KYBERNETIKA — VOLUME *43* (2007), NUMBER 2, PAGES 115 – 132

PRESERVATION OF PROPERTIES OF FUZZY RELATIONS DURING AGGREGATION PROCESSES

Józef Drewniak and Urszula Dudziak

Diverse classes of fuzzy relations such as reflexive, irreflexive, symmetric, asymmetric, antisymmetric, connected, and transitive fuzzy relations are studied. Moreover, intersections of basic relation classes such as tolerances, tournaments, equivalences, and orders are regarded and the problem of preservation of these properties by n-ary operations is considered. Namely, with the use of fuzzy relations $R_1, \ldots, R_n$ and n-argument operation F on the interval $[0, 1]$, a new fuzzy relation $R_F = F(R_1, \ldots, R_n)$ is created. Characterization theorems concerning the problem of preservation of fuzzy relations properties are given. Some conditions on aggregation functions are weakened in comparison to those previously given by other authors.

Keywords: fuzzy relation, fuzzy relation properties, fuzzy relation classes, $*$-transitivity, transitivity, aggregation functions, relation aggregation, triangular norms

AMS Subject Classification: 03E72, 68T37

1. INTRODUCTION

Fuzzy relations are often considered in a context of preservation of their properties in an aggregation process (cf. [4, 10, 13, 14, 15, 20]). There are many kinds of such properties. Especially, many authors deal with diverse types of transitivity properties (see in particular [20], where connections between the problem of preservation of the transitivity property and domination are investigated).

Until now, the problem of preservation of diverse fuzzy relations properties was examined for fixed aggregation functions separately (see e. g. [14, 15]). In our considerations we have another attitude towards the mentioned problem. Namely, we concentrate on a fixed property of fuzzy relations and characterize all aggregation functions preserving this property. Moreover, the aim of this paper is to give characterizations under the weakest assumptions on functions used for aggregation. Therefore, we start with an arbitrary n-ary operation without additional assumptions. In this way some conditions on aggregation functions in [14, 15, 20] are weakened. Our work is a continuation and a generalization of results obtained in [7, 8, 9]. We concentrate on basic properties of fuzzy relations, which are generalizations of crisp properties of binary relations. We also consider a few composed versions of these basic properties. However, it is worth mentioning that our considerations are restricted to fuzzy versions of standard relation properties and many other properties are omitted. Such property is reciprocity (see [5, 15, 21]).

In this paper, necessary and sufficient conditions for n-argument operation preserving fixed properties of fuzzy relations are presented. Results of such considerations can be applied in group choice theory and multiple-criteria decision making (cf. [15, 20]). Firstly (Section 2), we give basic definitions concerning n-ary operations on $[0,1]$ and examples of them. Next, we recall some information about fuzzy relations (Section 3), we present characterization theorems for basic properties (Section 4), and the composed versions of them (Section 5).

2. BASIC DEFINITIONS

We recall some information about operations on the interval $[0,1]$.

Definition 1. (Calvo et al. [3]) Let $n \geqslant 2$. An operation $F\colon [0,1]^n \to [0,1]$ is called an aggregation function if it is increasing in each of its arguments and fulfils the boundary conditions

$$F(0,\dots,0) = 0, \quad F(1,\dots,1) = 1. \tag{1}$$

There are many examples of aggregation functions. We present these examples, which will be used in the sequel (cf. [3, 8, 15]).

Example 1. Let $t_1,\dots,t_n,w_1,\dots,w_n \in [0,1]$. Aggregation functions are

- the weighted minimum

$$F(t_1,\dots,t_n) = \min_{1\leqslant k\leqslant n} \max(1-w_k,t_k), \text{ where } \max_{1\leqslant k\leqslant n} w_k = 1, \tag{2}$$

- the weighted maximum

$$F(t_1,\dots,t_n) = \max_{1\leqslant k\leqslant n} \min(w_k,t_k), \text{ where } \max_{1\leqslant k\leqslant n} w_k = 1, \tag{3}$$

- the projections

$$P_k(t_1,\dots,t_n) = t_k, \quad k \in \{1,\dots,n\}, \tag{4}$$

- the median value

$$\operatorname{med}(t_1,\dots,t_n) = \begin{cases} \frac{s_k+s_{k+1}}{2}, & \text{for } n = 2k, \\ s_{k+1}, & \text{for } n = 2k+1, \end{cases} \tag{5}$$

where $(s_1,\dots,s_n)$ is the increasingly ordered sequence of the values $t_1,\dots,t_n$, i.e. $s_1 \leqslant \dots \leqslant s_n$.

