KYBERNETIKA — VOLUME *43* (2007), NUMBER 3, PAGES 347 – 368

# MONOTONICITY OF MINIMIZERS IN OPTIMIZATION PROBLEMS WITH APPLICATIONS TO MARKOV CONTROL PROCESSES

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Firstly, in this paper there is considered a certain class of possibly unbounded optimization problems on Euclidean spaces, for which conditions that permit to obtain monotone minimizers are given. Secondly, the theory developed in the first part of the paper is applied to Markov control processes (MCPs) on real spaces with possibly unbounded cost function, and with possibly noncompact control sets, considering both the discounted and the average cost as optimality criterion. In the context described, conditions to obtain monotone optimal policies are provided. For the conditions of MCPs presented in the article, several controlled models including, in particular, two inventory/production systems and the linear regulator problem are supplied.

*Keywords:* monotone minimizer in an optimization problem, Markov control process, total discounted cost, average cost, monotone optimal policy

*AMS Subject Classification:* 90C40, 93E20

## 1. INTRODUCTION

This paper is divided into two parts.

Firstly, let $X \subset \mathbb{R}^n$ and $A \subset \mathbb{R}^m$, where $n$ and $m$ are positive integers, be nonempty Borel subsets. For each $x \in X$, let $A(x)$ be a nonempty subset of $A$. Let $G : \mathbb{K} \to \mathbb{R}$ be a function bounded below, where $\mathbb{K} = \{(x, a) : x \in X, a \in A(x)\}$.

Consider the following optimization problem:

$$\min_{a \in A(x)} G(x, a), \quad x \in X. \tag{1}$$

It will be assumed that the minimum in (1) is attained. For each $x \in X$, let $f(x)$ and $f'(x)$ denote the greatest and the least values of $a \in A(x)$, respectively, at which the minimum of (1) is reached (besides, it is supposed that $f(x) = \max\{a' \in \arg\min_{a \in A(x)} G(x, a)\}$ and $f'(x) = \min\{a' \in \arg\min_{a \in A(x)} G(x, a\}$ are well-defined).

In the present paper, conditions which imply that $f$ and $f'$ are monotone functions are presented. Basically, the conditions provided require the superadditivity (or subadditivity) of $G$ (see [13], pp. 103–104 or [19] Chapt. 10). (Some authors say that $G$ has increasing or isotone differences instead of saying that $G$ is superadditive, see [7, 19, 20], and [21].) It is also important to observe that neither the compactness of $A(x)$, $x \in X$, nor the upper boundedness of $G$ is necessary for the conditions provided. This fact allows to consider unbounded optimization problems. This is one of the main contributions of this article.

Topkis' paper [20] is an antecedent of this part of the present paper in the case of increasing minimizers. In [20] it is assumed that the constraint sets $A(x)$, $x \in X$, are compact sets. Besides, it is supposed that $G$ is both subadditive and submodular (however, it is important to mention that in [20] $X$ and $A$ are general sets, i. e. they are not subsets of Euclidean spaces).

Sundaram [19] and Topkis [21] study a maximization problem, similar to problem (1) (i. e. substituting "min" for "max" in (1)) and obtain increasing maximizers. In [19], $A$ is required to be a compact set, and $G$ to be both superadditive and supermodular. Topkis [21] presents a result, in which it is supposed that the constraint sets $A(x)$, $x \in X$, are finite, or they are compact subsets of $\mathbb{R}^m$, and $G$ is supermodular.

Secondly, the first part of the paper is applied to infinite horizon Markov control processes (MCPs) on real spaces with possibly unbounded cost function, and with possibly noncompact control sets, considering both the discounted and the average cost as optimality criterion (see [6]).

For such class of MCPs, different conditions which guarantee the existence of monotone optimal policies are established. The conditions considered are imposed on the elements of the Markov control model (see [6]), that is, on the state space $X$, the control set $A$, the restriction sets $A(x)$, $x \in X$, the transition probability law $Q$, and the cost function $c$. Furthermore, these conditions imply that both the state and control spaces $X$ and $A$ are non-numerable subsets of $\mathbb{R}$; in fact, they are intervals in $\mathbb{R}$. These are the other main contributions of the article.

Finally, several examples of MCPs which include two inventory/production systems and the linear regulator problem are provided.

Previous works on MCPs concerned with monotone optimal policies where the superadditivity (or subadditivity) of the dynamic programming operator is supposed directly, i. e. it is not given in terms of the elements of the Markov control model, are [9] and [16]. Also, in Hinderer [8], Porteus [12], and Topkis [21] there are considered discounted MCPs with finite horizon.

The monotonicity of optimal policies of MCPs is a widely known and studied feature which is of interest to people who make applications of MCPs to consumption-investment problems, inventory control systems, and queueing systems (see [4, 5, 7, 10, 11, 12, 13, 15], and [17]). The main reason for this interest is the fact that the monotonicity of an optimal policy reflects qualitative properties of the models studied.

On the other hand, in Puterman [13], for MCPs on finite spaces, the existence of an optimal policy which is strictly monotone is used in the following way. Its approximation, via the policy iteration algorithm, is improved in two ways: i) taking a monotone policy as the initial one, and ii) the convergence of this algorithm is accelerated (see [13], pp. 259–260; 428).

The paper is organized as follows. Section 2 provides basic concepts and results on lattices and on MCPs. In Section 3 conditions under which problem (1) has monotone minimizers are provided. In Section 4 the theory (without proofs) to establish the existence of monotone optimal policies is given. Several examples to illustrate the theory developed in Section 4 are presented in Section 5. Section 6 contains the proofs of Section 4. Finally, some remarks about the average cost MCPs and the conclusions are supplied in Sections 7 and 8, respectively.

## 2. PRELIMINARIES

### 2.1. Terminology and some results of lattice theory

A set $\hat{E}$ is said to be *partially ordered* if there is a binary relation "$\preceq$" that is reflexive, antisymmetric and transitive. This means that for each $x \in \hat{E}$, $x \preceq x$; for each $x, y \in \hat{E}$, $x \preceq y$ and $y \preceq x$ imply that $x = y$; for each $x, y, z \in \hat{E}$, $x \preceq y$, and $y \preceq z$ imply that $x \preceq z$.

The rest of this section contains concepts and results of the lattice theory (see [19, 20], and [21]), applied to a Euclidean space, for instance, $\mathbb{R}^n$, where $n$ is a positive integer. For such space the partial order $\preceq$ defined componentwise will be used, i.e., if $x$ and $y$ are vectors, then the inequality $x \preceq y$ is understood as $x_i \leq y_i$, for all $i$ (where $\leq$ is the usual order in $\mathbb{R}$). Moreover, $x \wedge y := \inf\{x, y\} = (\inf\{x_1, y_1\}, \dots, \inf\{x_n, y_n\})$ and $x \vee y := \sup\{x, y\} = (\sup\{x_1, y_1\}, \dots, \sup\{x_n, y_n\})$.

Let $\Gamma$ be a fixed subset of $\mathbb{R}^n$. Let $\Theta$ be a subset of $\Gamma$. $\hat{\gamma}$ is an *upper (lower) bound* for $\Theta$ if $\hat{\gamma} \in \Gamma$ and $\theta \preceq \hat{\gamma}$ ($\hat{\gamma} \preceq \theta$) for each $\theta \in \Theta$. $\hat{\gamma}$ is the *greatest (least) element* of $\Theta$ if $\hat{\gamma}$ is an upper (lower) bound for $\Theta$ and $\hat{\gamma} \in \Theta$. The *supremum* (*infimum*) of $\Theta$ is the least upper bound (greatest lower bound), when the set of upper (lower) bounds of $\Theta$ has a least (greatest) element. Besides, it is denoted by $\sup \Theta$ ($\inf \Theta$). The notation $\sup_\Gamma \Theta$ ($\inf_\Gamma \Theta$) is used as well if the set $\Gamma$ is not clear from the context.

$\Gamma$ is said to be a *lattice* if $\gamma_1 \wedge \gamma_2$ and $\gamma_1 \vee \gamma_2 \in \Gamma$, for all $\gamma_1, \gamma_2 \in \Gamma$.

Let $\Gamma$ be a lattice and let $\Theta$ be a subset of $\Gamma$. $\Theta$ is a *sublattice* of $\Gamma$ if $\Theta$ contains $\theta \wedge \theta'$ and $\theta \vee \theta'$ (with respect to $\Gamma$), for all $\theta, \theta' \in \Theta$. For a lattice $\Gamma$, $\pounds(\Gamma)$ denotes the set of all nonempty sublattices of $\Gamma$.

Let $\Theta$ be a sublattice of a lattice $\Gamma$. $\Theta$ is a *subcomplete* sublattice of $\Gamma$ if for each nonempty subset $\Psi$ of $\Theta$, $\sup \Psi$ and $\inf \Psi$ exist and are contained in $\Theta$. In fact, a lattice in which every nonempty subset has a supremum and infimum is *complete*.

