KYBERNETIKA — VOLUME *43* (2007), NUMBER 2, PAGES 197 – 208

SOME PROPERTIES OF B–OPERATIONS

Bohdan Butkiewicz

In the paper the problem of mathematical properties of B-operations and weak WB-operations introduced by the author for interpretation of connectives "and", "or", and "also" in fuzzy rules is considered. In previous author's papers some interesting properties of fuzzy systems with these operations were shown. These operations are weaker than triangular norms used commonly for a fuzzy system described by set of rules of the type if – then. Monotonicity condition, required for triangular norms, is replaced by condition of positivity (negativity), i. e. operations must be only positively (negatively) defined. Weak B-operations may not fulfill associativity condition.

Keywords: fuzzy system, connectives, weak operations, triangular norms

AMS Subject Classification: 03E72, 93C42, 20M14

1. INTRODUCTION

The problem of mathematical interpretation of connectives "and", "or", "also" in fuzzy rules were discussed at the beginning of fuzzy reasoning. Generally, triangular norms are used, especially algebraic product and minimum as t-norms and algebraic sum or maximum as t-conorms. Sometimes other operations were proposed, for instance ordered weighted average (OWA) operations [10], aggregation operators [6]. Recently an interesting paper was published concerning fuzzy connectives [8]. Also, the author introduced basic operations (B-operations) in [2]. They are something similar to triangular norms. Triangular norms and their properties were well described in [7]. B-operations were used for interpretation of connectives "and", "or", "also" in fuzzy control system. The name "basic" was introduced due to their properties. Let a point is the steady-state point of a system with linear or nonlinear plant and fuzzy proportional-derivative (FPD) controller for one pair of B-operations used for interpretation of the connectives "and", "or". Then, it was shown in [1, 2], and [3] that this point is the steady-state point for the system with any other pair of t-norms or B-operations used for interpretation of these connectives. Such theorem was enlarged after on weak B-operations (WB-operations) [4], where associativity is not necessary condition, and for larger class of fuzzy systems where a concept of a boundary state (some kind of steady-state) was introduced. In this paper the author shows some mathematical properties of B-operations and WB-operations.

2. BASIC OPERATIONS (B–OPERATIONS)

As it was explained earlier B-operations are generalization of triangular norms. They are used for fuzzy reasoning for interpretation of connectives "and", "or", "also", similarly as triangular norms.

Definition. (Basic operation B_{and}) Basic operation B_{and} is a binary operation $[0,1] \times [0,1] \to [0,1]$ satisfying for any $a, b, c \in [0,1]$ following conditions:

$aB_{\text{and}}b = bB_{\text{and}}a$	commutativity
$(aB_{\text{and}}b)B_{\text{and}}c = aB_{\text{and}}(bB_{\text{and}}c)$	associativity
$aB_{\text{and}}1 = a,\ aB_{\text{and}}0 = 0$	boundary
$aB_{\text{and}}b > 0$ if $a, b > 0$	is positively defined.

Operation B_{and} is similar as t-norm except of monotonicity. It is only positively defined. Evidently, each t-norm with no zero divisors is an B_{and} operation.

Definition. (Basic operation B_{or}) Basic operation B_{or} is a binary operation $[0,1] \times [0,1] \to [0,1]$ satisfying for $a, b, c \in [0,1]$ following conditions:

$aB_{\text{or}}b = bB_{\text{or}}a$	commutativity
$(aB_{\text{or}}b)B_{\text{or}}c = aB_{\text{or}}(bB_{\text{or}}c)$	associativity
$aB_{\text{or}}1 = 1,\ aB_{\text{or}}0 = a$	boundary
$aB_{\text{or}}b < 1$ if $a, b < 1$	is negatively defined.

Operation B_{or} is similar as t-conorm except of monotonicity. It is only negatively defined. Each t-conorm $S(a,b) < 1$ if $a, b < 1$ is B_{or} operation.

An explanation can be done. Operation B_{and} is assumed as positively defined from practical reason. If in a rule "if a is A and if b is B then c is C" both membership values $\mu_A(a)$, $\mu_B(b)$ are positive then the rule should be active and produce non zero output of fuzzy system therefore $\mu_C(c)$ must be positive. Similarly if a rule has a form "if a is A or if b is B then c is C" and both membership values $\mu_A(a), \mu_B(b) < 1$ then the rule should not be fully active and it follows than $\mu_C(c) < 1$.

