t)\,\mathrm{d}W_t^1 + b_{12}(t)\,\mathrm{d}W_t^2, \\ \mathrm{d}Y_t &= \beta\frac{X_tY_t}{N_t}\,\mathrm{d}t - \gamma Y_t\,\mathrm{d}t + b_{21}(t)\,\mathrm{d}W_t^1 + b_{22}(t)\,\mathrm{d}W_t^2, \end{aligned}$$

where (W^1, W^2) is a two-dimensional standard Wiener process and

$$B = B(X, Y) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} \frac{\beta}{N}XY & -\frac{\beta}{N}XY \\ -\frac{\beta}{N}XY & \frac{\beta}{N}XY + \gamma Y \end{pmatrix}^{\frac{1}{2}}.$$

Phillips–Saranson theorem [16, 12.12 Theorem, p. 134] proves that the equation has a unique strong solution and authors perform its sophisticated and instructive simulations. Note that the size of population N_t is a supermartingale in this case.

Obviously, while the (M2) model expects an epidemic or even a pandemic with only negligible fatal consequences, the model proposed by [2] may be relevant when the infections as HIV–AIDS in humans are studied.

[19] proved a unique existence theorem for a differential equation that, if generalized to $\mathbb{R}^3$, would cover the equation (M3).

2. KERMACK–McKENDRIC DETERMINISTIC EPIDEMIC

Here we shall treat (M1) deterministic model with an intenzity $\beta = \beta(z)$ which values are subject to changes dependent on the size of the *removals subpopulation*. Information on a solution $(x(t), y(t), z(t))$ to (M1) we are able recover in this case is summarized by

Theorem 1. Consider $\gamma > 0$ and assume $\beta = \beta(z)$ to be a nonnegative, bounded and locally Lipschitz continuous function on $\mathbb{R}^+$. Then (M1) has a unique solution $(x, y, z) \in C^1(\mathbb{R}^+, \mathbb{R}^3)$ that is positive on $(0, \infty)$ and such that

$$x = X(z), \quad y = Y(z) \quad \text{and} \quad \dot{z} = \gamma[n - z - X(z)], \quad z(0) = 0, \tag{4}$$

hold, where

$$X(z) = x_0 \exp\left\{-\frac{1}{\gamma}\int_0^z \beta(u)\,\mathrm{d}u\right\} \quad \text{and} \quad Y(z) = n - z - X(z), \quad 0 \le z \le n. \tag{5}$$

The number of susceptibles individuals $x(t)$ and the number of removals $z(t)$ is a positive nonincreasing and increasing function on $\mathbb{R}^+$, respectively, such that $0 < x(\infty) < n$ and $0 < z(\infty) \le n$ while $y(\infty)$ exists and equals to zero.

The limit $z(\infty)$ is a solution to the equation

$$n - z = X(z) \tag{6}$$

and if $\beta(z)$ is nonincreasing on $\mathbb{R}^+$ then it is a unique solution to the equation on the interval $[0, n]$.

If $\beta(0) > 0$, $\frac{\gamma}{\beta(0)} < x_0$ and $\beta(z)$ is again a nonincreasing function then the number of infectives has a unique maximum $y_+ = y(t_+)$, where

$$y_+ = n - z_+ - \frac{\gamma}{\beta(z_+)}, \quad t_+ = \frac{1}{\gamma}\int_0^{z_+} \frac{1}{Y(u)}\,\mathrm{d}u \tag{7}$$

and z_+ is a unique $0 < z < z(\infty)$ such that

$$\beta(z)X(z) = \gamma. \tag{8}$$

The number of infectives $y(t)$ is increasing on $[0, t_+]$ and decreasing on the interval $[t_+, \infty]$.

All the above statements are proved for a constant $\beta(z)$ in [7], for example.

Proof. The unique existence part follows from a more general Theorem 3 choosing there $\sigma = 0$. Also, apply (12), (13) and (14) in Lemma 1 to verify (4) and that $(x, y, z) > 0$ on $(0, \infty)$, hence $(x, y, z) \in [0, n]^3$ on $\mathbb{R}^+$. It follows that $\dot{z} = \gamma y > 0$, hence z is an increasing function on $\mathbb{R}^+$ with $z(\infty) \in (0, n]$. Similarly $\dot{x} \leq 0$ and x is seen as nonincreasing on $\mathbb{R}^+$ with $n > x_0 \geq x(\infty) = X(z(\infty)) > 0$. Thus, $y(t) = n - x(t) - z(t)$ has a finite limit $y(\infty)$. Assuming $y(\infty) > 0$ we conclude that $\lim_{t\to\infty} \dot{z}(t) > 0$ which implies $z(\infty) = \infty$, hence a contradiction. Finally, $z(\infty) = n - x(\infty) < n$.

The limit $z(\infty)$ solves (6) because $n - z(\infty) = x(\infty) = X(z(\infty))$, according to (4).

Assuming that $\beta(z)$ is a nonincreasing function we get $X(z)$ to be convex on $\mathbb{R}^+$. Hence if $0 < z_1 < z_2 \leq n$ are two distinct solutions to (6), then the graph of $X(z)$ on $[0, z_2]$ is bellow the segment that connects the points $(0, x_0)$ and $(z_2, n - z_2)$, which segment is further strictly bellow the segment that joins points $(0, n)$ and $(z_2, n - z_2)$. It follows that $n - z_1 > X(z_1)$ which is a contradiction.

Compute

$$Y'(z) = -1 + \frac{\beta(z)}{\gamma} X(z), \quad \text{and} \quad Y'(0^+) = -1 + \frac{\beta(0)}{\gamma} x_0 > 0. \tag{9}$$

It follows that there is a $z \in (0, z(\infty))$ such that $Y'(z) = 0$ as $Y(0) = y_0 > 0$ and $Y(z(\infty)) = 0$. Assuming that $Y'(z_1) = Y'(z_2) = 0$ for a pair $0 < z_1 < z_2 < z(\infty)$, or equivalently that (7) has not a unique solution in $(0, z(\infty))$, it follows that $1/X(z_1) = \beta(z_1)/\gamma \geq \beta(z_2)/\gamma = 1/X(z_2)$. This and the inequality $X(z_1) \geq X(z_2)$ imply that $X(z_1) = X(z_2)$ and further that $\beta(z) = 0$ on $[z_1, z_2]$. Hence, $\frac{1}{X(z_1)} = 0$ that contradicts the definition of $X(z)$. Thus, there is unique $z_+ \in (0, z(\infty))$ such that $-1 + \frac{\beta(z_+)}{\gamma} X(z_+) = Y'(z_+) = 0$ and $Y(z)$ increases and decreases on $(0, z_+)$ and $(z_+, z(\infty))$, respectively, $Y(z_+)$ being its unique maximum.