**Lemma 2.1.** (Topkis [21], Theorem 2.3.1) A sublattice of $\mathbb{R}^n$ is subcomplete if and only if it is compact.

Let $\Gamma$ be a lattice. Let $\Theta$ and $\Upsilon$ be subsets of $\Gamma$. $\Theta$ is *lower* than $\Upsilon$, written $\Theta \sqsubseteq \Upsilon$, if $\theta \wedge \upsilon \in \Theta$ and $\theta \vee \upsilon \in \Upsilon$ for all $\theta \in \Theta$ and $\upsilon \in \Upsilon$.

**Lemma 2.2.** (Topkis [20], Theorem 2.1 or Topkis [21], Theorem 2.4.1) If $\Gamma$ is a lattice with respect to the relation $\preceq$, then $\mathcal{L}(\Gamma)$ is a partially ordered set with respect to the relation $\sqsubseteq$.

Let $\Gamma$ be a lattice. Let $Z$ be a nonempty subset of $\mathbb{R}^m$, where $m$ is a positive integer. For $x \in Z$, let $\Gamma(x)$ be a nonempty sublattice of $\Gamma$. It will be said that the multifunction $x \to \Gamma(x)$ is *ascending* if $x \to \Gamma(x)$ is increasing with respect to the relation $\sqsubseteq$, i. e., $\Gamma(x) \sqsubseteq \Gamma(y)$, for $x \preceq y$ in $Z$. $x \to \Gamma(x)$ is *descending* if $x \to \Gamma(x)$ is decreasing with respect to the relation $\sqsubseteq$.

**Lemma 2.3.** (Topkis [21], Lemma 2.4.2) Let $\Gamma$ be a lattice, and let $\Theta$ and $\Upsilon$ be nonempty subsets of $\Gamma$, with $\Theta \sqsubseteq \Upsilon$.

a) If $\sup \Theta$ and $\sup \Upsilon$ exist, then $\sup \Theta \preceq \sup \Upsilon$.

b) If $\inf \Theta$ and $\inf \Upsilon$ exist, then $\inf \Theta \preceq \inf \Upsilon$.

Let $X$ and $A$ be fixed nonempty Borel subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. For each $x \in X$, let $A(x)$ be a nonempty (measurable) subset of $A$ (i. e., $x \to A(x)$ is a multifunction from $X$ to $A$). Suppose that $\mathbb{K} := \{(x, a) : x \in X,\ a \in A(x)\}$ is a measurable subset of $X \times A$.

A function $W : \mathbb{K} \to \mathbb{R}$ is *superadditive* (has isotone or increasing differences) on $\mathbb{K}$ if $W(y, a) + W(x, b) \leq W(y, b) + W(x, a)$ for all $x \preceq y$ in $X$ and $a \preceq b$, with $a, b \in A(x) \cap A(y)$. $W$ is called *subadditive* (has antitone or decreasing differences) on $\mathbb{K}$ if $-W$ is superadditive on $\mathbb{K}$.

Let $\mathbb{K}$ be a lattice. A function $\omega : \mathbb{K} \to \mathbb{R}$ is *supermodular* on $\mathbb{K}$ if $\omega(k) + \omega(k') \leq \omega(k \vee k') + \omega(k \wedge k')$, for each $k, k' \in \mathbb{K}$. $\omega$ is called *submodular* if $-\omega$ is supermodular.

**Lemma 2.4.** Let $V, W$ and $\omega$ be functions from $\mathbb{K}$ to $\mathbb{R}$.

a) If $V$ and $W$ are superadditive (subadditive) functions, then $V + W$ is superadditive (subadditive).

b) Let $\mathbb{K}$ be a lattice. If $W$ is subadditive on $\mathbb{K}$, then $W$ is submodular on $\mathbb{K}$.

c) Let $\mathbb{K}$ be a lattice. If $\omega(\cdot, \cdot)$ is submodular, then $\omega(x, \cdot)$ is also submodular, for each $x \in X$.

d) Let $w_1$ and $w_2$ be real-valued functions on $X$ and $A$, respectively. The function $W(x, a) = w_1(x) \cdot w_2(a)$, $(x, a) \in \mathbb{K}$, is subadditive if $w_1$ is an increasing monotone function and $w_2$ is a decreasing one, or viceversa.

Proof.

a) It follows directly from the properties of the usual order in $\mathbb{R}$.

b) This is a direct consequence of Theorem 10.12, in [19], because the statement "if $W$ is superadditive on $\mathbb{K}$, then $W$ is supermodular on $\mathbb{K}$" is equivalent to "if $-W$ is subadditive on $\mathbb{K}$, then $-W$ is submodular on $\mathbb{K}$" (this follows from the definitions of subadditive and submodular functions).

c) As $\omega(\cdot,\cdot)$ is submodular on $\mathbb{K}$, then for $x, y \in X$, $a \in A(x)$ and $b \in A(y)$, it results that

$$\omega(x \wedge y, a \wedge b) + \omega(x \vee y, a \vee b) \leq \omega(x, a) + \omega(y, b). \tag{2}$$

Therefore, the submodularity of $\omega$ in the second variable is a consequence of considering $y = x$ in (2).

d) Take $x, y \in X$ with $x \preceq y$, and $a, b \in A(x) \cap A(y)$ with $a \preceq b$. Then

$$\begin{aligned} & W(y,b) + W(x,a) - \Big[W(y,a) + W(x,b)\Big] \\ = \;& \Big[w_1(y) - w_1(x)\Big]\Big[w_2(b) - w_2(a)\Big] \leq 0, \end{aligned}$$

when $w_1$ is increasing and $w_2$ is decreasing, or viceversa. Therefore, the subadditivity of $W$ follows. □

Let $G : \mathbb{K} \to \mathbb{R}$ be a function, which is measurable and bounded below (for instance, nonnegative), and consider the following minimization problem:

$$\min_{a \in A(x)} G(x, a), \quad x \in X. \tag{3}$$

Also, for each $x \in X$, define $A^*(x)$ by

$$A^*(x) := \left\{ a \in A(x) : G(x, a) = \min_{a^* \in A(x)} G(x, a^*) \right\}. \tag{4}$$

**Assumption 2.1.**

a) $G$ is lower semicontinuous (l.s.c.) on $\mathbb{K}$.

b) $G$ is inf-compact on $\mathbb{K}$, that is, for every $x \in X$ and $\bar{s} \in \mathbb{R}$, the set $A_{\bar{s}}(x) := \{a \in A(x) : G(x, a) \leq \bar{s}\}$ is compact.

**Lemma 2.5.** (Rieder [14], Theorem 4.1) Assumption 2.1 implies that there exists a measurable function $g : X \to A$ such that $g(x) \in A^*(x)$, $x \in X$, i.e. $g$ is a minimizer for (3). In particular, observe that $A^*(x) \neq \emptyset$, for every $x \in X$.

**Remark 2.1.** Note that it is direct to verify that, for every $x \in X$, $A^*(x) \subset A_{G^*(x)}(x)$, where $G^*(x) := \min_{a^* \in A(x)} G(x, a^*)$.

**Lemma 2.6.** Assumption 2.1 implies that $A^*(x)$ is a compact set, for each $x \in X$.

Proof. Fix $x \in X$. Take a sequence $\{a_n\}$ in $A^*(x)$ such that $\lim_{n\to+\infty} a_n = a' \notin A^*(x)$. Observe that $G(x, a') > \min_{a^*\in A(x)} G(x, a^*)$; moreover, $G(x, a_n) = \min_{a^*\in A(x)} G(x, a^*)$, for each $n = 1, 2, \dots$. Hence, as $G$ is l.s.c. on $\mathbb{K}$, then $\min_{a^*\in A(x)} G(x, a^*) = \liminf_{n\to+\infty} G(x, a_n) \geq G(x, a')$, but this is a contradiction to the assumption above. Therefore, $a' \in A^*(x)$, i. e. $A^*(x)$ is closed.
Now, the compactness of $A^*(x)$ follows from Remark 2.1 and Assumption 2.1 b.

Since $x \in X$ is arbitrary, Lemma 2.6 follows. □

**Lemma 2.7.** (Topkis [20], Theorem 4.1) For each $x \in X$, suppose that $A(x)$ is a lattice and $G(x, \cdot)$ is submodular. Then, for every $x \in X$, $A^*(x)$ is a sublattice of $A(x)$.

**Lemma 2.8.** (Topkis [20], Theorem 6.1) Suppose that Assumption 2.1 holds. If $A$ is a lattice, $x \to A(x)$ is ascending, $A(y) \subset A(x)$ for $x \preceq y$ in X, $G(x, \cdot)$ is submodular, for each $x \in X$, and $G$ is subadditive on $\mathbb{K}$, then $x \to A^*(x)$ is ascending.

### 2.2. Markov control processes

Let $\{X, A, \{A(x) : x \in X\}, Q, c\}$ be a discrete-time, stationary *Markov control model* (see [6]), which consists of the *state space* $X$, the *action* or *control set* $A$, the *admissible control sets* $A(x)$, $x \in X$, the *transition law* $Q$, and the *one-stage cost* $c$.

$X$ and $A$ are assumed to be Borel spaces (i. e. Borel subsets of a complete separable metric space), with Borel $\sigma$-algebras $\mathcal{B}(X)$ and $\mathcal{B}(A)$, respectively. For each $x \in X$, let $A(x)$ be a nonempty Borel subset of $A$.