Example of B_{and}. (Mesiar [9])

$$B_{\text{and}}(a,b) = \begin{cases} \max(a,b) & \text{if } (a,b) \in [\frac{1}{3}, \frac{2}{3}] \\ \min(a,b) & \text{otherwise.} \end{cases} \tag{1}$$

Example of B_{or}. (Mesiar [9])

$$B_{\text{or}}(a,b) = \begin{cases} \min(a,b) & \text{if } (a,b) \in [\frac{1}{3}, \frac{2}{3}] \\ \max(a,b) & \text{otherwise.} \end{cases} \tag{2}$$

As it can be seen operations are composed from t-norm and a t-conorm , but after that composition they are not monotonic.

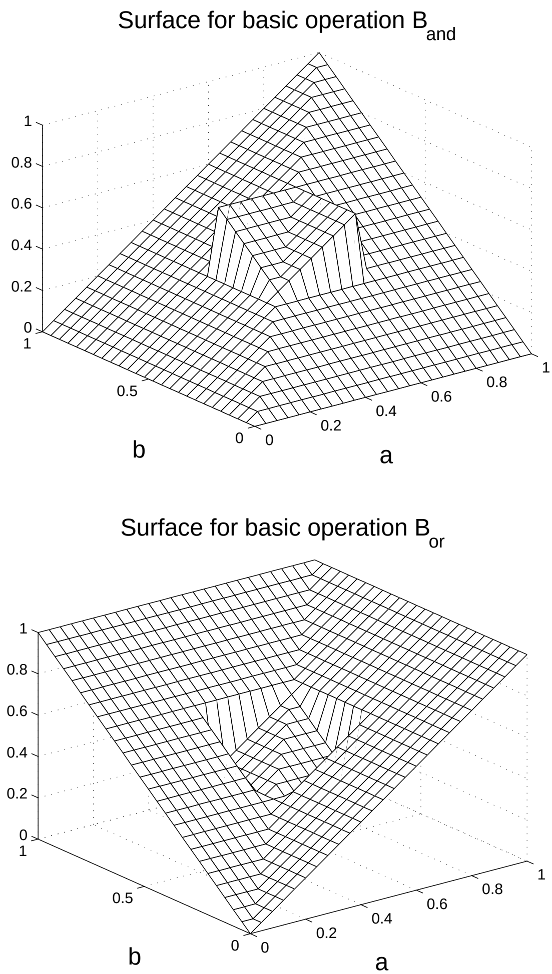

Fig. 1. Examples of B-operations: B_{and} (up), B_{or} (down).

3. PROPERTIES OF B–OPERATIONS

1. If B is defined as dual operation to B_{and}

$$B_{\text{or}}(a,b) = 1 - B_{\text{and}}(1-a, 1-b) \tag{3}$$

then it is B_{or}. Indeed, it is:

(a) symmetric

$$B_{\text{or}}(a,b) = 1-B_{\text{and}}(1-a,1-b) = 1 - B_{\text{and}}(1-b,1-a) = B_{\text{or}}(b,a) \tag{4}$$

(b) associative

$$
\begin{aligned}
& B_{\text{or}}(a, B_{\text{or}}(b,c)) = 1 - B_{\text{and}}(1-a, 1-[1-B_{\text{and}}(1-b,1-c)]) \\
= \; & 1 - B_{\text{and}}(1-a, B_{\text{and}}(1-b, 1-c) \\
= \; & 1 - B_{\text{and}}(B_{\text{and}}(1-a,1-b), 1-c) \qquad (5) \\
= \; & B_{\text{or}}(1 - B_{\text{and}}(1-a,1-b), c) = B_{\text{or}}(B_{\text{or}}(a,b), c)
\end{aligned}
$$

(c) fulfills boundary conditions

$$
\begin{aligned}
B_{\text{or}}(a,0) &= 1 - B_{\text{and}}(1-a,1) = 1-(1-a) = a \qquad (6) \\
B_{\text{or}}(a,1) &= 1 - B_{\text{and}}(1-a,0) = 1-0 = 1 \qquad (7)
\end{aligned}
$$

(d) negatively defined

$$
\text{if } a,b < 1 \text{ then } B_{\text{or}}(a,b) = 1 - B_{\text{and}}(1-a,1-b) < 1. \qquad (8)
$$

2. B_{and} and $B = B_{\text{or}}$ are mutually dual to each other

$$
\begin{aligned}
B_{\text{or}}(a,b) &= 1 - B_{\text{and}}(1-a,1-b) \qquad (9) \\
B_{\text{and}}(a,b) &= 1 - B_{\text{or}}(1-a,1-b) \qquad (10)
\end{aligned}
$$

3. $(B_{\text{and}}, B_{\text{or}}, N)$ where N is standard negation $N(a) = 1 - a$ forms de Morgan triple.