Finally, let $t = t(z)$ be the inverse function to $z = z(t)$. It follows by (M1) that $\frac{\mathrm{d}t}{\mathrm{d}z} = \frac{1}{\gamma Y(z)}$, hence $t(z) = \frac{1}{\gamma}\int_0^z \frac{1}{Y(u)}\,\mathrm{d}u$ and $t_+ = \frac{1}{\gamma}\int_0^{z_+} \frac{1}{Y(u)}\,\mathrm{d}u$ is the only argument of $\max_{t<\infty} y(t) = Y(z_+)$. □

Remark 1. The number of removals $z(t)$ is a unique solution to the differential equation (4). Having computed the integral

$$t = t(z) = \frac{1}{\gamma}\int_0^z \left[n - u - x_0 \exp\left\{ -\frac{1}{\gamma}\int_0^u \beta(v)\,\mathrm{d}v \right\} \right]^{-1} \mathrm{d}u, \tag{10}$$

and putting $z(t) = t^{-1}(z)$ we get the solution to (4).

Remark 2. The size of relative removal rate $\rho = \gamma/\beta$ is considered by epidemiologists as a good measure of the virulence of an epidemic: If x_0 exceeds ρ only by a small quantity then we model something as the common cold or a weak influenza. On the other hand, its values $\rho \sim x_0/2$ indicate the danger of a pandemic spread of infection.

Example 1. We may illustrate this by choosing a constant β, $n = 10000$, $x_0 = 9995$, $\gamma = \frac{1}{3}$ and $\rho^1 \sim x_0$, and $\rho^2 \sim \frac{1}{2}x_0$, respectively. See Figures 1 and 2 for the numerical illustration of the epidemics dynamics during first 60 days.

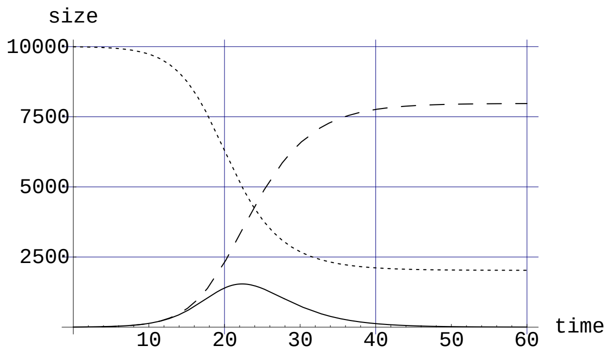

Fig. 1. The time dynamics of susceptibles (dotted line), infected (solid line) and removed (dashed line), $\rho = 4995$ (pandemic).

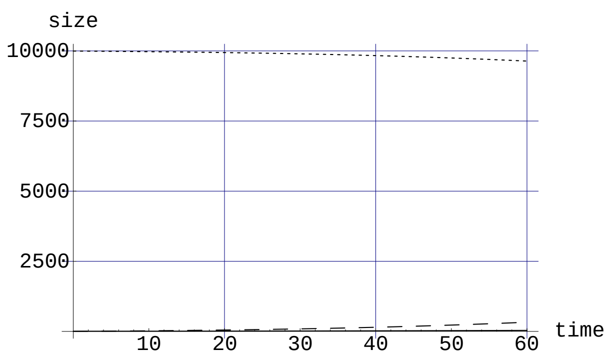

Fig. 2. The time dynamics of susceptibles (dotted line), infected (solid line) and removed (dashed line), $\rho = 8995$ (mild epidemic).

We get

ρ	$z(\infty)$	y_+	t_+
4995	7975	1540	22.33
8995	948	57	98.29

where $y_+ = \max_t y(t) = y(t_+)$. The limits $z(\infty) = z$ are given by (6), i. e. as the solutions to $n - z = x_0 \exp\{-z/\rho\}$, the maxima y_+ are equal to $n - z_+ - \rho$ where $z_+ = \rho \ln(x_0/\rho)$ by (7) and (8). To establish the arguments of maxima t_+ we apply the procedure (7) and compute

$$t_+ = \frac{1}{\gamma} \int_0^{z_+} \frac{1}{n - u - X(u)}\, \mathrm{d}u, \quad \text{where} \quad X(u) = x_0 \exp\left\{-\frac{u}{\rho}\right\}.$$

We have used program [9] in Mathematica® for numerical calculation (and interpolation) of $x(\cdot), y(\cdot), z(\cdot)$ and t_+.

Remark 3. Having postulated that

$$\beta(z) \geq 0 \quad \text{is a nonincreasing function supported by a compact} \quad [0, z_1],$$
$$\text{where} \quad z_1 > 0 \quad \text{and such that} \quad \frac{\gamma}{\beta(0)} < x_0,$$

we assume that the intensity of infection monotonously decreases with the increasing number of removals and becomes negligible at the time t_1 when $z(t_1) = z_1$. Note that $\rho(0) = \gamma/\beta(0) < x_0$ if and only if $\dot{y}(0+) > 0$, and the latter inequality is a condition necessary and sufficient for the outbreak of the epidemic. Also note that the number of susceptibles $x(t) = X(z(t))$ is decreasing on $[0, t_1]$ and then remains at the level $x(t_1) = X(z_1)$ for ever. The number of removals will be of course still increasing to $z(\infty) = n - X(z_1) \geq z_1$.

Example 2. Choose n, x_0, β_0, γ and z_1 positive such that $\rho_0 = \frac{\gamma}{\beta_0} < x_0$ and put $\beta(z) = \beta_0(1 - \frac{z}{z_1})$ for $z \in [0, z_1]$ and $\beta(z) = 0$ if $z \geq z_1$. We get $X(z) = x_0 \exp\{-\frac{1}{\rho_0} z(1 - \frac{z}{2z_1})\}$ and $z(\infty) = n - X(z_1)$. The time t_+ when the number of infectives achieves its maximum generally precedes the time t_1 of the first entry of $z(t)$ to z_1 since $\dot{y}(t_1) = -\gamma y(t_1) < 0$. The maximum y_+ and its argument t_+ are computed by the procedure suggested by (7) and (8) that are specified as

$$y_+ = n - z_+ - \frac{z_1\gamma}{(z_1 - z_+)\beta_0}, \quad t_+ = \frac{1}{\gamma}\int_0^{z_+} \frac{1}{Y(u)}\,\mathrm{d}u, \quad \frac{z_1 - z_+}{z_1}\beta_0 X(z_+) = \gamma.$$

The differential equation (M1) reads in this case as

$$\begin{aligned}
\dot{x}(t) &= \begin{cases} -\beta_0 \frac{z_1 - z(t)}{z_1} x(t)y(t), & z(t) < z_1 \\ 0, & z(t) \geq z_1 \end{cases} && x(0) = x_0 > 0,\\
\dot{y}(t) &= \begin{cases} \beta_0 \frac{z_1 - z(t)}{z_1} x(t)y(t) - \gamma y(t), & z(t) < z_1 \\ -\gamma y(t), & z(t) \geq z_1 \end{cases} && y(0) = y_0 = n - x_0 > 0,\\
\dot{z}(t) &= \gamma y(t), && z(0) = 0,
\end{aligned}$$

Note that $\dot{\beta}(z(t)) = -\frac{\beta_0\gamma}{z_1}\dot{y}(t)$ for $0 < t < t_1$ in this case.