- the quasi-linear means

$$F(t_1,\dots,t_n)=\varphi^{-1}\left(\sum_{k=1}^{n} w_k\varphi(t_k)\right), \tag{6}$$

where $\varphi\colon [0,1]\to\mathbb{R}$ is an increasing bijection and the weights $w_k>0$, $k=1,\dots,n$, fulfil the condition $\sum_{k=1}^{n} w_k=1$,

- the arithmetic mean

$$F(t_1,\dots,t_n)=\frac{1}{n}\sum_{k=1}^{n} t_k, \tag{7}$$

which we get if $w_k=\frac{1}{n}$ for each $k=1,\dots,n$ in (6).

Example 2. (Marichal [12]) The most popular members of the family of quasi-linear means on $[0,1]$ are presented in the following table:

$\varphi(t)$	weighted mean	type
t	$A(t_1,\dots,t_n)=\sum\limits_{k=1}^{n} w_k t_k$	arithmetic
t^2	$Q(t_1,\dots,t_n)=\sqrt{\sum\limits_{k=1}^{n} w_k t_k^2}$	quadratic
$\log t$	$G(t_1,\dots,t_n)=\prod\limits_{k=1}^{n} t_k^{w_k}$	geometric
t^{-1}	$H(t_1,\dots,t_n)=\begin{cases} 0, & \underset{1\leqslant k\leqslant n}{\exists} t_k=0 \\ \left(\sum\limits_{k=1}^{n}\frac{w_k}{t_k}\right)^{-1}, & \text{otherwise}\end{cases}$	harmonic
$t^r,\ r\neq 0$	$P_{(r)}(t_1,\dots,t_n)=\begin{cases} 0, & r<0,\ \underset{1\leqslant k\leqslant n}{\exists} t_k=0 \\ \left(\sum\limits_{k=1}^{n} w_k t_k^r\right)^{\frac{1}{r}}, & \text{otherwise}\end{cases}$	power
$e^{rt}, r\neq 0$	$E(t_1,\dots,t_n)=\frac{1}{r}\ln\left(\sum\limits_{k=1}^{n} w_k e^{rt_k}\right)$	exponential

Definition 2. (Klement et al. [11], p. 4) A triangular norm $T\colon [0,1]^2\to[0,1]$ (triangular conorm $S\colon [0,1]^2\to[0,1]$) is an arbitrary associative, commutative operation having a neutral element $e=1$ ($e=0$), which is increasing in each of its arguments.

In our further considerations we will use the abbreviation ‘t-norm’ when speaking of a triangular norm.

Example 3. (Klement et al. [11], pp. 4, 11) The four well-known examples of t-norms T and corresponding t-conorms S are:

$$T_M(s,t) = \min(s,t), \qquad T_P(s,t) = st, \qquad T_L(s,t) = \max(s+t-1,0),$$
$$S_M(s,t) = \max(s,t), \qquad S_P(s,t) = s+t-st, \qquad S_L(s,t) = \min(s+t,1),$$

$$T_D(s,t) = \begin{cases} s, & t=1 \\ t, & s=1 \\ 0, & \text{otherwise}, \end{cases} \qquad S_D(s,t) = \begin{cases} s, & t=0 \\ t, & s=0 \\ 1, & \text{otherwise} \end{cases}$$

for $s,t \in [0,1]$.