4. Generally, B-operations are not distributive, i. e. equation $aB_{\text{and}}(bB_{\text{or}}c) = (aB_{\text{and}}b)B_{\text{or}}(aB_{\text{and}}c)$ need not be true.

5. Associativity allows to extend each B-operation in a unique way to an n-ary operation by induction.

$$
B_{i=1}^{n} a_i = B(B_{i=1}^{n-1} a_i, a_n) = \cdots = B(a_1, a_2, \ldots, a_n) \qquad (11)
$$

6. Inequalities
Operation B_{and} fulfils inequality

$$
T_D(a,b) \le B_{\text{and}}(a,b) \le B^*_{\text{and}}(a,b) \qquad (12)
$$

where T_D is drastic product and $B^*_{\text{and}}(a,b)$ equals 1 in open unit square $(0,1)^2$ and min(a,b) else.

P r o o f. Operation T_D is weakest binary operation on $[0,1]^2$ satisfying the boundary conditions. $B^*_{\text{and}}(a,b)$ is supremum of B_c operators on open unit square as constant c, and min on the boundary. For c in open $(0,1)^2$ these all are B_{and} operators. Obviously T_D and B^*_{and} are not B_{and} operators, but they are the best possible bounds. □

Similarly B_{or} fulfils inequality

$$B_{\text{or}}^{+} \leq B_{\text{or}}(a,b) \leq S_D(a,b) \tag{13}$$

where S_D is drastic sum and B_{or}^{+} equals 0 if $a,b \in (0,1)$ and $\max(a,b)$ else.

Proof. Similarly, S_D is strongest binary operation on $[0,1]^2$ satisfying the boundary conditions. Lower bound can be justified using property of duality. Likewise as before, S_D and B_{and}^{+} are not B_{or} operators. □

7. B-operations form Abelian semigroups (X, B_{and}) and (X, B_{or}) on the closed interval $X = [0,1]$ which are commutative and associative (by definition). (X, B_{and}) has neutral element 1 because for each element $a \in [0,1]$ we have $aB_{\text{and}}1 = 1B_{\text{and}}a = a$. Annihilator (zero element) is 0 because $aB_{\text{and}}0 = 0B_{\text{and}}\,a = 0$. Thus (X, B_{and}) is monoid. Similar properties hold for (X, B_{or}).

4. GENERATION OF B–OPERATIONS

It can be shown that B-operations can be generated in a similar way as t-norms.

Theorem 1. Take any bijection

$$\beta : [0,1] \to [0,\infty] \tag{14}$$

with:

AND type:	$\beta(1) = 0$	$\beta(0) = \infty$
OR type:	$\beta(1) = \infty$	$\beta(0) = 0.$

Then the following function

$$B : [0,1] \times [0,1] \to [0,1]$$

is a B-operation (B_{and} type or B_{or} type respectively) given by

$$B(a,b) = \beta^{-1}[\beta(a) + \beta(b)] \tag{15}$$

is a B-operation where β^{-1} denotes inverse of β.

Proof. The proof is very similar to the proof of Theorem 3.23 for triangular norms in the book of Klement, Mesiar, and Pap [7]. Let $a,b,c \in [0,1]$. For bijection β exists an inversion β^{-1}. Therefore, following properties can be proved:

a) Commutativity

$$B(b,a) = \beta^{-1}[\beta(b) + \beta(a)] = \beta^{-1}[\beta(a) + \beta(b)] = B(a,b). \tag{16}$$

b) Boundary conditions

Consider AND type:

$$B_{\text{and}}(a,1) = \beta^{-1}[\beta(a)+\beta(1)] = \beta^{-1}[\beta(a)+0] = a \tag{17}$$
$$B_{\text{and}}(a,0) = \beta^{-1}[\beta(a)+\beta(0)] = \beta^{-1}[\beta(a)+\infty] = 0. \tag{18}$$