For a numerical illustration we have chosen similar values as in Example 1, in particular $n = 10000$, $\gamma = \frac{1}{3}$ and $\rho_0 = 4995$. We have chosen $z_1 = 4000$. The time course of the epidemics is shown in Figure 3. Compare Figure 3 with decreasing intensity of infection and Figure 1 to see that the total number of infected is much smaller now.

We have again used Mathematica® to calculate the limit $z(\infty) \doteq 3303$ and $y_+ \doteq 1133$, $z_+ = 1344$ and $t_+ = 19.0402$.

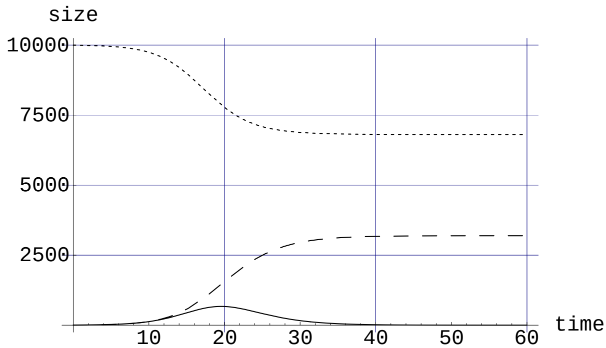

Fig. 3. The time dynamics of susceptibles (dotted line), infected (solid line) and removed (dashed line), $\beta(z)$ decreasing function.

Example 3. Kendall [11] proposes (see also [7]) a more sophisticated choice for $\beta(z)$ in the form

$$\beta(z) = \begin{cases} \frac{2\beta_0}{(1-z/\rho_0)+(1-z/\rho_0)^{-1}}, & 0 \le z \le \rho_0, \quad \rho_0 = \frac{\gamma}{\beta_0}, \gamma, \beta_0 > 0. \\ = 0, & z \ge \rho_0, \end{cases}$$

Note that $\beta(z)$ is chosen such that the assumptions of Remark 3 are satisfied, in particular $z_1 = \rho_0$, $\beta(0) = \beta_0$, and $z(\infty) = n - X(\rho_0) = (x_0/\rho - 1 + \alpha)\rho^2/x_0$, with α given by (1).

3. KERMACK–McKENDRICK WITH A STOCHASTIC EMIGRATION AND IMMIGRATION

A random three dimensional dynamics $L_t = (X_t, Y_t, Z_t)$, $t > 0$ of the number X_t, Y_t and Z_t of susceptibles, infectives and removals, respectively, with the size of population $N_t = X_t + Y_t + Z_t$ whose stochastic nature is caused by small random emigration-immigration perturbations is proposed by the stochastic differential equations (M2) where $\beta(x, y, z)$ and $\sigma(n)$ are suitable functions on $\mathbb{R}^3$ and $\mathbb{R}^1$, respectively. Note that the size N_t is a solution to the Engelbert–Schmidt stochastic differential equation (2). The nature of $\beta(x, y, z)$, or perhaps more simply of $\beta(z)$, and the constant $\gamma > 0$ that enter (M2) is explained in Section 1. The diffusion coefficient $\sigma(n)$ is designed to control the global size of the population N_t inside reasonable bounds. It can be seen, for example, by choosing $\sigma(n)$ as in (3) since $a \le N_t \le b$ holds almost surely in this case. Note that choosing $a = b = n_0$ and $\sigma = 0$ (M2) becomes (M1).

Let us agree that $L_t = (X_t, Y_t, Z_t)$ is a solution to (M2) if (M2) makes sense and if it holds for a standard Brownian motion W_t defined on a complete probability space $(\Omega, \mathcal{F}, P)$, i. e. if X, Y and Z are continuous $\mathcal{F}_t$-semimartingales that satisfy (M2), having denoted by $\mathcal{F}_t$ the augmented canonical filtration of the process W_t.

Recall that (M2) is said to have a unique strong solution if there is an almost surely determined solution on arbitrary $(\Omega, \mathcal{F}, P, W)$. Also recall that (M2) is said to have a unique weak solution if there is a setting $(\Omega, \mathcal{F}, P, W)$ on which a solution X, Y, Z may be constructed and if $\mathcal{L}(X,Y,Z) = \mathcal{L}(X', Y', Z')$ whatever solutions to(M2) (X, Y, Z) and (X', Y', Z') might be chosen.

By Itô formula, or more precisely by [10, Proposition 21.2] on Doleáns equation, we prove easily the following lemma.

Lemma 1. Assume $\beta(z, y, z)$ and $\sigma(n)$ bounded. Then (X, Y, Z) is a solution to (M2) if and only if

$$N_t = n_0 \exp\left\{\int_0^t \sigma(N_u)\,\mathrm{d}W_u - \frac{1}{2}\int_0^t \sigma^2(N_u)\,\mathrm{d}u\right\}, \tag{11}$$

$$X_t = \frac{x_0}{n_0}\exp\left\{-\int_0^t \beta(X_u, Y_u, Z_u)Y_u\,\mathrm{d}u\right\} \cdot N_t, \tag{12}$$

$$Y_t = \frac{y_0}{n_0}\exp\left\{\int_0^t \beta(X_u, Y_u, Z_u)X_u\,\mathrm{d}u - \gamma t\right\} \cdot N_t, \tag{13}$$

$$Z_t = \gamma \int_0^t \frac{Y_u}{N_u}\,\mathrm{d}u \cdot N_t \tag{14}$$

hold for all $t \geq 0$ almost surely. Especially, processes X,Y and N are positive on $\mathbb{R}^+$ and Z is a process positive on $(0, \infty)$.

Theorem 2. Assume β nonnegative and bounded, let σ satisfy (3) and consider arbitrary solution (X, Y, Z) to (M2). Then the following statements hold;

(i) The size of population N_t is a bounded martingale such that $a \leq N \leq b$ holds and such that the limit $N_\infty := \lim_{t\to\infty} N_t$ exists. Especially, $\mathrm{E}N_t = n_0$ holds for all $t \geq 0$.

(ii) If $\mathrm{supp}(\sigma) = (a, b)$ and $\int_a^{a^+} \frac{1}{u^2\sigma^2(u)}\,\mathrm{d}u = \int_{b-}^{b} \frac{1}{u^2\sigma^2(u)}\,\mathrm{d}u = \infty$, the limit N_∞ equals to a and b with probability $\frac{b-n_0}{b-a}$ and $\frac{n_0-a}{b-a}$, respectively.

(iii) The number of susceptibles $0 < X \leq b$ and the number of removals $0 \leq Z \leq b$ is a supermartingale and submartingale, respectively. The process Z is positive on $(0, \infty)$. Especially, $\mathrm{E}X_s \geq \mathrm{E}X_t$ and $\mathrm{E}Z_s \leq \mathrm{E}Z_t$ holds for $s \leq t$.