Lemma 1. (Klement et al. [11], pp. 6–7) For arbitrary t-norm T one has

$$T_D \leqslant T \leqslant T_M. \tag{8}$$

Moreover

$$T_D \leqslant T_L \leqslant T_P \leqslant T_M. \tag{9}$$

Now some facts concerning operations on $[0,1]$ will be presented. Directly from the definition of increasing bijections we get

Lemma 2. (Drewniak and Dudziak [8]) If $\varphi\colon [0,1] \to [0,1]$ is an increasing bijection, then for arbitrary $s,t \in [0,1]$ one has

$$\varphi(t) = 0 \Leftrightarrow t = 0, \qquad \varphi(t) = 1 \Leftrightarrow t = 1, \tag{10}$$

$$\varphi(t) > 0 \Leftrightarrow t > 0, \qquad \varphi(t) < 1 \Leftrightarrow t < 1, \tag{11}$$

$$\varphi(\min(s,t)) = \min(\varphi(s),\varphi(t)), \qquad \varphi(\max(s,t)) = \max(\varphi(s),\varphi(t)). \tag{12}$$

Definition 3. (Saminger et al. [20], Definition 2.5) Let $m, n \in \mathbb{N}$. Operation $F\colon [0,1]^m \to [0,1]$ dominates operation $G\colon [0,1]^n \to [0,1]$ ($F \gg G$) if for an arbitrary matrix $[a_{ik}] = A \in [0,1]^{m\times n}$ the following inequality holds

$$\begin{aligned} &F(G(a_{11},\ldots,a_{1n}),\ldots,G(a_{m1},\ldots,a_{mn})) \\ \geqslant\ &G(F(a_{11},\ldots,a_{m1}),\ldots,F(a_{1n},\ldots,a_{mn})). \end{aligned} \tag{13}$$

Theorem 1. (Saminger et al. [20], Proposition 5.1) A function $F\colon [0,1]^n \to [0,1]$, which is increasing in each of its arguments dominates minimum iff

$$F(t_1,\ldots,t_n) = \min(f_1(t_1),\ldots,f_n(t_n)), \ \ t_1,\ldots,t_n \in [0,1], \tag{14}$$

where $f_k\colon [0,1] \to [0,1]$ is increasing with $k = 1,\ldots,n$.

For further results on domination, specially domination between t-norms, we recommend [11], pp. 152–156.

Example 4. There are a few useful examples of functions (14):

if $f_k(t) = t$, $k = 1, \ldots, n$, then $F = \min$,

if for a certain $k \in \{1, \ldots, n\}$, function $f_k(t) = t$ and $f_i(t) = 1$ for $i \neq k$, then $F = P_k$, (cf. (4)),

if $f_k(t) = \max(1 - v_k, t)$, $v_k \in [0, 1]$, $k = 1, \ldots, n$, $\max_{1 \leqslant k \leqslant n} v_k = 1$, then F is the weighted minimum (2).

Example 5. (Drewniak and Dudziak [7], Saminger et al. [20]) The weighted geometric mean (cf. Example 2) dominates t-norm T_P. The weighted arithmetic mean (cf. Example 2) dominates t-norm T_L. The function

$$F(t_1, \ldots, t_n) = \frac{p}{n} \sum_{k=1}^{n} t_k + (1 - p) \min_{1 \leqslant k \leqslant n} t_k \tag{15}$$

dominates T_L, where $p \in (0, 1)$.

From the definitions of t-norms and t-conorms and by the properties of minimum and maximum it follows that

Lemma 3. Let T be an arbitrary t-norm, S an arbitrary t-conorm. Thus for any $a, b, c \in [0, 1]$ one has

$$T(\max(a, b), \max(a, c)) \leqslant \max(a, T(b, c)), \tag{16}$$

$$T(\min(a, b), \min(a, c)) \leqslant \min(a, T(b, c)), \tag{17}$$

$$S(\max(a, b), \max(a, c)) \geqslant \max(a, S(b, c)), \tag{18}$$

$$S(\min(a, b), \min(a, c)) \geqslant \min(a, S(b, c)). \tag{19}$$

In virtue of Lemma 3 we obtain

Theorem 2. The weighted minimum (2) dominates every t-norm T.