Consider OR type:

$$B_{\text{or}}(a,1) = \beta^{-1}[\beta(a)+\beta(1)] = \beta^{-1}[\beta(a)+\infty] = 1 \tag{19}$$
$$B_{\text{or}}(a,0) = \beta^{-1}[\beta(a)+\beta(0)] = \beta^{-1}[\beta(a)+0] = a. \tag{20}$$

c) Associativity

$$\begin{aligned} B[B(a,b),c] &= \beta^{-1}\{\beta\circ\beta^{-1}[\beta(a)+\beta(b)]+\beta(c)\} \\ &= \beta^{-1}[\beta(a)+\beta(b)+\beta(c)] \\ &= \beta^{-1}\{\beta(a)+\beta\circ\beta^{-1}[\beta(b)+\beta(c)]\} = B[a,B(b,c)]. \end{aligned} \tag{21}$$

d) Positivity (Negativity)

If $a,b>0$ then $\beta(a)<\infty$, $\beta(b)<\infty$, $B_{\text{and}}(a,b)=\beta^{-1}[\beta(a)+\beta(b)]>0$.

If $a,b<1$ then $\beta(a)<\infty$, $\beta(b)<\infty$, $B_{\text{or}}(a,b)=\beta^{-1}[\beta(a)+\beta(b)]<1$. □

Example of B_{and}. (Mesiar [9]) Let bijection, shown in Figure 2, be defined as follows

$$\beta(a) = \begin{cases} \infty & \text{if } a=0 \\ (1-a)/a & \text{if } 0<a\le 1/2 \\ 2a-1 & \text{if } 1/2<a<1 \\ 0 & \text{if } a=1. \end{cases} \tag{22}$$

Application of generation method for B_{and} is shown in Figure 2. Inversion of $\beta(a)$ is

$$\beta^{-1}(b) = \begin{cases} 1/(1+b) & \text{for } b\in[1,\infty] \\ (1+b)/2 & \text{for } b\in(0,1) \\ 1 & \text{for } b=0. \end{cases} \tag{23}$$

If $0<a,b\le 1/2$ then

$$\begin{aligned} B_{\text{and}}(a,b) &= \beta^{-1}\left(\frac{1-a}{a}+\frac{1-b}{b}\right) = \beta^{-1}\left(\frac{a+b-2ab}{ab}\right) \\ &= \frac{1}{1+\frac{a+b-2ab}{ab}} = \frac{ab}{a+b-ab}. \end{aligned} \tag{24}$$

Similarly B_{and} can be calculated for other cases. Operations defining B_{and} in different regions of $a,b\in[0,1]$ are shown in the next table. Operation fulfills boundary conditions.

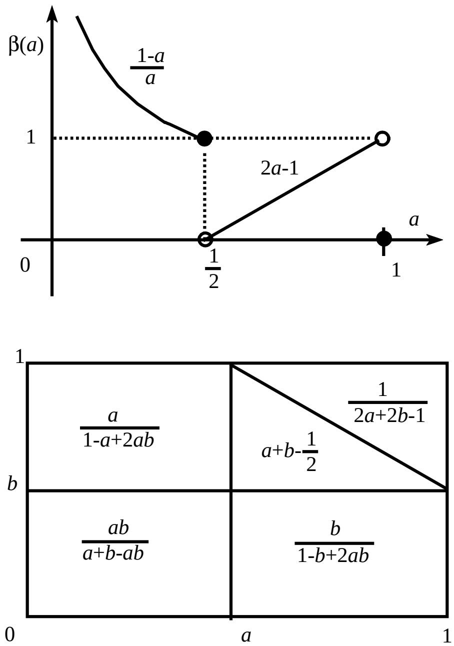

Fig. 2. Bijection $\beta(a)$ (up) and Table 1 describing B_{and} for $a, b \in (0, 1)$ (down).

5. FUZZY SYSTEM IN BOUNDARY CONDITIONS

The author proposed in [2] a definition of special conditions, which are something similar to steady-state of a fuzzy system.