Define $\mathbb{K} := \{(x, a) : x \in X,\ a \in A(x)\}$. It will be assumed that $\mathbb{K}$ is measurable in $X \times A$. The transition law $Q(B|x, a)$, $B \in \mathcal{B}(X)$, $x \in X$, and $a \in A(x)$ is a *stochastic kernel* on $X$, given $\mathbb{K}$, that is, $Q(\cdot|x, a)$ is a probability measure on $X$, for every $(x, a) \in \mathbb{K}$, and $Q(B|\cdot)$ is a measurable function on $\mathbb{K}$, for every $B \in \mathcal{B}(X)$. Finally, $c : \mathbb{K} \to \mathbb{R}$ is a measurable function.

Let $\mathbb{F}$ be the set of *decision functions* or *measurable selectors*, i. e., the set of all measurable functions $\varrho : X \to A$, such that $\varrho(x) \in A(x)$, for all $x \in X$. A sequence $\pi = \{\varrho_t\}$, such that, for each $t$, $\varrho_t \in \mathbb{F}$ is called a *Markov policy.* A *stationary policy* is a Markov policy $\pi$ such that $\varrho_t = \varrho$, for all $t = 0, 1, \dots$. In fact, a Markov policy $\pi = \{\varrho_t\}$ is a special kind of a general control policy $\overline{\pi}$ defined as a (measurable, possibly randomized) rule for choosing controls, and at each $t = 0, 1, \dots$, $\overline{\pi}$ may depend on the current state as well as on the history of previous states and controls (see [6]). The set of all policies will be denoted by $\Pi$.

Given the initial state $x_0 = x$ and any policy $\pi$, there is a probability measure $\mathcal{P}_x^\pi$, induced by the pair $(\pi, x)$, on the space $(X \times A)^\infty$ with $\mathcal{F}$ as the product $\sigma$-algebra, in a canonical way (see [6]). The corresponding expectation operator will be denoted by $\mathrm{E}_x^\pi$; $\{x_t\}$ and $\{a_t\}$ denote the state and control sequences, respectively.

A policy $\pi$ and an initial state $x_0 = x$ determine a stochastic process $(\Omega, \mathcal{F}, \mathcal{P}_x^\pi, \{x_t\})$ called a *Markov control process.*

In many applications, the evolution of Markov control processes (MCPs) is given by a transition probability law $Q$ induced by a difference equation of the type:

$$x_{t+1} = F(x_t, a_t, \xi_t), \tag{5}$$

$t = 0, 1, \ldots$, where $x_0 = x \in X$ is the initial state, $\{\xi_t\}$ is a sequence of independent and identically distributed (i.i.d.) random variables (r.v.), taking values in some Borel space $S$. Let $\xi$ denote a generic element of the sequence $\{\xi_t\}$ ($\xi$ will be used in the paper to specify some assumptions related to the sequence $\{\xi_t\}$) and let $F : X \times A \times S \to X$ be a measurable function.

In the article the following objective functions (6) and (7) will be taken into account. In both cases, the initial state is $x_0 = x$, and $\pi$ is the policy that drives the system.

The *expected total discounted cost* is given by:

$$V_\alpha(\pi, x) := \mathrm{E}_x^\pi \left[ \sum_{t=0}^{+\infty} \alpha^t c(x_t, a_t) \right], \tag{6}$$

where the number $\alpha \in (0, 1)$ is the discount factor.

The *expected average cost* is defined as follows:

$$J(\pi, x) := \limsup_{n \to +\infty} \frac{1}{n} \mathrm{E}_x^\pi \left[ \sum_{t=0}^{n} c(x_t, a_t) \right]. \tag{7}$$

Fix an objective function $T(\pi, x)$, given by (6) or (7). A policy $\pi^*$ will be called *optimal* if

$$T(\pi^*, x) = \inf_{\pi \in \Pi} T(\pi, x),$$

for all $x \in X$ and the minimum cost $T^*(x) := T(\pi^*, x)$, $x \in X$, is referred to as the *optimal value function*. Specifically, on one hand, if $T(\pi, x) = V_\alpha(\pi, x)$, $x \in X$, $\pi \in \Pi$, the minimum over all $\pi$ is denoted by $V_\alpha^*(x)$, for each $x \in X$. On the other hand, if $T(\pi, x) = J(\pi, x)$, $x \in X$, $\pi \in \Pi$, the minimum over all $\pi$ is denoted by $J^*(x)$, for each $x \in X$.

## 3. MONOTONE MINIMIZERS

In this section the problem (3) stated in Section 2 will be referred to.

### 3.1. Decreasing minimizers of superadditive functions

**Theorem 3.1.** If $A$ is a lattice, $x \to A(x)$ is descending (in particular, $A(x)$ is a sublattice of $A$, for each $x \in X$), $A(y) \subset A(x)$ for $x \preceq y$ in $X$, $G$ is superadditive, $G(x, \cdot)$ is submodular, for each $x \in X$, and Assumption 2.1 holds, then for $f(x) := \sup A^*(x)$, $x \in X$, it is obtained that $f(y) \preceq f(x)$, with $x \preceq y$ in $X$. Besides, $f(x) \in A^*(x)$, for every $x \in X$, i. e., $f$ is a minimizer for (3).

Proof. Take $x, y \in X$ with $x \preceq y$, and take $a \in A^*(x) \subset A(x)$ and $b \in A^*(y) \subset A(y) \subset A(x)$. Then

$$G(x, a \vee b) - G(x, a) \le G(x, b) - G(x, a \wedge b) \le G(y, b) - G(y, a \wedge b), \tag{8}$$

where the first inequality holds by the submodularity of $G(x, \cdot)$, and the second one is due to the superadditivity of $G$ on $\mathbb{K}$ (note that $a \wedge b \in A(y)$, because $A(y) \sqsubseteq A(x)$, and $a \wedge b, a \vee b \in A(x)$, due to the fact that $A(x)$ is a sublattice of $A$). Hence, by (8) and the optimality of $a$ and $b$,

$$0 \le G(x, a \vee b) - G(x, a) \le G(y, b) - G(y, a \wedge b) \le 0. \tag{9}$$

Consequently, the equality in (9) holds, and $a \wedge b \in A^*(y)$ and $a \vee b \in A^*(x)$, that is, $A^*(y) \sqsubseteq A^*(x)$, for $x \preceq y$.

Now, since $G(x, \cdot)$ is submodular on the lattice $A(x)$, for all $x \in X$, and using Lemma 2.7, it results that $A^*(x)$ is a sublattice of $A(x)$, for each $x \in X$. Therefore, $x \to A^*(x)$ is descending.

Moreover, Assumption 2.1 yields that, for each $x \in X$, $A^*(x)$ is a compact set on $\mathbb{R}^m$ (see Lemma 2.6). Then, from Lemma 2.1 it follows that, for each $x \in X$, $A^*(x)$ contains both a supremum and an infimum. Define $f(x) := \sup A^*(x)$, $x \in X$. Take $x, y \in X$, with $x \preceq y$. As $A^*(y) \sqsubseteq A^*(x)$, from Lemma 2.3 a, it results that $f(y) \preceq f(x)$. Then, Theorem 3.1 follows. □

**Remark 3.1.** In the proof of Theorem 3.1 it is possible to consider $f'(x) := \inf A^*(x)$, $x \in X$, and also to demonstrate (using Lemma 2.3 b) that $f'$ is a decreasing minimizer for (3).

**Example 3.1.** Let $X$ be a nonempty Borel subset of $\mathbb{R}^2$. Take $A = A(x) = \mathbb{R}$, $x \in X$, and $G(x, a) = \kappa(x) + \nu(a)$, $(x, a) \in \mathbb{K}$, where $\kappa$ and $\nu$ are real-valued functions defined on $X$ and $A$, respectively.

**Assumption 3.1.**

a) $\kappa$ and $\nu$ are nonnegative and continuous.

b) $\lim_{a \to +\infty} \nu(a) = \lim_{a \to -\infty} \nu(a) = +\infty$.

**Lemma 3.1.** Under Assumption 3.1, Example 3.1 satisfies the assumptions of Theorem 3.1. (Therefore, $f(x) := \sup A^*(x)$, $x \in X$ is a decreasing minimizer.)