(iv) All limits X_∞, Y_∞ and Z_∞ exist, $Y_\infty = 0$ and $X_\infty > 0$ if and only if $N_\infty > 0$, hence assuming $a > 0$ we get X_∞ as a positive random variable.

Observe that all the above statements, equalities and inequalities are meant to hold almost surely, for example $X > 0$ is to be read as $X_t > 0$ holds for all $t \geq 0$ almost surely. The stated martingale and semimartingale properties are w.r.t. the augmented canonical filtration of W_t denoted by $\mathcal{F}_t$. For example, the statement X is a supermartingale is precisely as X is an $\mathcal{F}_t$-supermartingale.

P r o o f. The size of population N_t is a positive local martingale by definition and by (11). It is a constant on a bounded interval (s,t) if and only if its quadratic variation $[N]_t = \int_0^t N(u)^2\sigma^2(N_u)\,\mathrm{d}u$ has the same property [16, 30.4 Theorem, p. 54]. Since $[\sigma^2 > 0] = (a,b)$ we reason that $a \le N \le b$ and therefore N_t is a bounded martingale. Thus, according to [15, Theorem 69.1, p. 176] $a \le N_\infty \le b$ exists. We have proved (i).

The proof of (ii) will be postponed until Theorem 5.

Since N_t is a martingale and $-\int_0^t \beta(X_u,Y_u,Z_u)Y_u\,\mathrm{d}u$ a nonincreasing process we get $0 \le X \le b$ as a bounded supermartingale by (12) in Lemma 1. Similarly, we verify that $0 \le Z_t \le b$ is a bounded submartingal by (14). This concludes the proof of (iii).

It follows by the supermartingale and submartingale property of bounded processes X_t and Z_t, respectively, applying [15, Th. 69.1] again, that X_∞ and Z_∞ exist. To prove that $Y_\infty = N_\infty - X_\infty - Z_\infty = 0$ we shall verify that $\int_0^\infty Y_u\,\mathrm{d}u < \infty$: Observe first that $\int_0^t Z_u^2\sigma^2(N_u)\,\mathrm{d}u < \infty$ holds for all $t > 0$. This implies that $\int_0^t Z_u\sigma(N_u)\,\mathrm{d}W_u$ defines an L_2-martingale. Hence, $\mathrm{E}\int_0^t Y_u\,\mathrm{d}u = \frac{1}{\gamma}\mathrm{E}Z_t$ by (M2) and $\mathrm{E}\int_0^\infty Y_u\,\mathrm{d}u = \frac{1}{\gamma}\mathrm{E}Z_\infty \le \frac{b}{\gamma}$. In particular, $\int_0^\infty Y_u\,\mathrm{d}u < \infty$. Finally, observing that $\int_0^\infty \beta(X_u,Y_u,Z_u)Y_u\,\mathrm{d}y \le c\int_0^\infty Y_u\,\mathrm{d}u < \infty$ for a $c \in (0,\infty)$, it follows by (12) that $X_\infty > 0$ if and only if $N_\infty > 0$ and (iv) is proved completely. □

A unique strong solution to (M2) exists under fairly mild requirements on the coefficients $\beta(x,y,z)$ and $\sigma(n)$. Putting $L = (X,Y,Z)$ we write (M2) as

$$\mathrm{d}L_t = b(L_t)\,\mathrm{d}t + a(L_t)\,\mathrm{d}B_t, \quad L_0 = (x_0,y_0,0), \tag{15}$$

where

$$b(x,y,z) = \begin{pmatrix} b_1(x,y,z) \\ b_2(x,y,z) \\ b_3(x,y,z) \end{pmatrix} = \begin{pmatrix} -xy\beta(x,y,z) \\ xy\beta(x,y,z) - \gamma y \\ \gamma y \end{pmatrix} \quad : \mathbb{R}^3 \to \mathbb{R}^3,$$

$$a(x,y,z) = \begin{pmatrix} a^{11}(x,y,z) & 0 & 0 \\ a^{21}(x,y,z) & 0 & 0 \\ a^{31}(x,y,y) & 0 & 0 \end{pmatrix} = \begin{pmatrix} x\sigma(x+y+z) & 0 & 0 \\ y\sigma(x+y+z) & 0 & 0 \\ z\sigma(x+y+z) & 0 & 0 \end{pmatrix} : \mathbb{R}^3 \to \mathbb{M}^3$$

and $B_t = (W_t^1, W_t^2, W_t^3)$ is a three dimensional Brownian motion with $W_t^1 = W_t$. By $\mathbb{M}^3$ we have denoted the space of the 3×3-matrices endowed with the Eucleidian norm. A standard of stochastic analysis [16, V.12.1 Theorem] says that (15) has a unique strong solution provided that the maps b and a are locally Lipschitz and of a linear growth. Even though our particular case (M2) involves a coefficient $b = (b_1,b_2,b_3)$ that is not of a linear growth whatever nontrivial bounded $\beta(x,y,z)$ may be chosen, we are able to prove

Theorem 3. Let $\sigma(n)$ be a Lipschitz continuous function such that (3) holds for some $a \le b$. Further, assume that $\beta(x,y,z)$ is a nonnegative function that is Lipschitz continuous on

$$\Delta_{ab} := \left\{(x,y,z) \in [0,b]^3, \quad a \le x+y+z \le b\right\}.$$

Then (M2) has a unique strong solution $L = (X, Y, Z)$ that is a positive process on $(0, \infty)$ such that $a \le N = X + Y + Z \le b$.

Proof. Choose bounded locally Lipschitz $\varphi_i(x, y, z) : \mathbb{R}^3 \to \mathbb{R}$ such that

$$\varphi_1(x, y, z) = -\beta(x, y, z)y, \quad \varphi_2(x, y, z) := \beta(x, y, z)x \quad \text{holds for} \quad (x, y, z) \in \Delta_{ab}$$

and consider the stochastic differential equations

$$\begin{aligned} \mathrm{d}X_t &= \varphi_1(X_t, Y_t, Z_t)X_t\,\mathrm{d}t + X_t\sigma(N_t)\,\mathrm{d}W_t, & X_0 &= x_0 \\ \mathrm{d}Y_t &= \varphi_2(X_t, Y_t, Z_t)Y_t\,\mathrm{d}t + Y_t\sigma(N_t)\,\mathrm{d}W_t - \gamma Y_t\,\mathrm{d}t, & Y_0 &= y_0 \\ \mathrm{d}Z_t &= \gamma Y_t\,\mathrm{d}t + Z_t\sigma(N_t)\,\mathrm{d}W_t, & Z_0 &= 0, \end{aligned} \tag{16}$$

where $N_t = X_t + Y_t + Z_t$. This is the case of equation (15) that has coefficients $b(x, y, z)$ and $a(x, y, z)$ that are locally Lipschitz and of a linear growth. Hence, by [16, V.12.1 Theorem], (16) has a unique strong solution $L = (X, Y, Z)$ that is positive on $(0, \infty)$:

$$\begin{aligned} X_t &= x_0 \exp\left\{\int_0^t \varphi_1(X_u, Y_u, Z_u) - \frac{1}{2}\sigma^2(N_u)\,\mathrm{d}u + \int_0^t \sigma(N_u)\,\mathrm{d}W_u\right\}, \\ Y_t &= y_0 \exp\left\{\int_0^t \varphi_2(X_u, Y_u, Z_u) - \frac{1}{2}\sigma^2(N_u)\,\mathrm{d}u - \gamma t + \int_0^t \sigma(N_u)\,\mathrm{d}W_u\right\}, \end{aligned}$$

and

$$\begin{aligned} Z_t = \exp&\left\{\int_0^t \sigma(N_u)\,\mathrm{d}W_u - \frac{1}{2}\int_0^t \sigma^2(N_u)\,\mathrm{d}u\right\} \times \\ &\int_0^t \exp\left\{-\int_0^u \sigma(N_w)\,\mathrm{d}W_w + \frac{1}{2}\int_0^u \sigma^2(N_w)\,\mathrm{d}w\right\} \cdot \gamma Y_u\,\mathrm{d}u \end{aligned}$$

by [10, Theorem 21.2] again. Denote by $\tau := \inf\{t > 0 : L_t \notin \Delta_{ab}\}$ the first entry of $L = (X, Y, Z)$ to the complement of Δ_{ab}. Obviously, (16) yields equations

$$\begin{aligned} X_{t\wedge\tau} &= x_0 + \int_0^{t\wedge\tau} -\beta(X_u, Y_u, Z_u)X_uY_u\,\mathrm{d}u + \int_0^{t\wedge\tau} X_u\sigma(N_u)\,\mathrm{d}W_u, \\ Y_{t\wedge\tau} &= y_0 + \int_0^{t\wedge\tau} \beta(X_u, Y_u, Z_u)X_uY_u - \gamma Y_u\,\mathrm{d}u + \int_0^{t\wedge\tau} Y_u\sigma(N_u)\,\mathrm{d}W_u \\ Z_{t\wedge\tau} &= \gamma\int_0^{t\wedge\tau} Y_u\,\mathrm{d}u + \int_0^{t\wedge\tau} Z_u\sigma(N_u)\,\mathrm{d}W_u. \end{aligned} \tag{17}$$

Hence, $N_t = n_0 + \int_0^t N_u\sigma(N_u)\,\mathrm{d}W_u$ for $t \le \tau$ and therefore $a \le N_t \le b$ for any such t as $\mathrm{supp}(\sigma) \subset [a, b]$. Obviously there is no $t > 0$ for which either $N_t \notin [a, b]$ or $\min(X_t, Y_t, Z_t) < 0$ would hold, consequently there is no $t > 0$ such that $\max(X_t, Y_t, Z_t) > b$. Hence $\tau = \infty$. Reading (17) again we conclude that (X, Y, Z) solves (M2).

Finally, if (X, Y, Z) is a solution to (M2) it follows from Theorem 2 that $(X_t, Y_t, Z_t) \in \Delta_{ab}$ at any time t. It yields that (X, Y, Z) is a solution to (16). The stochastic differential equation (M2) has a unique strong solution since (16) has the property. □

Later on we shall appreciate even a more general result.

Theorem 4. Consider (M2) model with a time dependent intensity $\beta(x,y,z,t) \geq 0$ that is Lipschitz continuous on Δ_{ab} uniformly for $t \geq 0$ (especially bounded on Δ_{ab}). Also let a Lipschitz continuous $\sigma(n)$ to satisfy (3) for some $a \leq b$. Then (M2) has a unique strong solution and any solution $L = (X, Y, Z)$ to (M2) is a positive process on $(0, \infty)$ with $a \leq N = X + Y + Z \leq b$.

The proof goes along the lines of the reasoning employed in the proof of of Theorem 3 only a finer construction of Lipschitz extensions φ_i has to be performed: Define $\varphi_i : C(\mathbb{R}^+, \mathbb{R}^3) \times \mathbb{R} \to \mathbb{R}$ for $t \geq 0$, $(x_., y_., z_.) \in C(\mathbb{R}^+, \mathbb{R}^3)$ by

$$\varphi_1(x_., y_., z_., t) = -\beta(x_{t\wedge\tau}, y_{t\wedge\tau}, z_{t\wedge\tau}, t\wedge\tau) \cdot y_{t\wedge\tau}$$
$$\varphi_2(x_., y_., z_., t) = \beta(x_{t\wedge\tau}, y_{t\wedge\tau}, z_{t\wedge\tau}, t\wedge\tau) \cdot x_{t\wedge\tau},$$

where $\tau = \tau(x_., y_., z_.) = \inf\{t > 0 : (x_t, y_t, z_t) \notin \Delta_{ab}\} : C(\mathbb{R}^+, \mathbb{R}^3) \to [0, \infty]$ is time of the first entry of the $C(\mathbb{R}^+, \mathbb{R}^3)$-canonical process $l_t = (x_t, y_t, z_t)$ to the complement of Δ_{ab}. It easy to check that φ_1 and φ_2 are bounded progressive path functionals on $C(\mathbb{R}^+, \mathbb{R}^3) \times \mathbb{R}^+$ that are Lipschitz continuous on $C(\mathbb{R}^+, \mathbb{R}^3)$, i. e. such that

$$|\varphi_i(l_., t) - \varphi_i(l'_., t)| \leq C \cdot \|l - l'\|_t, \quad l, l' \in C(\mathbb{R}^+, \mathbb{R}^3), \quad t \geq 0 \tag{18}$$

holds for a constant C and $\|l\|_t$ denotes the sup-norm in $C([0,t], \mathbb{R}^3)$. Putting

$$b(l_., t) = \begin{pmatrix} \varphi_1(l_., t)x_t \\ \varphi_2(l_., t)y_t - \gamma y_t \\ \gamma y_t \end{pmatrix}, \quad a(l_., t) = \begin{pmatrix} x_t\sigma(x_t + y_t + z_t) & 0 & 0 \\ y_t\sigma(x_t + y_t + z_t) & 0 & 0 \\ z_t\sigma(x_t + y_t + z_t) & 0 & 0 \end{pmatrix}$$

for $l_t = (x_t, y_t, z_t) \in C(\mathbb{R}^+, \mathbb{R}^3)$ and $t \geq 0$ we propose a three dimensional SDE

$$\mathrm{d}L_t = b(L_t, t)\,\mathrm{d}t + a(L_t, t)\,\mathrm{d}B_t, \qquad L_0 = (x_0, y_0, 0) \tag{19}$$

where $L_t = (X_t, Y_t, Z_t)$ and $B_t = (W_t, W_t^2, W_t^3)$ is a three dimensional standard Brownian motion as before. Because b and a are progressive path functionals