Definition. (System in boundary conditions) Let fuzzy system is described by set of rules of the type

$$R_l\text{: if } (S_{l1}) \text{ and } (S_{l2}) \text{ and } \ldots \text{and } (S_{ln_l}) \text{ then } u \text{ is } U_l$$

which are joined by sentence connective "or", where:

S_{lj} are sentences of the type "x_j is A_{lj}", $j = 1, \ldots, n_l$,

x_j are considered, but not necessarily, as input values,

A_{lj} is any fuzzy set, defined on x_j axis, of linguistic variable X_j.

The system is in boundary conditions if it fulfils two conditions:

(1) All values x_j except one, say x_1, attain such values that appropriate membership functions $\mu_{A_{lj}}(x_j)$ $j = 2, \ldots, n_l$ in all rules of the system are equal to 0 or 1.

(2) If there are some rules with the same conclusion U_l then only one rule is active, i. e. only one of these conclusions has membership value different from zero.

Example. The system is described by differential equation of finite order. In the steady-state of the system all derivatives are equal to zero

$$\frac{dy}{dt} = 0, \quad \frac{d^2y}{dt^2} = 0, \quad \ldots \quad \frac{d^ny}{dt^n} = 0. \tag{25}$$

If output $y(t)$ of the system with two physical variables $x(t)$ and $y(t)$ depends only on $x(t)$ and derivatives of $y(t)$ then for time $t \to \infty$ the system can tends to boundary conditions. Very popular fuzzy controllers: PD, PI, PID, where input variables are proportional to error and its derivative and/or integral, tends in the steady-state to boundary conditions.

The author in [2] has shown that for a system in boundary conditions and with triangular norms or B-operations, used for interpretation of "and" and "or", following theorem can be proved.

Theorem 2. (Behavior of a system in boundary conditions) Let fuzzy system described before by set of rules

$$\mathrm{R}_l\text{: if } (S_{l1}) \text{ and } (S_{l2}) \text{ and} \ldots \text{and } (S_{lj}) \text{ and } \ldots \text{and } (S_{ln_l}) \text{ then } u \text{ is } U_l$$

joined with connective "or", where (S_{lj}) are sentences of the type "x_j is A_{lj}", $j = 1, \ldots, n_l$, be in boundary conditions. Suppose that a pair of B-operations is used for mathematical interpretation of "and", "or". If B-operations are exchanged into another pair of B-operations then the system outputs will not change, i. e. this point is also the point of the same boundary conditions. The result can depend on defuzzification method.

If the reader is interesting in a proof of this theorem he can see the proof of the Theorem 3.

6. WEAK BASIC OPERATIONS

In the previous section connectives "and", "or" were considered as associative. Generally it is reasonable, but for some systems this supposition is not true. For instance, some fuzzy microcontrollers, realized as integrated circuits, do not fulfill such supposition, because the result depends on order of rules. Therefore, the author proposes operations with weaker properties than triangular norms (and triangular conorms) and B-operations.

Definition. (Weak basic operation WB_{and}) Weak basic operation WB_{and} is a binary operation $[0,1] \times [0,1] \to [0,1]$ satisfying for $a, b \in [0,1]$ conditions:

$aWB_{\mathrm{and}}1 = a$, $1WB_{\mathrm{and}}a = a$	boundary condition
$aWB_{\mathrm{and}}0 = 0$, $0WB_{\mathrm{and}}a = 0$	boundary condition
$aWB_{\mathrm{and}}b > 0$ and $bWB_{\mathrm{and}}a > 0$ if $a, b > 0$	is positively defined.

Definition. (Weak basic operation WB_{or}) Weak basic operation WB_{or} is a binary operation $[0,1] \times [0,1] \to [0,1]$ satisfying for $a, b \in [0,1]$ conditions:

$aWB_{\text{or}}1 = 1$, $1WB_{\text{or}}a = 1$	boundary condition
$aWB_{\text{or}}0 = a$, $0WB_{\text{or}}a = a$	boundary condition
$aWB_{\text{or}}b < 1$ and $bWB_{\text{or}}a < 1$ if $a, b < 1$	is negatively defined.

It will be shown that without commutativity and associativity condition the theorem is also right under some additional suppositions.