Proof. Notice that $A = \mathbb{R}$ is a lattice, $x \to A(x)$ is descending, $A(y) \subset A(x)$ for $x \preceq y$ in $X$ (in fact, $x \to A(x)$ is a constant multifunction, i. e., $A(x) = A$, for all $x$), and $G$ is l.s.c. on $\mathbb{K}$ and nonnegative, due to Assumption 3.1 a. Moreover, if $x, y \in X$, and $a, b \in A(y) = \mathbb{R}$ with $x \preceq y$ and $a \le b$, then $G(y, b) + G(x, a) - [G(y, a) + G(x, b)] = 0$. Therefore, $G$ is superadditive. Observe that, for each $x \in X$, $G(x, \cdot)$ is submodular as a consequence of that $A(x) = \mathbb{R}$. Now it will be verified that $G$ is inf-compact (see Assumption 2.1 b). Fix $\bar{s} \in \mathbb{R}$ and $x \in X$. Notice that if $\bar{s} - \kappa(x) < 0$, then $A_{\bar{s}}(x) = \emptyset$ (recall that from Assumption 3.1 a, $\nu$ is nonnegative). Note also that if $\bar{s} - \kappa(x) \geq 0$, then $A_{\bar{s}}(x)$ is closed since $G$ is continuous. Furthermore, $A_{\bar{s}}(x)$ must be bounded. To prove this, let $\{a_n\}$ be a sequence in $A_{\bar{s}}(x)$ such that $a_n \uparrow +\infty$. Observe that

$$\kappa(x) + \nu(a_n) \leq \bar{s}, \tag{10}$$

for all $n$. Hence letting $n \to +\infty$ in (10), and using Assumption 3.1 b, it results that $\bar{s} \geq +\infty$, which is a contradiction. Therefore, $A_{\bar{s}}(x)$ is upper bounded. In a similar way it is possible to show that $A_{\bar{s}}(x)$ is lower bounded. Since $A_{\bar{s}}(x) \subset \mathbb{R}$, it follows that $A_{\bar{s}}(x)$ is compact. As $\bar{s}$ and $x$ are arbitrary, it results that $G$ is inf-compact on $\mathbb{K}$. Therefore, Assumption 2.1 holds. □

### 3.2. Increasing minimizers of subadditive functions

Now, a result which allows to obtain increasing minimizers in unbounded optimization problems will be presented. This result extends, in the context of Euclidean spaces, a previous one obtained by Topkis [20] (see Theorem 6.2 in [20]).

**Theorem 3.2.** Suppose that $A$ and $\mathbb{K}$ are lattices. If $x \to A(x)$ is ascending (in particular, for each $x \in X$, $A(x)$ is a sublattice of $A$), $A(y) \subset A(x)$ for $x \preceq y$ in X, $G$ is subadditive on $\mathbb{K}$, and Assumption 2.1 holds, then for $f(x) := \sup A^*(x)$, $x \in X$, it is obtained that $f(x) \preceq f(y)$, for all $x \preceq y$. Besides, $f(x) \in A^*(x)$, for every $x \in X$, i. e., $f$ is a minimizer for (3).

Proof. Since $\mathbb{K}$ is a lattice, and $G$ is subadditive, from Lemma 2.4 b and Lemma 2.4 c, it follows that $G(x,\cdot)$ is submodular on the lattice $A(x)$, for each $x \in X$. Moreover, as $A$ is a lattice, Assumption 2.1 holds, $x \to A(x)$ is ascending, $A(y) \subset A(x)$ for $x \preceq y$ in X, $G$ is subadditive, and using Lemma 2.8, it results that $x \to A^*(x)$ is ascending (in particular, for each $x \in X$, $A^*(x)$ is a sublattice of $A(x)$). Furthermore, it is obtained from Lemma 2.6 that $A^*(x)$, $x \in X$, is a compact set on $\mathbb{R}^m$. Therefore, from Lemma 2.1, it follows that $A^*(x)$, $x \in X$, contains both a supremum and an infimum. The rest of the proof follows from Lemma 2.3 a. □

**Remark 3.2.** For Theorem 3.2, the function $f'(x) := \inf A^*(x)$, $x \in X$, also works as an increasing minimizer for (3), using Lemma 2.3 b.

**Example 3.2.** Consider $X = A = \mathbb{Z}$ (where $\mathbb{Z}$ is the set of integers). Take $A(x) = [x, \infty) \cap \mathbb{Z}$, $x \in X$, and define $G(x,a) = \mathrm{e}^{a-x}$, $(x,a) \in \mathbb{K}$.

**Lemma 3.2.** Example 3.2 satisfies the assumptions of Theorem 3.2. (Therefore, $f(x) := \sup A^*(x)$, $x \in X$ is an increasing minimizer.)

Proof. Note that $A$ and $\mathbb{K}$ are trivially lattices. Take $x, y \in X$ with $x \leq y$, $a \in A(x)$, and $b \in A(y)$. To verify that $x \to A(x)$ is ascending, it is sufficient to consider the following cases: $a \in [x, y) \cap \mathbb{Z}$, $a \in [y, b) \cap \mathbb{Z}$ or $a \in [b, \infty) \cap \mathbb{Z}$, and the fact that $A(x)$, $x \in X$, is a sublattice of $A$ (recall that for each $x \in X$, $A(x)$ is a subset of $\mathbb{Z}$). If $a \in [y, b) \cap \mathbb{Z}$ or $a \in [b, \infty] \cap \mathbb{Z}$, then $a \wedge b \in A(y) \subset A(x)$, and $a \vee b \in A(y)$; if $a \in [x, y) \cap \mathbb{Z}$, then $a \wedge b = a \in A(x)$, and $a \vee b = b \in A(y)$. Therefore, $A(x) \sqsubseteq A(y)$.

Observe also that $G$ is a subadditive function on $\mathbb{K}$ as a consequence of Lemma 2.4 d. Clearly, $G$ is as well positive and continuous.

On the other hand, fix $x \in X$ and $\bar{s} \in \mathbb{R}$. If $\bar{s} \leq 0$, then $A_{\bar{s}}(x) = \emptyset$ is compact. Now, suppose that $0 < \bar{s} < 1$. Take $a \in A_{\bar{s}}(x)$ and note that $\mathrm{e}^{a-x} \leq \bar{s}$ implies that $a - x \leq \ln \bar{s} < 0$, i. e., $a < x$, but this is a contradiction, because $a \in [x, \infty) \cap \mathbb{Z}$. Thus, $A_{\bar{s}}(x) = \emptyset$ and, therefore, it is also compact.

Finally, suppose that $\bar{s} \geq 1$ and take $a \in A_{\bar{s}}(x)$. Observe that $\mathrm{e}^{a-x} \leq \bar{s}$ and $a \in [x, \infty) \cap \mathbb{Z}$ imply that $a \in [x, x + \ln \bar{s}] \cap \mathbb{Z}$. So, the compactness of $A_{\bar{s}}(x)$ follows from the facts that $A_{\bar{s}}(x)$ is a closed set (recall that $G$ is continuous), and that $A_{\bar{s}}(x) \subset [x, x + \ln \bar{s}] \cap \mathbb{Z}$. Therefore, since $x$ and $\bar{s}$ are arbitrary, $G$ is inf-compact. □

## 4. MONOTONE OPTIMAL POLICIES OF DISCOUNTED MCPs

Let $\{X, A, \{A(x), x \in X\}, Q, c\}$ be a fixed Markov control model. Here and in the following two sections, $\alpha \in (0, 1)$ will be considered fixed.

**Assumption 4.1.** (Hernández-Lerma and Lasserre [6], p. 46, Assumptions 4.2.1 and 4.2.2)

a) The one-stage cost $c : \mathbb{K} \to \mathbb{R}$ is nonnegative, l.s.c., and inf-compact on $\mathbb{K}$.

b) The transition law $Q$ is strongly continuous.

c) There is a policy $\pi$ such that $V_\alpha(\pi, x) < +\infty$, for all $x \in X$.

**Lemma 4.1.** (Hernández-Lerma and Lasserre [6], p. 46, Theorem 4.2.3 part a, b) Under Assumption 4.1, the discounted cost optimal value function $V_\alpha^*$ satisfies the *discounted cost optimality equation* (DCOE), i. e., for all $x \in X$,

$$V_\alpha^*(x) = \min_{a \in A(x)} \left[ c(x, a) + \alpha \int V_\alpha^*(z) Q(\mathrm{d}z | x, a) \right]. \tag{11}$$

Also there is $g_d \in \mathbb{F}$, such that

$$V_\alpha^*(x) = c(x, g_d(x)) + \alpha \int V_\alpha^*(z) Q(\mathrm{d}z | x, g_d(x)), \quad x \in X, \tag{12}$$

and $g_d$ is optimal. Conversely, if $g_d$ is stationary optimal, then it satisfies (12).

Define the function $G_1$ as

$$G_1(x, a) := c(x, a) + \alpha \int V_\alpha^*(z) Q(\mathrm{d}z | x, a), \tag{13}$$

$(x, a) \in \mathbb{K}$, which corresponds to the function that is minimized in (11). (13) will be called the *discounted dynamic programming operator* (DDPO), applied to $V_\alpha^*(\cdot)$ (see [16]). (In fact, (13) is the connection with the minimization problem presented in (3).)

**Remark 4.1.** Note that the set $A^*(x)$, $x \in X$, defined in (4), in the context of discounted MCPs, is denoted by $A_d^*(x)$, and is given by

$$A_d^*(x) := \left\{ a \in A(x) : G_1(x, a) = \min_{a^* \in A(x)} G_1(x, a^*) \right\}, \quad x \in X, \tag{14}$$

with $G_1$ defined in (13). Also, observe that due to Lemma 4.1, for each $x \in X$, $A_d^*(x)$ is nonempty and represents the set of minimizers of DCOE, for $x$.