$$b : C(\mathbb{R}^+, \mathbb{R}^3) \times \mathbb{R}^+ \to \mathbb{R}^3, \quad a : C(\mathbb{R}^+, \mathbb{R}^3) \times \mathbb{R}^+ \to \mathbb{M}^3$$

such that

$$|b(l_., t) - b(l'_., t)| + |a(l_., t) - a(l'_., t)| \leq C_N \cdot \|l - l'\|_t,$$

holds for $\|l\| \leq N$, $\|l'\| \leq N$, $N \in \mathbb{N}$ and $t \geq 0$ and

$$|b(l_., t)| + |a(l_., t)| \leq C \cdot (1 + \|l\|)$$

holds for all $l \in C(\mathbb{R}^+, \mathbb{R}^3)$ and $t \geq 0$ according to (18), where C_N and C are constants. We have denoted by $|b|$, $|a|$, the Eucleidian norms in $\mathbb{R}^3$ and $\mathbb{M}^3$, respectively and $\|l\|$ the sup-norm in $C(\mathbb{R}^+, \mathbb{R}^3)$. It follows by [16, V.12.1 Theorem], p. 131 that (19) has a unique strong solution. Choosing a solution $L = (X, Y, Z)$ to (19) we proceed the proof exactly as in the proof of Theorem 3 putting there $\tau = \tau(L)$ where $\tau : C(\mathbb{R}+, \mathbb{R}^3) \to [0, \infty]$ is defined as the first entry of $l_.$ to $\mathbb{R}^3 \setminus \Delta_{ab}$.

Remark 4. Saying that $b(l_{\cdot}, t)$ is a progressively measurable map we mean that b is an $\mathcal{C}_t$-progressive process on $C(\mathbb{R}^+, \mathbb{R}^3)$ where $\mathcal{C}_t$ is the minimal right continuous filtration built up on the top of the canonical filtration $\mathcal{H}_t$ in $C(\mathbb{R}^+, \mathbb{R}^3)$. It is easy to check that V.12.1 Theorem of [16], that is formulated for $\mathcal{H}_t$-progressive (previsible) coefficients, is applicable more generally for the $\mathcal{C}_t$-progressive path functionals a and b. Our reason for this enrichment is that τ, being a $\mathcal{C}_t$-Markov time, lacks this property with respect to the canonical filtration $\mathcal{H}_t$.

4. WEAK SOLUTIONS TO KERMACK–McKENDRICK STOCHASTIC EPIDEMIC MODEL

The following theorem suggests a two-step procedure for obtaining a solution (X, Y, Z) to (M2).

Theorem 5. A semimartingale $L_t = (X_t, Y_t, Z_t)$ is a solution to (M2) if and only if

$$X_t = x(t) \cdot N_t, \quad Y_t = y(t) \cdot N_t, \quad Z_t = z(t) \cdot N_t, \quad t \geq 0, \quad \text{almost surely,} \tag{20}$$

where N_t is is a solution to Engelbert–Schmidt stochastic differential equation (2) and $l(t) = \big(x(t), y(t), z(t)\big)$ is a semimartingale that solves almost surely the ordinary differential equation with random coefficients (M3) where $\alpha(x, y, z, n) = n \cdot \beta(nx, ny, nz)$ for $(x, y, z, n) \in \mathbb{R}^4$.

Proof. Let $L = (X, Y, Z)$ to solve (M2) and put $l(t) = (x(t), y(t), z(t)) := \frac{1}{N_t}(X_t, Y_t, Z_t)$ where $N = X + Y + Z$. Then N_t is a solution to (2) and it follows by (12), (13) and (14) in Lemma 1 that $(x(t), y(t), z(t))$ is a solution to (M3) almost surely.

On the other hand, consider a solution N_t to (2) and $\big(x(t), y(t), z(t)\big)$ a solution to (M3). Note that $x(t) + y(t) + z(t) = 1$ and put $(X_t, Y_t, Z_t) := \big(x(t), y(t), z(t)\big) \cdot N_t$. It follows that

$$\begin{aligned} x(t) &= x(0) \cdot \exp\left\{-\int_0^t \alpha(x(u), y(u), z(u), N_u) \cdot y(u)\,\mathrm{d}u\right\} \\ &= \frac{x_0}{n_0} \cdot \exp\left\{-\int_0^t \beta(X_u, Y_u, Z_u) \cdot Y_u\,\mathrm{d}u\right\}, \\ y(t) &= y(0) \cdot \exp\left\{\int_0^t \alpha(x(u), y(u), z(u), N_u) \cdot x(u)\,\mathrm{d}u - \gamma \cdot t\right\} \\ &= \frac{y_0}{n_0} \cdot \exp\left\{\int_0^t \beta(X_u, Y_u, Z_u) \cdot X_u\,\mathrm{d}u - \gamma t\right\} \end{aligned}$$

and

$$z(t) = \gamma \int_0^t y(u)\,\mathrm{d}u = \gamma \int_0^t \frac{Y_u}{N_u}\,\mathrm{d}u,$$

that is exactly as to say that (X_t, Y_t, Z_t) is a solution to (M2) according to Lemma 1. □

We need sufficient conditions for (M3) to have a unique solution almost surely. Observe the following requirements on $\beta(x,y,z)$ and $\alpha(x,y,z,n)$, respectively.

(i) $\beta(x,y,z)$ is a Lipschitz continuous function on Δ_{ab}.

(ii) $\alpha(x,y,z,n)$ is Lipschitz continuous on $\Delta_{11} = \{(x,y,z) \in [0,1]^3,\ x+y+z=1\}$ uniformly for $n \in [a,b]$.

(iii) For arbitrary $N_. \in C(\mathbb{R}^+)$ such that $a \le N_. \le b$ the function $\alpha(x,y,z,N_t)$ is Lipschitz continuous on Δ_{11} uniformly for $t \ge 0$.

(iv) For arbitrary $N \in C(\mathbb{R}^+)$ such that $a \le N \le b$ and $N_0 = n_0$, the equation (M3) has a unique solution $l(N,t) = (x(N,t), y(N,t), z(N,t)) \in C(\mathbb{R}^+, \mathbb{R}^3)$.

It is obvious that

$$\text{(i)} \quad \Longrightarrow \quad \text{(ii)} \quad \Longrightarrow \quad \text{(iii)} \quad \Longrightarrow \quad \text{(iv)}, \tag{21}$$

where the last implication is proved by Theorem 4 with $a=b=1$, especially with $\sigma = 0$. Consequently, the proposed two-step construction of a solution to (M2) is good one from the mathematical point of view.

Corollary 1. Let either $\beta(x,y,z)$ to satisfy (i) or $\alpha(x,y,z,n)$ to satisfy (ii) and assume that $\sigma(n)$ follows (3) for some $a \le b$. Then (M2) has a unique strong solution if and only if the equation (2) has the property. Moreover, assuming that (2) has a unique weak solution (M2) inherits the property.