P r o o f. Let $s_k, t_k, w_k \in [0, 1]$, $k = 1, \ldots, n$. According to (13) for $n = 2$ we have to show that

$$\min_{1 \leqslant k \leqslant n} \max(1 - w_k, T(s_k, t_k)) \geqslant T\left(\min_{1 \leqslant k \leqslant n} \max(1 - w_k, s_k), \min_{1 \leqslant k \leqslant n} \max(1 - w_k, t_k)\right). \tag{20}$$

We will apply the following inequality

$$\min_{1 \leqslant k \leqslant n} T(s_k, t_k) \ \geqslant \ T(\min_{1 \leqslant k \leqslant n} s_k, \min_{1 \leqslant k \leqslant n} t_k), \tag{21}$$

which may be obtained by induction with respect to $n \in \mathbb{N}$ and follows from the fact that $T_M \gg T$ (cf. [11], p. 152). As a result, by (21), (16) and by the fact that minimum is increasing, we get

$$T\left(\min_{1\leqslant k\leqslant n} \max(1-w_k, s_k), \min_{1\leqslant k\leqslant n} \max(1-w_k, t_k)\right)$$
$$\leqslant \min_{1\leqslant k\leqslant n} T(\max(1-w_k, s_k), \max(1-w_k, t_k))$$
$$\leqslant \min_{1\leqslant k\leqslant n} \max(1-w_k, T(s_k, t_k)),$$

which proves the inequality (20). $\square$

3. FUZZY RELATIONS

We recall basic properties of fuzzy relations.

Definition 4. (Zadeh [22]) A fuzzy relation on a set $X \neq \emptyset$ is an arbitrary function $R : X \times X \to [0,1]$. The family of all fuzzy relations on X is denoted by $FR(X)$.

Example 6. Let $X = [0, 120]$ be the length of a human life. A relation $R \in FR(X)$ 'a person x is much older than a person y' may be described by a function

$$R(x,y) = \begin{cases} 0, & x-y \leqslant 0 \\ \frac{x-y}{30} & 0 < x-y < 30, \qquad x,y \in X. \\ 1, & x-y \geqslant 30 \end{cases}$$

Remark 1. If $\operatorname{card} X = n$, $X = \{x_1, \ldots, x_n\}$, then $R \in FR(X)$ may be presented by a matrix $R = [r_{ik}]$, where $r_{ik} = R(x_i, x_k)$, $i, k = 1, \ldots, n$.

Definition 5. (Zadeh [22]) Let $*\colon [0,1]^2 \to [0,1]$. A sup-$*$-composition of relations $R, S \in FR(X)$ is the relation $(R \circledast S) \in FR(X)$ such that

$$(R \circledast S)(x,z) = \sup_{y\in X} R(x,y) * S(y,z), \quad (x,z) \in X \times X. \tag{22}$$

For $* = \min$ we write

$$(R \circ S)(x,z) = \sup_{y\in X} \min(R(x,y), S(y,z)), \quad (x,z) \in X \times X. \tag{23}$$

If $\operatorname{card} X = n$, $R = [r_{ij}]$, $S = [s_{jk}]$ (cf. Remark 1), then

$$R \circledast S = [t_{ik}], \text{ where } t_{ik} = \max_{1\leqslant j\leqslant n} (r_{ij} * s_{jk}), \quad i,k = 1, \ldots, n, \quad n \in \mathbb{N}. \tag{24}$$

Let $A \subset X$, $A \neq \emptyset$. Now we pose a fact connected with the extension of a relation $R_0 \in FR(A)$ to the relation $R \in FR(X)$, where

$$R(x,y) = \begin{cases} R_0(x,y), & x,y \in A \\ 0, & \text{otherwise.} \end{cases} \tag{25}$$

Theorem 3. Let $R_0, S_0 \in FR(A)$, $\emptyset \neq A \subset X$, $R, S \in FR(X)$ be of the form (25), respectively. If an operation $*\colon [0,1]^2 \to [0,1]$ has zero element $z = 0$, then

$$(R \circledast S)(x,y) = \begin{cases} (R_0 \circledast S_0)(x,y), & x,y \in A \\ 0, & \text{otherwise.} \end{cases} \tag{26}$$