### 4.1. Superadditive or subadditive DDPO

In this subsection, sufficient conditions are given for $G_1$ in (13) to be a superadditive or subadditive function (see C1 – C4 below). Such conditions have in common the following assumption.

**Assumption 4.2.**

a) $X$ and $A$ are intervals on $\mathbb{R}$.

b) $(1 - \lambda)a + \lambda a' \in A((1 - \lambda)x + \lambda x')$, for all $x, x' \in X$, $a \in A(x)$, $a' \in A(x')$, $\lambda \in [0, 1]$, $A(y) \subset A(x)$, for $x \le y$ in $X$, and $A(x)$ is convex for all $x \in X$.

c) $c$ is convex on $\mathbb{K}$.

**Condition 1. (C1)**

a) $Q$ is given by $x_{t+1} = \gamma x_t + \delta a_t + \xi_t$, $t = 0, 1, \ldots$, $\gamma, \delta > 0$, and $\{\xi_t\}$ is a sequence of i.i.d. r.v. which take values in $S \subset \mathbb{R}$. (Obviously, assuming that $\gamma x + \delta a + s \in X$, for all $x \in X$, $a \in A(x)$, and $s \in S$.)

b) $x \to A(x)$ is descending.

c) $c$ is superadditive on $\mathbb{K}$.

**Condition 2. (C2)**

a) $\mathbb{K}$ is a lattice.

b) Consider C1 a, but changing $\gamma, \delta > 0$ by $\gamma > 0$, $\delta < 0$.

c) $x \to A(x)$ is ascending.

d) $c$ is subadditive on $\mathbb{K}$.

**Condition 3. (C3)**

a) $x \to A(x)$ is descending.

b) For $x \leq y$ in $X$, $c(x,a) \leq c(y,a)$, for each $a \in A(y)$.

c) $c$ is superadditive on $\mathbb{K}$.

$Q$ is given by $x_{t+1} = F(x_t, a_t, \xi_t)$, $t = 0, 1, \ldots$, as in (5), with $S \subset \mathbb{R}$. Moreover,

d) if $x \leq y$ in $X$, then $F(x,a,s) \leq F(y,a,s)$, for each $a \in A(y)$ and $s \in S$.

e) $F(x,\cdot,s)$ is increasing, for each $x \in X$ and $s \in S$.

f) $F(\cdot,\cdot,s)$ is convex on $\mathbb{K}$, for each $s \in S$.

g) $F(\cdot,\cdot,s)$ is superadditive on $\mathbb{K}$, for each $s \in S$.

**Condition 4. (C4)**

a) $\mathbb{K}$ is a lattice.

b) Same as C3 b, d, f.

c) $x \to A(x)$ is ascending.

d) $c$ is subadditive on $\mathbb{K}$.

$Q$ is given by $x_{t+1} = F(x_t, a_t, \xi_t)$, $t = 0, 1, \ldots$, as in (5), with $S \subset \mathbb{R}$. Furthermore,

e) $F(x,\cdot,s)$ is decreasing, for each $x \in X$ and $s \in S$.

f) $F(\cdot,\cdot,s)$ is subadditive on $\mathbb{K}$, for each $s \in S$.

**Remark 4.2.** The conditions C1 – C4 given here have been inspired by the results on Table 1, p. 312 in [20], and by the results on monotonicity and convexity of the optimal value function for discounted MCPs provided in Lemmas 6.1 and 6.2 in [2].

Observe that C1 – C4 are imposed on the elements on the corresponding Markov control model, and they require (essentially on these elements) properties of monotonicity, convexity and superadditivity or subadditivity. Besides, note that C1 and C2 are proposed for linear models, while C3 and C4 allow to work with nonlinear models.

### 4.2. Main results

The proofs of the following theorems will be given in Section 6.

**Theorem 4.1.** Suppose that Assumption 4.1 and Assumption 4.2 hold. Then there is a decreasing stationary optimal policy under each C1 and C3.

**Theorem 4.2.** Suppose that Assumption 4.1 and Assumption 4.2 hold. Then there is an increasing stationary optimal policy under each C2 and C4.

## 5. EXAMPLES

### 5.1. Examples for C1 and C2

**Example 5.1.** The first example of an inventory/production system. (See Example 4.5 in [2] and Example 1.3.3 in [6].)

Take $X = \mathbb{R}$ and $A = A(x) = [0, +\infty)$, $x \in X$. The dynamic of this system is given by $x_{t+1} = x_t + a_t - \xi_t$, $t = 0, 1, \ldots$. Here $\xi_0, \xi_1, \ldots$ are i.i.d. r.v. taking values in $S = [0, +\infty)$ and with common density $\Delta$. The cost is given by

$$c(x, a) = \beta a + \hat{h}\,\mathrm{E}\,[\max(0, x + a - \xi)] + \hat{p}\,\mathrm{E}\,[\max(0, \xi - x - a)] \tag{15}$$

with nonnegative constants $\hat{h}$, $\hat{p}$, and $\beta$.

**Assumption 5.1.** $\Delta$ is continuous and $\mathrm{E}[\xi]$ is finite.

**Lemma 5.1.** Suppose that Assumption 5.1 holds. Then Example 5.1 has a decreasing stationary optimal policy. (In fact, it will be shown that Example 5.1 satisfies Assumptions 4.1 and 4.2, and C1.)

Proof. In [2] it is shown that under Assumption 5.1, Example 5.1 satisfies Assumption 4.1 (see Lemma 4.7 in [2]; note that here the strict convexity of $c$ is not necessary, so in the proof of that lemma it is possible to take $\varphi(a) = \beta a$, $a \in A(x)$, $x \in X$).

On the other hand, in [2] it is also proved that $c$ is convex, and to verify that Example 5.1 satisfies C1 and Assumption 4.2, it is enough to show that $c$ is superadditive, because Assumptions 4.2 a, b, and C1 a and C1 b trivially hold.

Now, the proof that $c$ is superadditive on $\mathbb{K}$ will be given. Using the fact that $\max(l, l') = \frac{l+l'+|l-l'|}{2}$, for $l, l' \in \mathbb{R}$, it results that the cost function given in (15) has the following form:

$$c(x, a) = \beta a + (\hat{h} - \hat{p})\frac{x + a}{2} - (\hat{h} - \hat{p})\frac{\mathrm{E}(\xi)}{2} + (\hat{h} + \hat{p})\frac{\mathrm{E}[|x + a - \xi|]}{2}, \tag{16}$$

where $(x, a) \in \mathbb{K}$ and the expectation in (15) and (16) is with respect to $\xi$.

Using (16) it is obtained that

$$\begin{aligned} c(y, a) - c(x, a) & \\ &= \frac{1}{2}(\hat{h} + \hat{p}) \int \Big[|y + a - s| - |x + a - s|\Big]\Delta(s)\,\mathrm{d}s + \frac{1}{2}(\hat{h} - \hat{p})(y - x), \end{aligned} \tag{17}$$

$x, y \in X$ and $a \in A = A(y)$. Note that, from (17), in order to obtain that $c$ is superadditive, it suffices to verify that for $x, y \in \mathbb{R}$, with $x < y$ and $s \in [0, +\infty)$, $\Theta(a) = |y + a - s| - |x + a - s|$, $a \in A$, is increasing.

Fix $x, y \in \mathbb{R}$, $x < y$ and $s \in [0, +\infty)$. Take $a, b \in [0, +\infty)$, with $a < b$. Observe that there are three cases: $y + a - s < x + b - s$, $y + a - s = x + b - s$ and $y + a - s > x + b - s$. Suppose the first one holds, i. e., $y + a - s < x + b - s$, this yields that $x + a - s < y + a - s < x + b - s < y + b - s$. Since the absolute value function is convex, using twice inequality (i) in Lemma 4.42 of [18] (firstly, this inequality will be used for $x + a - s < y + a - s < x + b - s$, and secondly, it will be applied to $y + a - s < x + b - s < y + b - s$), it results that

$$\begin{aligned} \frac{|y+a-s| - |x+a-s|}{y-x} &\le \frac{|x+b-s| - |x+a-s|}{b-a} \\ &\le \frac{|x+b-s| - |y+a-s|}{x+b-(y+a)}, \end{aligned} \tag{18}$$

$$\begin{aligned} \frac{|x+b-s| - |y+a-s|}{x+b-(y+a)} &\le \frac{|y+b-s| - |y+a-s|}{b-a} \\ &\le \frac{|y+b-s| - |x+b-s|}{y-x}. \end{aligned} \tag{19}$$

Combining (18) and (19) it follows that $\Theta(a) \le \Theta(b)$, $a < b$.

In a similar way, it is possible to prove that $\Theta(a) \le \Theta(b)$ for the cases $y+a-s = x+b-s$ and $y+a-s > x+b-s$. Since $x$ and $y$ are arbitrary, Lemma 5.1 follows. □

**Example 5.2.** The linear regulator problem (Section 4.7, [6]).
Take $X = A = A(x) = \mathbb{R}$, for every $x \in X$. The equation that describes the dynamic of this system is given by $x_{t+1} = \gamma x_t + \delta a_t + \xi_t$, $t = 0, 1, \ldots$. The cost at each time in which the process is observed is given by $c(x, a) = qx^2 + ra^2$, for $(x, a) \in \mathbb{K}$.