Theorem 3. (System in boundary conditions with *WB*-operations) Let fuzzy system described before by set of rules

$$R_l\text{: if } (S_{l1}) \text{ and } (S_{l2}) \text{ and } \ldots \text{and } (S_{lj}) \text{ and } \ldots \text{and } (S_{ln_l}) \text{ then } u \text{ is } U_l$$

joined with connective "or", where (S_{lj}) are sentences of the type "x_j is A_{jk}", be in boundary conditions. Suppose that a pair of *WB*-operations is used for mathematical interpretation of "and", "or". If *WB*-operations are exchanged to another pair of *WB*-operations and order of the terms (S_{lj}) in the rules and order of the rules in the system is not changed then the system outputs will not change, i. e. this point is also the point of the same boundary conditions. The outputs can depend on defuzzification method.

P r o o f. Consider at the beginning an active rule where weights of all statements "x_j is A_{lj}" are different from zero. The result obtained for membership function $\mu_l(u)$ for any fuzzy set U_l of output u equals $\mu_l(u_1) = \mu_{l1}(x_1)WB_{\text{and}}, \ldots, WB_{\text{and}}$ $\mu_{ln_l}(x_{n_l})$ where $\mu_{l1}(x_1), \ldots, \mu_{ln}(x_{n_l})$ denote the weights of all statements "x_j is A_{lj}". If the rule is active in boundary conditions then only one term, say $\mu_{l1}(u)$, is different from 1. Therefore, the weight of the rule R_l is equal to $\mu_l = \mu_{l1}(x_1)$. The result does not depend on interpretation of WB_{and}. For not active rules the result is 0. Rules with the same conclusion set U_l can be aggregated using connective "or". The rules are joined by WB_{or} operations, but in boundary conditions only one rule is active, say the rule R_k. Then $\mu_k(u) = \mu_1(x_1)WB_{\text{or}}0WB_{\text{or}}0WB_{\text{or}} \cdots WB_{\text{or}}0 = \mu_1(x_1)$. The result does not depend on the interpretation of WB_{or}. However, it can depend on the order of terms in the rules, on order of rules, and on defuzzification method. $\square$

Remark. From mathematical point of view Theorem 2 and 3 are right also if as "and" operation is used any function on unit square coinciding in boundary with minimum, and as "or" operation any function on unit square coinciding in boundary with maximum. Such formulation of the theorems shown here is given for practical reasons. Fuzzy systems with *B* operations and *WB* operations compose different classes of systems. Operations used in fuzzy controllers ought to have at least such properties as *WB* operations. For example, let there is a rule "*if Temperature is Very High and Pressure is Very High then stop Heating*", which prevents catastrophic operation of a boiler system. If both statements are true then the rule must be active. If "and" does not fulfill positivity condition then possibility of the catastrophe is very high or it certainly become.

The main reason for dealing here with different operations applied, as connectives, were previous researches of the author who looked for the best reasoning methods for fuzzy control system. The system was built as closed loop with fuzzy controller of the type PD, PI or PID (proportional-integral-derivative), described by *if* $\cdots$ *then* rules, and a plant. More than 30 different reasoning methods were applied with pairs of t-norms and t-conorms as well as pairs of B-operations. Among t-norms were chosen: logic, algebraic, Hamacher, Yager, Łukasiewicz, drastic, and some others. Using each time 6 different plans: 1st order with delay, linear 2nd and 3rd order, nonlinear 2nd order, fuzzy dynamic, and inverted pendulum, the author shown that very popular Mamdani reasoning with minimum and maximum is not good solution. Very often there are many operations giving better system behavior. The results were published in [5].

Example. As an example, consider the plant of second order described by transfer function

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{26}$$

The membership functions used for system error, derivative of error and output of fuzzy PD (proportional derivative) controller were shown in Figure 3 while the results of control were presented in Figure 4.

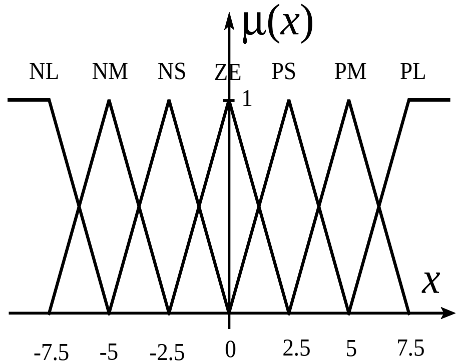

Fig. 3. Membership functions for fuzzy PD controller.

The result obtained for logic operations (Mamdani reasoning) are compared with three pairs of B-operations. Here, the best result B 2 is for the pair from Figure 2. The controller parameters were chosen in order to show better the differences. Reader can observe the same steady state and similar behavior of the system with t-norms as well as with B-operations.