P r o o f. Let $x, y \in X$, $B = X \setminus A$. By the associativity of the supremum on $[0,1]$ we may consider the following cases:
if $(x,y) \in A \times A$, then

$$(R \circledast S)(x,y) = \sup_{w \in X} R(x,w) * S(w,y)$$

$$= \max(\sup_{w \in A} R(x,w) * S(w,y), \sup_{w \in B} R(x,w) * S(w,y))$$

$$= \max(\sup_{w \in A} R_0(x,w) * S_0(w,y), \sup_{w \in B} 0 * 0) = \max((R_0 \circledast S_0)(x,y), 0) = (R_0 \circledast S_0)(x,y),$$

if $(x,y) \in A \times B$, then

$$(R \circledast S)(x,y) = \sup_{w \in X} R(x,w) * S(w,y)$$

$$= \max(\sup_{w \in A} R(x,w) * S(w,y), \sup_{w \in B} R(x,w) * S(w,y))$$

$$= \max(\sup_{w \in A} R_0(x,w) * 0, \sup_{w \in B} 0 * 0) = \max(0,0) = 0,$$

if $(x,y) \in B \times A$, then

$$(R \circledast S)(x,y) = \sup_{w \in X} R(x,w) * S(w,y)$$

$$= \max(\sup_{w \in A} R(x,w) * S(w,y), \sup_{w \in B} R(x,w) * S(w,y))$$

$$= \max(\sup_{w \in A} 0 * S_0(w,y), \sup_{w \in B} 0 * 0) = \max(0,0) = 0,$$

if $(x,y) \in B \times B$, then

$$(R \circledast S)(x,y) = \sup_{w \in X} R(x,w) * S(w,y) = \sup_{w \in X} (0 * 0) = 0.$$

As a result $R \circledast S$ is of the form (26). □

Next we present properties of fuzzy relations, which will be discussed in our further considerations.

Definition 6. (Zadeh [22]) Let $*\colon [0,1]^2 \to [0,1]$. Relation $R \in FR(X)$ is:

- reflexive, if $\underset{x\in X}{\forall} R(x,x) = 1,$ (27)
- irreflexive, if $\underset{x\in X}{\forall} R(x,x) = 0,$ (28)
- symmetric, if $\underset{x,y\in X}{\forall} R(x,y) = R(y,x),$ (29)
- asymmetric, if $\underset{x,y\in X}{\forall} \min(R(x,y), R(y,x)) = 0,$ (30)
- antisymmetric, if $\underset{x,y,x\neq y\in X}{\forall} \min(R(x,y), R(y,x)) = 0,$ (31)
- totally connected, if $\underset{x,y\in X}{\forall} \max(R(x,y), R(y,x)) = 1,$ (32)
- connected, if $\underset{x,y,x\neq y\in X}{\forall} \max(R(x,y), R(y,x)) = 1,$ (33)
- transitive, if $\underset{x,y,z\in X}{\forall} \min(R(x,y), R(y,z)) \leqslant R(x,z),$ (34)
- $*$-transitive, if $\underset{x,y,z\in X}{\forall} R(x,y) * R(y,z) \leqslant R(x,z).$ (35)

Lemma 4. A relation $R \in FR(X)$ is $*$-transitive iff

$$R \circledast R \leqslant R. \tag{36}$$

P r o o f. Let $R \in FR(X)$. Directly by (22) and (35)

$$R \circledast R \leqslant R \Leftrightarrow \underset{x,z\in X}{\forall} \sup_{y\in X} R(x,y) * R(y,z) \leqslant R(x,z)$$

$$\Leftrightarrow \underset{x,y,z\in X}{\forall} R(x,y) * R(y,z) \leqslant R(x,z),$$

which finishes the proof. □

4. PRESERVATION OF BASIC PROPERTIES

Now we turn to the main part of our considerations.

Definition 7. (Drewniak and Dudziak [8], Definition 4) Let $n \in \mathbb{N}$, $n \geqslant 2$, $R_1, \ldots, R_n \in FR(X)$. A property of fuzzy relations is preserved by an operation $F\colon [0,1]^n \to [0,1]$, if for fuzzy relations $R_1, \ldots, R_n$ having this property, the fuzzy relation $R_F \in FR(X)$,

$$R_F(x,y) = F(R_1(x,y), \ldots, R_n(x,y)), \quad x,y \in X$$

also has this property.