**Assumption 5.2.**

a) $\gamma > 0$ and $\delta < 0$ (see Remark 5.1). Both $q$ and $r$ are positive.

b) The disturbances $\xi_t$, $t = 0, 1, \ldots$, are i.i.d. r.v. with values in $S = \mathbb{R}$. Moreover, $\xi$ has a continuous density $\Delta$, $\mathrm{E}[\xi] = 0$ and $0 < \mathrm{Var}[\xi] = \mathrm{E}[\xi^2] < +\infty$.

**Lemma 5.2.** Suppose that Assumption 5.2 holds. Then Example 5.2 satisfies Assumptions 4.1 and 4.2, and C2, i. e., it has an increasing stationary optimal policy.

Proof. In Example 4.8 in [2], it is proved that under Assumption 5.2, Example 5.2 satisfies Assumption 4.1.

It is not difficult to verify that Assumption 4.2, C2 a, C2 b, and C2 c hold. Moreover, observe that $c$ is a subadditive function on $\mathbb{K}$, because for $x \le y$ in $X$ and $a \le b$ in $A(y) = \mathbb{R}$, it is obtained that $c(y, b) + c(x, a) = qy^2 + rb^2 + qx^2 + ra^2 = c(y, a) + c(x, b)$. This concludes the proof of Lemma 5.2. □

**Remark 5.1.** In a similar way it is possible to verify that if in Assumption 5.2 a $\delta > 0$, then Example 5.2 has a decreasing stationary optimal policy.

### 5.2. Examples for C3 and C4

**Example 5.3.** The second example of an inventory/production system. (See Example 4.1 in [3] and Example 1.3.3 in [6].)

Let $\widetilde{M}$ be a fixed positive constant and consider $X = A = [0, \widetilde{M}]$ and $A(x) = [0, \widetilde{M} - x]$, $x \in X$. The dynamic of the system is given by $x_{t+1} = [x_t + a_t - \xi_t]^+$, $t = 0, 1, \ldots$, where $j^+ := \max\{0, j\}$. Here, $\xi_t$, $t = 0, 1, \ldots$, are i.i.d. r.v. taking values on $S = [0, +\infty)$, and with common density $\Delta$. The cost function is given as in (15) with $\mathrm{E}\,[\max(0, \xi - x - a)] = 0$, i.e., $c(x, a) = \beta a + \hat{h}\,\mathrm{E}\,[\max(0, x + a - \xi)]$, $(x, a) \in \mathbb{K}$. (Observe that, since for each $t \geq 0$, $x_{t+1} \geq 0$,

$$\begin{aligned} c(x_t, a_t) &= \beta a_t + \hat{h}\,\mathrm{E}\,[\max(0, x_{t+1})] + \hat{p}\,\mathrm{E}\,[\max(0, -x_{t+1})] \\ &= \beta a_t + \hat{h}\,\mathrm{E}\,[x_{t+1}] = \beta a_t + \hat{h}\,\mathrm{E}\,[\max(0, x_t + a_t - \xi_t)]\,.) \end{aligned}$$

**Assumption 5.3.** $\Delta$ is continuous and bounded.

**Lemma 5.3.** Suppose that Assumption 5.3 holds. Then Example 5.3 has a decreasing stationary optimal policy, because it satisfies Assumptions 4.1 and 4.2, and C3.

Proof. Clearly, $c$ is nonnegative. The inf-compactness of $c$ is a direct consequence of its continuity (see the proof of Lemma 4.7 a in [2]), and of the compactness of $A(x)$, $x \in X$. Since $c$ is bounded (recall that $c$ is continuous and that $\mathbb{K}$ is a compact set), it follows that $0 \leq V_\alpha(\pi, x) \leq \widehat{M}/(1 - \alpha) < +\infty$, for all $x \in X$ and $\pi \in \Pi$, where $\widehat{M}$ is a bound for $c$ and $\alpha$ is the discount factor. Under Assumption 5.3, in Example 4.1 in [3], it has been proved that Assumption 4.1 b holds.

$c$ is a convex function on $\mathbb{K}$ (see Lemma 4.7 a in [2]), and an elementary computation permits to obtain that $(1-\lambda)a + \lambda a' \in A((1-\lambda)x + \lambda x') = [0, \widetilde{M} - ((1-\lambda)x + \lambda x')]$, if $x, x' \in X$, $a \in A(x) = [0, \widetilde{M} - x]$, $a' \in A(x') = [0, \widetilde{M} - x']$, $\lambda \in [0, 1]$. Also, note that, for each $x \in X$, $A(x)$ is convex and $A(x) \supset A(y)$, for $x \leq y$ in $X$. This, in combination with the fact that $X$ and $A$ are intervals in $\mathbb{R}$, allows to conclude that Assumption 4.2 holds.

Now, C3 will be verified. It is not difficult to show that Example 5.3 satisfies that $c$ is increasing in the first variable. The proof of the superadditivity of $c$ is a consequence of the superadditivity of the cost function (15) (see Lemma 5.1, considering that $\max(0, \xi - x - a) = 0$, $(x, a) \in \mathbb{K}$.

Take $x, y \in X$ with $x \leq y$, $a \in A(x)$, and $b \in A(y)$. To prove that $A(y) \sqsubseteq A(x)$ it is sufficient to consider the following three cases: $a \in [0, b]$, $a \in (b, \widetilde{M} - y]$, or $a \in (\widetilde{M} - y, \widetilde{M} - x]$. If $a \in [0, b]$ or $a \in (b, \widetilde{M} - y]$, then $a \wedge b \in A(y)$ and $a \vee b \in A(y) \subset A(x)$; if $a \in (\widetilde{M} - y, \widetilde{M} - x]$, then $a \wedge b = b \in A(y)$ and $a \vee b = a \in A(x)$. Since $A(x)$, $x \in X$, is a sublattice, it follows that $x \to A(x)$ is descending.

On the other hand, C3 d and C3 e hold as a consequence of the facts that $\eta(j) := j^+$, $j \in \mathbb{R}$, is non-decreasing and $\sigma(x, a, s) = x + a - s$ is an increasing function of $x$ and of $a$, for all $s \in [0, +\infty)$.

Now, the convexity of $F(\cdot,\cdot,s)$ on $\mathbb{K}$, for each $s \in [0,+\infty)$, is due to $F(x,a,s) = \eta(\sigma(x,a,s))$, $(x,a) \in \mathbb{K}$, $s \in [0,+\infty)$, with $\eta$ and $\sigma$ as above, the convexity and the increasing monotonicity of $\eta$, and the convexity of $\sigma(\cdot,\cdot,s)$ on $\mathbb{K}$, for each $s \in [0,+\infty)$.

The proof that $F(\cdot,\cdot,s)$ is superadditive on $\mathbb{K}$, for each $s \in S$, is similar to the proof that $c$, given in (15), is superadditive (see Lemma 5.1; in fact, now the absolute value function in (18) and (19) will be substituted by the positive part function). □

The following example is very similar to Example 4.1 in [2].

**Example 5.4.** Let $X = \mathbb{R}$ and $A = A(x) = [0,+\infty)$, $x \in X$, and consider $x_{t+1} = x_t + \mathrm{e}^{-a_t} + \xi_t$, $t = 0,1,\ldots$, where $\{\xi_t\}$ is a sequence of i.i.d. r.v. taking values in $S = \mathbb{R}$. The cost function is given by $c(x,a) = \mathrm{e}^x + \varphi(a)$, where $(x,a) \in \mathbb{K}$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}$ is a function defined as $\varphi(a) = a^2 - 1$, for $a > 1$, and $\varphi(a) = 0$, for $a \in [0,1]$.

**Assumption 5.4.**

a) $\xi$ has a continuous density $\Delta$.

b) $k := \int \mathrm{e}^s \, \Delta(s) \, \mathrm{d}s$ is finite and it satisfies $0 < \alpha \, k \, \mathrm{e} < 1$, where e is the basis of natural logarithm and $\alpha$ is the discount factor.

**Lemma 5.4.** Suppose that Assumption 5.4 holds. Then Example 5.4 satisfies Assumptions 4.1 and 4.2 and C4, i.e., it has an increasing stationary optimal policy.

Proof. Under Assumption 5.4, it is possible to verify that Example 5.4 satisfies Assumption 4.1 making little changes in the proof of Lemma 4.2 in [2]. Furthermore, it is not difficult to prove that Assumption 4.2 and C4 hold. □

## 6. PROOFS OF THEOREMS 4.1 AND 4.2

**Lemma 6.1.** Assumption 4.1 implies that Assumption 2.1 (for $G_1$) holds. Therefore, $A_d^*(x)$ is a nonempty compact set, for every $x \in X$ (see Remark 4.1).