7. CONCLUSIONS

Some properties of operations used as connectives in fuzzy system described by rules were shown. Weaker operations than triangular norms are proposed. Operations,

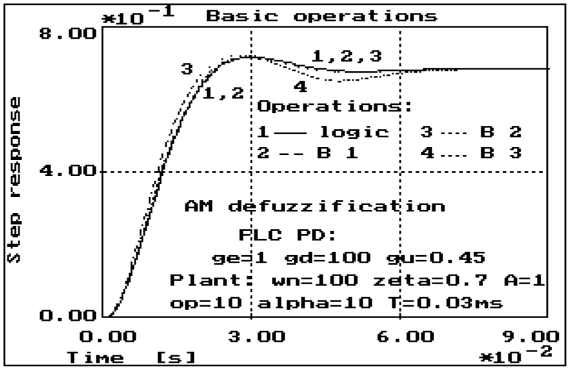

Fig. 4. Example of results for fuzzy PD control.

which do not satisfy associativity condition, are applied in integrated circuits of microcontrollers. It is the main reason to deal with such operations in this paper. Theorems 2 and 3 about behavior of a system in boundary conditions show that steady state of fuzzy control system can be exactly the same for different reasoning methods. Moreover, experimental with different plants and reasoning operations allow to draw the conclusion that not only steady state but also dynamics of the system behavior are very similar. Something as "fuzzy theorem" (notion defined in [5]) can be formulated, which describes dynamic behavior of the closed loop fuzzy PID control systems with such operations.

ACKNOWLEDGEMENT

The author would like to thank very much Professor Radko Mesiar for his suggestions and nice examples prepared during the same conference session in Zittau, where the author presented first version of this paper. Moreover, the author would like to thank the anonymous reviewers for insightful comments and helpful suggestions to improve the quality, which have been incorporated in the manuscript.

(Received November 15, 2006.)

REFERENCES

[1] B. S. Butkiewicz: Steady-state error of a system with fuzzy controller. IEEE Trans. System Man Cybernet., Part B: Cybernetics *28* (1998), 6, 855–860.

[2] B. S. Butkiewicz: Fuzzy control system with *B*-operations. In: Advances in Soft Computing, Developments in Soft Computing (R. John and R. Birkenhead, eds.), Physica–Verlag, Heidelberg – New York 2001, pp. 48–55.

[3] B. S. Butkiewicz: About robustness of fuzzy logic PD and PID controller under changes of reasoning methods. In: Advances in Computational Intelligence and Learning, Methods and Applications (H.-J. Zimmermann, G. Tselentis, M. van Someren, and G. Dounias, eds.), Kluwer Academic Publishers, Boston – London 2001, pp. 307–318.

[4] B. S. Butkiewicz: Fuzzy systems with weak basic operations. In: Proc. East-West Fuzzy Colloquium 2005 12$^{\text{th}}$ Zittau Fuzzy Colloquium. Hochschule Zittau/Gorlitz 2005, pp. 73–78.
[5] B. S. Butkiewicz: About the best reasoning method for fuzzy control. In: 6th World Congress on Intelligent Control and Automation, Dalian, China, Conf. Proc. Vol. 5, pp. 3920–2924, IEEE Catalog No 06EX1358, 2006.
[6] T. Calvo and R. Mesiar: Aggregation operators: ordering and bounds. Fuzzy Sets and Systems *139* (2003), 685–697.
[7] E. P. Klement, R. Mesiar, and E. Pap: Triangular Norms. Kluwer Academic Publishers, Dordrecht 2000.
[8] R. Mesiar, B. Reusch, and H. Thielle: Fuzzy equivalence relations and fuzzy partitions. J. Multiple-Valued Logic & Soft Computing *12* (2006), 167–181.
[9] R. Mesiar: Personal communication on East-West Fuzzy Colloquium, Zittau 2005.
[10] R. R. Yager: On ordered weighted operators in multicriteria decisionmaking. IEEE Trans. System Man Cybernet. *18* (1988), 183–190.

Bohdan S. Butkiewicz, Warsaw University of Technology, Institute of Electronic Systems, Nowowiejska 15/19, 00-665 Warsaw. Poland.
e-mail: bb@ise.pw.edu.pl