P r o o f. The first assertion is mostly a simple consequence of Theorem 5 applying the chain of implications (21). It remains to assume that (M2) has a unique strong solution and prove that $N = N'$ almost surely for any pair of solutions to (2). Let $l(t) = (x(t), y(t), z(t))$ be the unique solution to (M3) generated by N_t and $l'(t) = (x'(t), y'(t), z'(t))$ the unique solution to (M3) generated by N'_t. Then $l(t) \cdot N_t = l'(t) N'_t$ holds for $t \ge 0$ almost surely. Hence, $N'_t = c(t) \cdot N_t$, where $c(t) := \frac{x(t)}{x'(t)}$, $c(0) = 1$, is a continuous process of finite variation. Since N_t is a martingale, it follows by Itô per partes formula that $\mathrm{E} \int_0^t N_u \, \mathrm{d}c(u) = 0$ for all $t \ge 0$ almost surely, hence c is a process with constant trajectories. Thus, $c(t) = 1$ for all t and therefore $N = N'$.

Note that (iv) defines a map

$$F : \{N \in C(\mathbb{R}^+), \quad a \le N \le b, \quad N_0 = n_0\} \to C(\mathbb{R}^+, \mathbb{R}^3)$$

that sends any such N to $F(N)(t) = l(N,t)$, i.e. to a unique solution to (M3) generated by N. The map F is easily seen to be Borel measurable and therefore the probability distribution of N_t uniquely determines the probability distribution of $N \cdot F(N)$ that is a solution to (M2) according to Theorem 5. □

We suspect that there might be situations when (M2) has a unique weak solution while there need not to exist a unique strong solution to the Engelbert–Schmidt equation (2).

Example 4. Choose $\beta(x,y,z)$ such that $\beta(x,y,z) = \frac{B}{A(x+y+z)+z}$ holds on Δ_{ab}. Then $\alpha(x,y,z,n) = \frac{B}{A+z}$ is Lipschitz on Δ_{11} and the equation (M3) adopts the deterministic form of Kermack–McKendric model scrutinized in Theorem 1.

Write $p_0 = x_0/n_0$ and $q_0 = y_0/n_0 = 1 - p_0$. We get (M1) as

$$\begin{aligned} \dot{x} &= -\frac{B}{A+z} \cdot x \cdot y, & x(0) &= p_0 \\ \dot{y} &= \frac{B}{A+z} \cdot x \cdot y - \gamma y, & y(0) &= q_0 \\ \dot{z} &= \gamma \cdot y, & z(0) &= 0. \end{aligned}$$

We easily compute

$$X(z) = p_0 \cdot \exp\left\{-\frac{1}{\gamma}\int_0^z \alpha(u)\,\mathrm{d}u\right\} = p_0\left(\frac{A}{A+z}\right)^{B/\gamma}$$

and

$$Y(z) = 1 - z - p_0\left(\frac{A}{A+z}\right)^{B/\gamma}.$$

Choosing $B = \gamma$ and $A = \frac{p_0}{2}$ we get $\frac{\gamma}{\alpha(0)} = A < p_0$ and observe that Theorem 1 is applicable completely with

$$\alpha(z) = \frac{2\gamma}{p_0 + 2z}, \quad X(z) = \frac{p_0^2}{p_0 + 2z} \quad \text{and} \quad Y(z) = 1 - z - \frac{p_0^2}{p_0 + 2z}.$$

The above differential equation has a unique solution $(x(t), y(t), z(t))$ on $\mathbb{R}^+$, $z(t)$ being computed by solving

$$\dot{z} = \gamma \cdot \left(1 - z - \frac{p_0^2}{p_0 + 2z}\right), \quad z(0) = 0.$$

There is no explicit solution to this differential equation nevertheless some features of the solution can be stated. The limit $z_\infty = \lim_{t\to\infty} z(t) \in [0,1]$ may be computed by solving $1 - z_\infty = X(z_\infty) = p_0^2/(p_0 + 2z_\infty)$. We get

$$z_\infty = \frac{2 - p_0 + \sqrt{4 + 4p_0 - 7p_0^2}}{4}, \qquad x_\infty = \frac{2 + p_0 - \sqrt{4 + 4p_0 - 7p_0^2}}{4}.$$

Also solve the equation $2p_0^2 = (p_0 + 2z_+)^2$ to establish a $z_+ \in (0,\infty)$ and compute

$$y_+ = 1 - z_+ - \frac{p_0^2}{p_0 + 2z_+}, \quad t_+ = \frac{1}{\gamma}\int_0^{z_+} \frac{2}{1 - u - p_0^2(p_0 + 2u)^{-1}}\,\mathrm{d}u$$

to get

$$y_+ = \max y(t) = 1 - p_0\left(\sqrt{2} - \frac{1}{2}\right), \quad z_+ = p_0\frac{\sqrt{2}-1}{2}, \quad x_+ = p_0\frac{\sqrt{2}}{2}$$

and $t_+ > 0$ such that $y_+ = y(t_+)$.

We summarize that having (M2) model with $\beta(x,y,z) = \frac{2\gamma}{p_0(x+y+z)+2z}$ and a Lipschitz continuous function $\sigma(n)$ that satisfies (10) for a pair $a \le b$, we get a model that has a unique strong solution

$$X_t = x(t)\cdot N_t, \quad Y_t = y(t)\cdot N_t, \quad z_t = z(t)\cdot N_t,$$

where N_t solves the equation (2). We easily compute that

$$\mathrm{E}X_t = x(t)\cdot n_0, \quad \mathrm{E}Y_t = y(t)\cdot n_0, \quad \mathrm{E}Z_t = z(t)\cdot n_0.$$

The above two-step procedure obviously generates a couple of problems:

(A) To solve Engelbert–Schmidt equation (10) by which we mean to receive an information as complex as possible about the probability distribution of its solution N_t with the aim to deliver simulations of its trajectories and formulas for the moments $\mathrm{E}N_t^k$. To complicate things even more epidemiologists prefer models with possibly non-Lipschitz *weight functions* $\sigma(n)$

$$\sigma(n) = K(n-a)^{\mu}\cdot(b-n)^{\nu}, \quad a \le n \le b, \quad 0 < a < n_0 < b < \infty, \quad \mu,\nu > 0.$$

(B) Having done this we are of course expected to solve the ordinary equation (M3) with the similar aims as proposed by (A).