Proof. Take $(x,a) \in \mathbb{K}$. Let $\{(x_l,a_l)\}_{l \geq 0}$ be a sequence in $\mathbb{K}$ such that $(x_l,a_l) \to (x,a)$. Let $\{u_n\}$ be a sequence of measurable bounded functions on $X$ such that $u_n \uparrow V_\alpha^*$ (observe that $V_\alpha^*(\cdot) \geq 0$ as a consequence of the fact that the cost function $c$ is nonnegative; moreover, $V_\alpha^*$ is measurable due to Assumption 4.1). Then, for each $n = 1,2,\ldots,$

$$\begin{aligned}\liminf_{l\to+\infty} \int V_\alpha^*(z) Q(\mathrm{d}z|x_l,a_l) &\geq \liminf_{l\to+\infty} \int u_n(z) Q(\mathrm{d}z|x_l,a_l) \\ &= \int u_n(z) Q(\mathrm{d}z|x,a).\end{aligned} \tag{20}$$

The equality in (20) is a consequence of the strong continuity of $Q$. Letting $n$ tend to infinity in (20) yields that (by the Monotone Convergence Theorem)

$$\liminf_{l\to+\infty} \int V_\alpha^*(z)Q(\mathrm{d}z|x_l, a_l) \geq \int V_\alpha^*(z)Q(\mathrm{d}z|x, a).$$

Therefore, $\int V_\alpha^*(z)Q(\mathrm{d}z|\cdot,\cdot)$ is l.s.c.

Now, as $c(\cdot,\cdot)$ is also l.s.c., and $c$ and $\int V_\alpha^*(z)Q(\mathrm{d}z|\cdot,\cdot)$ are bounded below, then from Proposition A.3 (a) in [6], it results that $G_1(\cdot,\cdot)$ is l.s.c. and bounded below.

Now, to prove that $G_1$ is inf-compact on $\mathbb{K}$, firstly note that since $G_1(\cdot,\cdot)$ is l.s.c., it follows that, for each $x \in X$, $G_1(x,\cdot)$ is l.s.c. Hence from Proposition A.1 (c) in [6] (p. 170) it results that $A_{\bar{s}}(x)$ is closed, for each $x \in X$ and $\bar{s} \in \mathbb{R}$.

Secondly, the compactness of $\{a \in A(x) : G_1(x,a) \leq \bar{s}\}$, for each $x \in X$ and $\bar{s} \in \mathbb{R}$, follows directly from the compactness of $\{a \in A(x) : c(x,a) \leq \bar{s}\}$, for each $x \in X$ and $\bar{s} \in \mathbb{R}$ (see Assumption 4.1 a), and the fact that $\{a \in A(x) : G_1(x,a) \leq \bar{s}\} \subseteq \{a \in A(x) : c(x,a) \leq \bar{s}\}$, for each $x \in X$ and $\bar{s} \in \mathbb{R}$ (recall that $V_\alpha^*(x) \geq 0$).

Finally, applying Lemma 2.6 it is obtained that $A_d^*(x)$, $x \in X$, is a nonempty compact set. □

**Lemma 6.2.** (Cruz-Suárez et al. [2], Lemma 6.1 and Lemma 6.2) Suppose that Assumption 4.1 holds.

a) Each C3 and C4 implies that $V_\alpha^*$ is increasing.

b) Suppose that Assumption 4.2 holds. Each Ci, i=1,...,4 implies that $V_\alpha^*$ is convex.

**Lemma 6.3.** Let $H : \mathbb{R} \to \mathbb{R}$ be a convex function. Let $\hat{a}, \hat{b}, \hat{c} \in \mathbb{R}$, with $\hat{a} < \hat{c} < \hat{b}$. Then $H(\hat{a} + \hat{b} - \hat{c}) \leq H(\hat{a}) + H(\hat{b}) - H(\hat{c})$.

P r o o f. Observe that it is possible to represent $\hat{c}$ as $\hat{c} = r\hat{a} + s\hat{b}$ with $r + s = 1$ due to $\hat{a} < \hat{c} < \hat{b}$. Then

$$\begin{aligned} H(\hat{a} + \hat{b} - \hat{c}) &= H((1-r)\hat{a} + (1-s)\hat{b}) \leq (1-r)H(\hat{a}) + (1-s)H(\hat{b}) \\ &\leq H(\hat{a}) + H(\hat{b}) - H(r\hat{a} + s\hat{b}) = H(\hat{a}) + H(\hat{b}) - H(\hat{c}), \end{aligned}$$

where both inequalities are a consequence of the fact that $H$ is a convex function.□

**Remark 6.1.** From Lemma 6.1, for each $x \in X$, $\sup A_d^*(x)$ and $\inf A_d^*(x)$ are well defined, i.e., $\sup A_d^*(x), \inf A_d^*(x) \in A_d^*(x) \subset A \subset \mathbb{R}$, for all $x \in X$. Thus, it is possible to define $f_d : X \to A$, the maximum minimizer that satisfies the DCOE, in the following way:

$$f_d(x) := \sup A_d^*(x), \quad x \in X. \tag{21}$$

Moreover, $f'_d : X \to A$, the minimum minimizer that satisfies the DCOE, can be defined as follows:

$$f'_d(x) := \inf A_d^*(x), \quad x \in X. \tag{22}$$

**Lemma 6.4.** Suppose that Assumptions 4.1 and 4.2 hold. Each C1 and C3 implies that $f_d(x)$ and $f'_d(x)$, $x \in X$, (defined in (21) and (22), respectively), are decreasing.

Proof. Theorem 3.1 and Remark 3.1 will be used to show that $f_d$ and $f'_d$ are decreasing, respectively. For this, observe that $A$ is trivially a lattice, because it is an interval of $\mathbb{R}$, $x \to A(x)$ is descending due to C1 or C3, $A(y) \subset A(x)$, for $x \leq y$ in $X$ by Assumption 4.2 b, and for each $x \in X$, $G_1(x, \cdot)$ is trivially submodular, as a consequence of that $A(x) \subset \mathbb{R}$; Assumption 4.1 implies that Assumption 2.1 holds (see Lemma 6.1), and it remains to prove that each C1 and C3 implies that the function $G_1$, in (13), is superadditive.

Assume that C3 holds. As $F(\cdot, \cdot, s)$ is superadditive on $\mathbb{K}$, for each $s \in S$, and using that $V^*_\alpha$ is increasing (see Lemma 6.2 a), it results that

$$V^*_\alpha(F(x, b, s)) \leq V^*_\alpha(F(y, b, s) + F(x, a, s) - F(y, a, s)), \tag{23}$$

for $x \leq y$ in $X$, $a \leq b$ in $A(y)$ and $s \in S$. Now, from C3 d, C3 e, the convexity of $V^*_\alpha$ (see Lemma 6.2 b), and Lemma 6.3, it follows that

$$\begin{aligned} V^*_\alpha(F(y, b, s) &+ F(x, a, s) - F(y, a, s)) \\ &\leq V^*_\alpha(F(y, b, s)) + V^*_\alpha(F(x, a, s)) - V^*_\alpha(F(y, a, s)), \end{aligned} \tag{24}$$

given that $F(x, a, s) \leq F(y, a, s) \leq F(y, b, s)$, with $x \leq y$, $a \leq b$ and $s \in S$.

Then, combining (23) and (24), it results that

$$V^*_\alpha(F(y, a, s)) + V^*_\alpha(F(x, b, s)) \leq V^*_\alpha(F(y, b, s)) + V^*_\alpha(F(x, a, s)),$$

for every $x \leq y$ in $X$, $a \leq b$ in $A(y)$, and $s \in S$. This means that $V^*_\alpha(F(\cdot, \cdot, s))$ is superadditive on $\mathbb{K}$, for each $s \in S$.

On the other hand, the monotonicity and the linearity of the integral yield that

$$\begin{aligned} \alpha \int V^*_\alpha(F(y, a, s))\, \Delta(s)\, \mathrm{d}s &+ \alpha \int V^*_\alpha(F(x, b, s))\, \Delta(s)\, \mathrm{d}s \\ &\leq \alpha \int V^*_\alpha(F(y, b, s))\, \Delta(s)\, \mathrm{d}s + \alpha \int V^*_\alpha(F(x, a, s))\, \Delta(s)\, \mathrm{d}s, \end{aligned}$$

for $x \leq y$ in $X$, and $a \leq b$ in $A(y)$. Therefore, the integral $\alpha \int V^*_\alpha(F(\cdot, \cdot, s)) \triangle(s)\, \mathrm{d}s$ is superadditive on $\mathbb{K}$. Now, using the fact that $c$ is superadditive and that the sum of two superadditive functions is also superadditive (see Lemma 2.4 a), it follows that $G_1$, given by (13), is superadditive.

Now suppose that C1 holds. As in this case the dynamic of the system is linear, the equality in (23) is attained. The rest of the proof is similar to the previous one exposed, considering $F(x, a, s) = \gamma x + \delta a + s$, $(x, a) \in \mathbb{K}$ and $s \in S$. $\square$

**Lemma 6.5.** Suppose that Assumptions 4.1 and 4.2 hold. Each C2 and C4 implies that $f_d(x)$ and $f'_d(x)$, $x \in X$ (defined in (21) and (22), respectively) are increasing.

Proof. Theorem 3.2 and Remark 3.2 will be used to show that $f_d$ and $f'_d$ are increasing. For this, observe that $A$ is trivially a lattice, because it is an interval of $\mathbb{R}$.