Example 7. Every property of fuzzy relations is preserved by the projections (4), because for $F = P_k$ we get $R_F = R_k$.

Many authors contributed to the problem of preservation of fuzzy relations properties by aggregation functions (e. g. [15, 20]). Our aim is to consider arbitrary operations on the interval $[0, 1]$ and weaken some conditions used for aggregation functions in the above mentioned works. We want to characterize all n-ary operations preserving fixed properties of fuzzy relations. We also recall some partial results obtained before for concrete examples of operations in [8].

Example 8. (Drewniak and Dudziak [8]) Preservation of basic properties of fuzzy relations by aggregation functions (cf. Definition 1).

Property \ Aggreg.	Arbitrary	Min	Arithm. (7)	Quasi-lin. (6)	Max
Reflexivity	+	+	+	+	+
Irreflexivity	+	+	+	+	+
Symmetry	+	+	+	+	+
Asymmetry	—	+	—	—	—
Antisymmetry	—	+	—	—	—
Connectedness	—	—	—	—	+
Total connectedness	—	—	—	—	+
Transitivity	—	+	—	—	—
$*$–Transitivity	—	+	—	—	—

In the above table symbol "+" means that the function from the chosen column preserves the property from the chosen row, and symbol "—" means that there exists a counter-example.

Now we will give characterizations of operations preserving properties (27) – (35). Some results were published before and this is why we recall only respective theorems without proofs.

Theorem 4. (Drewniak and Dudziak [9]) Let $R_1, \ldots, R_n \in FR(X)$ be reflexive (respectively irreflexive). The relation R_F is reflexive (respectively irreflexive), iff the function F satisfies the condition (37) (respectively (38)), where

$$F(1, \ldots, 1) = 1, \tag{37}$$

$$F(0, \ldots, 0) = 0. \tag{38}$$

Example 9. Any aggregation function and any idempotent function F,

$$F(t, \dots, t) = t, \qquad \text{for } t \in [0, 1], \tag{39}$$

fulfil the conditions (37) and (38).

Symmetry is the most stable property of fuzzy relations because we have

Theorem 5. (Drewniak and Dudziak [9]) Let $R_1, \dots, R_n \in FR(X)$ be symmetric. For every function F the fuzzy relation R_F is also symmetric.

Theorem 6. (Drewniak and Dudziak [9]) Let $\operatorname{card} X \geqslant 2$. Operation F preserves asymmetry (antisymmetry) iff it satisfies the condition

$$\underset{s,t\in[0,1]^n}{\forall} \left(\underset{1\leqslant k\leqslant n}{\forall} \min(s_k, t_k) = 0 \right) \Rightarrow \min\left(F(s), F(t)\right) = 0. \tag{40}$$

Now we give examples of operations preserving asymmetry and antisymmetry.

Example 10. The operation $F = \min$ preserves asymmetry (antisymmetry). A simple condition sufficient for (40) is connected with the zero element $z = 0$ of the operation F with respect to a certain co-ordinate:

$$\underset{1\leqslant k\leqslant n}{\exists} \ \underset{i\neq k}{\forall} \ \underset{t_i\in[0,1]}{\forall} F(t_1, \dots, t_{k-1}, 0, t_{k+1}, \dots, t_n) = 0.$$

In particular, the weighted geometric mean fulfils (40). As another example we may consider the median function (5). If function F fulfils the condition

$$\underset{t\in[0,1]^n}{\forall} \operatorname{card}\{k : t_k = 0\} > \frac{n}{2} \Rightarrow F(t) = 0, \tag{41}$$

then we also get (40) (e. g. the median fulfils (41)). However, the above condition is not necessary for (40), because it does not cover the projections.