To this end precise requirements (2) as

$$\sigma \ge 0 \quad \text{bounded with } \operatorname{supp}(\sigma) = (a,b), \quad 0 < a < n_0 < b < \infty, \tag{22}$$

$$\int_a^{a^+} \frac{1}{n^2\sigma^2(n)}\,\mathrm{d}n = \int_{b^-}^{b} \frac{1}{n^2\sigma^2(n)}\,\mathrm{d}n = \infty \tag{23}$$

and note that (23) is satisfied for Lipschitz continuous $\sigma(n)$ that satisfy (22). Consider a Brownian motion $B(t)$ such that $B(0) = n_0$, denote

$$A(s) = \int_0^s \frac{1}{B(u)^2\sigma^2(B(u))}\,\mathrm{d}u, \quad s \ge 0, \quad \tau_t = \inf\{s > 0 : A(s) > t\}, \quad t \ge 0$$

and recall *Engelbert–Schmidt* Theorem [10, Theorem 23.1, Lemma 23.2, p. 451–2] to get (looking also through the proofs) a simple

Corollary 2. Assume (22). Then the condition (23) is necessary and sufficient for the equation (2) to have a unique weak solution. Arbitrary solution N_t to (2) is equally distributed in $C(\mathbb{R}^+)$ as the continuous stochastic process $B(\tau_t)$. Moreover,

$$\tau_\infty := \inf\{s \ge 0 : A(s) = \infty\} = \lim_{t\to\infty} \tau_t = \inf\{s \ge 0 : B(s) \in \{a,b\}\}. \tag{24}$$

In a combination with Corollary 1 we also prove

Corollary 3. Assume (22) and (23) for $\sigma(n)$ and consider arbitrary $\alpha(x,y,z,n) = n\cdot\beta(nx,ny,nz)$ that is Lipschitz continuous on

$$\Delta_{11} = \{(x,y,z)\in[0,1]^3 : x+y+z=1\}$$

uniformly for $n\in[a,b]$. Then the equation (M2) has a unique weak solution.

Engelbert–Schmidt Theorem in a more detailed form of Corollary 2 also proves the (ii) statement in Theorem 2:

Consider arbitrary solution N_t to the equation (2). It follows by (24) that the limit N_∞ and transformation $B(\tau_\infty)$ are equally distributed random variables. Because $B(\tau_\infty)$ is a $\{a,b\}$-valued random variable almost surely and τ_∞ a Markov time such that $B(t)\in[a,b]$ if $\tau_\infty\ge t$ we may apply the Wald equalities

$$a\mathrm{P}\left[B(\tau_\infty)=a\right]+b\,\mathrm{P}\left[B(\tau_\infty)=b\right]=n_0,\quad a^2\mathrm{P}\left[B(\tau_\infty)=a\right]+b^2\mathrm{P}\left[B(\tau_\infty)=b\right]=\mathrm{E}\tau_\infty$$

to verify (ii) in Theorem 2.

The reader may find in [17, 18] a source of information relevant to the problems proposed by (A).

Example 5. Consider the same $\beta(x,y,z)$ as in Example 4, where $B=\gamma=1/4$, $p_0=0.995$ and $A=p_0/2$. Recall that p_0 is the initial ratio of the susceptibles in the population. The numerical solution of the population ratio of the susceptibles, infected and removed dynamics is shown in Figure 4. We can also calculate

z_∞	x_∞	y_+	z_+	t_+
0.5074	0.4926	0.0904	0.2061	19.42

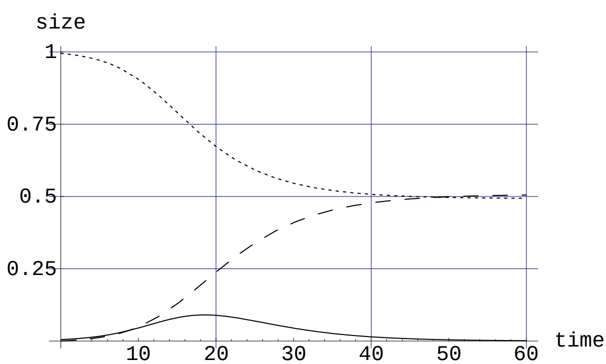

Fig. 4. The time dynamics of susceptibles (dotted line), infected (solid line) and removed (dashed line) – proportions of the population size.

For the population size itself we use the Engelbert–Schmidt equation (2), where

$$\sigma(n)=6\sqrt{\frac{(n-a)(b-n)}{a+b}},\quad a=1000,\ b=1050,\ n_0=1025. \tag{25}$$

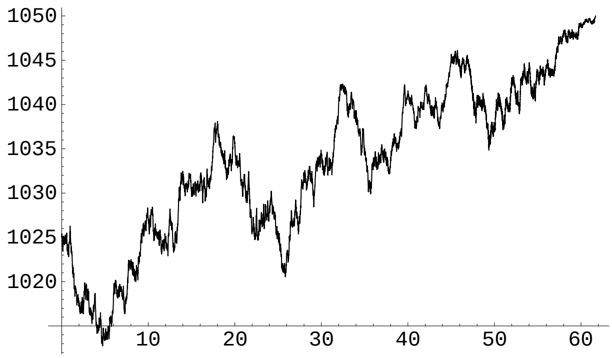

Fig. 5. The simulated population size.

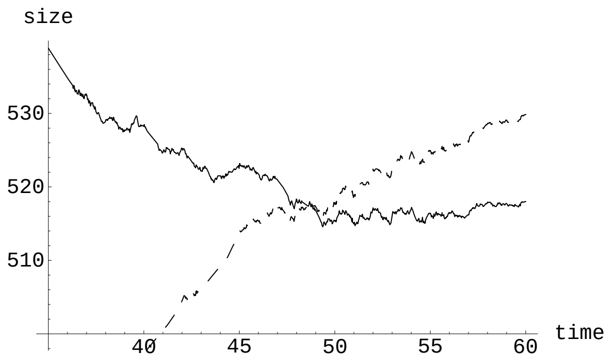

Fig. 6. The susceptibles and removed at the end of the epidemic.

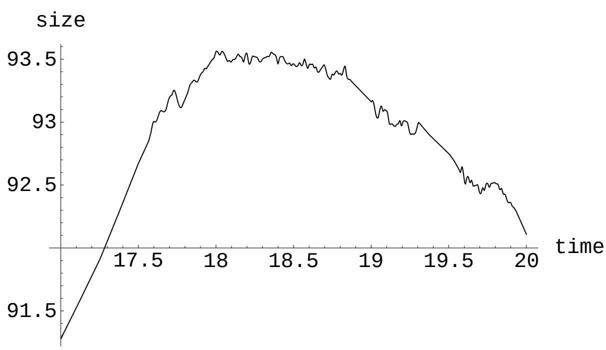

Fig. 7. The infected during the period of highest infection.

A simulated sample path of such a population up to the time 60 is given in Figure 3. Parts of the simulated solutions of the stochastic model (M2), (X_t, Y_t, Z_t), are shown in Figures 6 and 7. The population dynamics is, however, quite small with respect to the dynamics of the size of susceptibles, infected and removed. The population size is between 1,000 and 1,050 to keep the emmigration/immigration dynamics realistic. On the other hand the number of susceptibles changes from 995 to 500 quite fast.

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Josef Štěpán and Daniel Hlubinka, Department of Probability and Statistics, Faculty of Mathematics and Physics – Charles University, Sokolovská 83, 186 75 Praha 8. Czech Republic.
e-mails: stepan@karlin.mff.cuni.cz, hlubinka@karlin.mff.cuni.cz