Moreover, $x \to A(x)$ is ascending, and $\mathbb{K}$ is a lattice due to C2 or C4. $A(y) \subset A(x)$, for $x \leq y$ in $X$ by Assumption 4.2 b. Assumption 4.1 implies that Assumption 2.1 holds (see Lemma 6.1), and it only remains to prove that each C2 and C4 implies that the function $G_1$ in (13) is subadditive.

The proof that C4 implies that $G_1$ given by (13) is subadditive is made in a similar way to the proof of Lemma 6.4, with the obvious changes.

The proof that C2 implies that $G_1$ is subadditive, is analogue to the proof of Lemma 6.4, using that $F(x, a, s) = \gamma x + \delta a + s$ is increasing in $x \in X$ and decreasing in $a \in A(x)$, for each $s \in S$ (see C2 b). Thus, the proof of Lemma 6.5 is concluded. □

Proof of Theorems 4.1 and 4.2. Since $f_d$ and $f'_d$ are monotone functions (see Lemmas 6.4 and 6.5) and $X, A \subset \mathbb{R}$, it follows from the well-known results that both $f_d$ and $f'_d$ are continuous almost everywhere (see Theorem 4.3.1 in [1], and the paragraph just next to the end of the proof of this theorem); hence, they are measurable (see Remark 6.3.4 in [18]). Therefore, $f_d$ and $f'_d$ are stationary policies. Besides, since $f_d(x), f'_d(x) \in A^*_d(x)$, for all $x$, it follows that $V^*_\alpha(x) = V^*_\alpha(f_d(x), x) = V^*_\alpha(f'_d(x), x)$, for all $x \in X$ (the proof of this is similar to the proof of Theorem 4.2.3 b, pp. 50–51 in [6]). □

## 7. REMARKS ON MONOTONE OPTIMAL POLICIES OF AVERAGE MCPs

In this section, conditions that ensure the existence of monotone optimal policies for average MCPs are commented. The idea is to use the so-called vanishing discount approach (see [6]). This approach is based on discounted MCPs with a variant discount factor $\alpha \in (0, 1)$.

**Lemma 7.1.** Under certain assumptions (see Remark 7.1 below), it follows that

i) there is a constant $\rho^*$ and a function $h : X \to \mathbb{R}$ such that the pair $(\rho^*, h)$ is a solution to the *average cost optimality equation* (ACOE), i. e.,

$$\rho^* + h(x) = \min_{a \in A(x)} \left[ c(x, a) + \int h(z) Q(\mathrm{d}z|x, a) \right], \quad x \in X. \tag{25}$$

The function $h$ can be represented as follows:

$$h(x) = \lim_{n \to +\infty} (V^*_{\alpha_n}(x) - V^*_{\alpha_n}(\bar{x})), \quad x \in X, \tag{26}$$

where $\bar{x}$ is a fixed state, and $\{\alpha_n\}$ is a sequence of discount factors such that $\alpha_n \uparrow 1$;

ii) there is $g_a \in \mathbb{F}$ such that

$$\rho^* + h(x) = c(x, g_a(x)) + \int h(z) Q(\mathrm{d}z|x, g_a(x)), \quad x \in X, \tag{27}$$

and $g_a$ is average optimal; in fact, any stationary policy $g_a$ which satisfies (27) is average optimal;

iii) $J^*(x) = \rho^*$, for all $x \in X$.

**Remark 7.1.** See Assumptions 4.2.1 and 5.5.1, and Theorem 5.5.4 in [6].

Consider an MCP for which Lemma 7.1 holds. Define the function $G_2$, for each $(x,a) \in \mathbb{K}$, as

$$G_2(x,a) := c(x,a) + \int h(z)Q(\mathrm{d}z|x,a), \tag{28}$$

which corresponds to the function that is minimized in (25).

For each $x \in X$, define $f_a(x) := \sup A_a^*(x)$ and $f_a'(x) := \inf A_a^*(x)$, where

$$A_a^*(x) := \left\{ a \in A(x) : G_2(x,a) = \min_{a^* \in A(x)} G_2(x,a^*) \right\}.$$

**Remark 7.2.**

a) Each Ci, i=1,...,4, Assumptions 4.1 and 4.2, for each $\alpha_n$, $n = 1, 2, \ldots$, and (26) imply that $h$ is a convex function (see Lemma 6.2 b).

b) Similar to Theorem 4.1, under Assumptions 4.1 and 4.2, and each C1 and C3, it is possible to prove that $f_a$ and $f_a'$ are decreasing stationary optimal policies.

c) Similar to Theorem 4.2, under Assumptions 4.1 and 4.2, and each C2 and C4, it is possible to prove that $f_a$ and $f_a'$ are increasing stationary optimal policies.

d) Example 5.2 from the previous section satisfies Assumptions 4.1 and 4.2 (see Lemma 5.2). Assumptions for Lemma 7.1 have been proved in Example 5.4.2 and Remark 5.5.3 b in [6]. Therefore, Example 5.2 has an increasing average optimal policy if $\delta < 0$; otherwise, Example 5.2 has a decreasing average optimal policy if $\delta > 0$.

## 8. CONCLUSIONS

After the discussion about the monotonicity of both discounted and average optimal policy, it is possible to conclude that, although our conditions are not exhaustive, they cover most of the examples that appear in MCPs on non-numerable real spaces. (See Sections 5 and 7.)

Now, the information about the existence of monotone optimal policies allows to explore how to improve the convergence in the policy iteration algorithm. This has been studied by Puterman [13] for the case of MCPs on finite spaces (see [13], pp. 259–260; 428). Meanwhile, research for the case of discounted MCPs on non-numerable real spaces is still in progress.

## ACKNOWLEDGEMENT

This work was partially supported by Programa de Mejoramiento del Profesorado (PROMEP) under grant UATLAX-152 and by Consejo Nacional de Ciencia y Tecnología (CONACyT) under grant 51222.

(Received January 11, 2007.)

## REFERENCES


[1] R. B. Ash: Real Variables with Basic Metric Space Topology. IEEE Press, New York 1993.

[2] D. Cruz-Suárez, R. Montes-de-Oca, and F. Salem-Silva: Conditions for the uniqueness of optimal policies of discounted Markov decision processes. Math. Methods Oper. Res. *60* (2004), 415–436.

[3] D. Cruz-Suárez, R. Montes-de-Oca, and F. Salem-Silva: Pointwise approximations of discounted Markov decision processes to optimal policies. Internat. J. Pure Appl. Math. *28* (2006), 265–281.

[4] M. C. Fu, S. I. Marcus, and I-J Wang: Monotone optimal policies for a transient queueing staffing problem. Oper. Res. *48* (2000), 327–331.

[5] E. Gallish: On monotone optimal policies in a queueing model of M/G/1 type with controllable service time distribution. Adv. in Appl. Probab. *11* (1979), 870–887.

[6] O. Hernández-Lerma and J. B. Lasserre: Discrete-Time Markov Control Processes. Springer-Verlag, New York 1996.

[7] D. P. Heyman and M. J. Sobel: Stochastic Models in Operations Research, Vol. II. Stochastic Optimization. McGraw-Hill, New York 1984.

[8] K. Hinderer and M. Stieglitz: Increasing and Lipschitz continuous minimizers in one-dimensional linear-convex systems without constraints: The continuous and the discrete case. Math. Methods Oper. Res. *44* (1996), 189–204.

[9] D. Kalin: A note on 'monotone optimal policies for Markov decision processes'. Math. Programming *15* (1978), 220–222.

[10] R. Mendelssohn and M. Sobel: Capital accumulation and the optimization of renewable resource models. J. Econom. Theory *23* (1980), 243–260.

[11] A. O. Pittenger: Monotonicity in a Markov decision process. Math. Oper. Res. *13* (1988), 65–73.

[12] E. L. Porteus: Foundations of Stochastic Inventory Theory. Stanford University Press, Stanford, Calif. 2002.

[13] M. L. Puterman: Markov Decision Processes: Discrete Stochastic Dynamic Programming. Wiley, New York 1994.

[14] U. Rieder: Measurable selection theorems for optimization problems. Manuscripta Math. *24* (1978), 115–131.

[15] S. M. Ross: Introduction to Stochastic Dynamic Programming. Academic Press, San Diego 1983.

[16] R. F. Serfozo: Monotone optimal policies for Markov decision processes. Math. Programming Stud. *6* (1976), 202–215.

[17] Sh. Stidham and R. R. Weber: Monotonic and insensitive optimal policies for control of queues with undiscounted costs. Oper. Res. *37* (1989), 611–625.

[18] K. R. Stromberg: An Introduction to Classical Real Analysis. Wadsworth International Group, Belmont 1981.

[19] R. K. Sundaram: A First Course in Optimization Theory. Cambridge University Press, Cambridge 1996.

[20] D. M. Topkis: Minimizing a submodular function on a lattice. Oper. Res. *26* (1978), 305–321.

[21] D. M. Topkis: Supermodularity and Complementarity. Princeton University Press, Princeton, N. J. 1988.

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