Theorem 7. (Drewniak and Dudziak [9]) Let $\operatorname{card} X \geqslant 2$. Operation F preserves connectedness (total connectedness) iff it satisfies the condition

$$\underset{s,t\in[0,1]^n}{\forall} \left(\underset{1\leqslant k\leqslant n}{\forall} \max(s_k, t_k) = 1 \right) \Rightarrow \max\left(F(s), F(t)\right) = 1. \tag{42}$$

Example 11. Examples of functions fulfilling (42) are $F = \max$, $F = \operatorname{med}$ or operations F with zero element $z = 1$ with respect to a certain co-ordinate:

$$\underset{1\leqslant k\leqslant n}{\exists} \ \underset{i\neq k}{\forall} \ \underset{t_i\in[0,1]}{\forall} F(t_1, \dots, t_{k-1}, 1, t_{k+1}, \dots, t_n) = 1.$$

The dual property for (41) have the form

$$\underset{t\in[0,1]^n}{\forall} \quad \mathrm{card}\{k : t_k = 1\} > \frac{n}{2} \Rightarrow F(t) = 1. \tag{43}$$

Now, some facts useful for our further considerations of the preservation of transitivity property will be presented. In virtue of Theorem 3 we get

Lemma 5. Let $A \subset X$, $A \neq \emptyset$, $*\colon [0,1]^2 \to [0,1]$ and operation $*$ has zero element $z = 0$. If relation $R_0 \in FR(A)$ is $*$-transitive in A, then relation $R \in FR(X)$ described by (25) is $*$-transitive in X.

P r o o f. Let $(x, y) \in A \times A$. By Theorem 3, Lemma 4 and by $*$-transitivity of relation R_0 in A we get

$$(R \circledast R)(x, y) = (R_0 \circledast R_0)(x, y) \leqslant R_0(x, y) = R(x, y), \quad x, y \in A.$$

If $(x, y) \in X \times X \setminus A \times A$, then $(R \circledast R)(x, y) = 0 \leqslant R(x, y)$. As a result relation R is $*$-transitive in X. □

We will give theorems (Theorems 8, 9) providing a necessary and a sufficient condition for the preservation of $*$-transitivity (cf. also [20]).

Theorem 8. Let $\mathrm{card}\, X \geqslant 3$, and operation $*\colon [0,1]^2 \to [0,1]$ have zero element $z = 0$. If operation $F\colon [0,1]^n \to [0,1]$ preserves $*$-transitivity, then it dominates $*$ ($F \gg *$), it means that (cf. (13))

$$\underset{(s_1,\ldots,s_n),(t_1,\ldots,t_n)\in[0,1]^n}{\forall} F(s_1 * t_1, \ldots, s_n * t_n) \geqslant F(s_1, \ldots, s_n) * F(t_1, \ldots, t_n). \tag{44}$$

P r o o f. Let us fix $s_k, t_k \in [0,1]$, $k = 1, \ldots, n$ and take $u, v, w \in X$. By assumption that $\mathrm{card}\, X \geqslant 3$, we may write $u \neq v$, $v \neq w$, $u \neq w$. We also denote $A = \{u, v, w\}$. Firstly, we will create fuzzy relations S_k, $k = 1, \ldots, n$ in the set A described by the matrices (cf. Remark 1)

$$S_k = \begin{bmatrix} 0 & s_k & s_k * t_k \\ 0 & 0 & t_k \\ 0 & 0 & 0 \end{bmatrix}, \quad k = 1, \ldots, n.$$

In other words

$$S_k(u, v) = s_k, \qquad S_k(v, w) = t_k, \qquad S_k(u, w) = s_k * t_k. \tag{45}$$

By Lemma 4 relations S_k are $*$-transitive in A because

$$S_k \circledast S_k = \begin{bmatrix} 0 & 0 & s_k * t_k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \leqslant S_k, \quad k = 1, \ldots, n.$$

In virtue of Lemma 5 relations $R_k \in FR(X)$, $k = 1, \ldots, n$,

$$R_k(x,y) = \begin{cases} S_k(x,y), & x, y \in A \\ 0, & \text{otherwise} \end{cases} \tag{46}$$

are $*$-transitive in X. By assumption F preserves $*$-transitivity of fuzzy relations so relation $R_F = F(R_1, \ldots, R_n)$ is also $*$-transitive for $x, y, z \in X$

$$F(R_1(x,y), \ldots, R_n(x,y)) * F(R_1(y,z), \ldots, R_n(y,z)) \leqslant F(R_1(x,z), \ldots, R_n(x,z)).$$

In particular, for elements $u, v, w \in X$ the above inequality is also fulfilled. Thus applying the notations (45) and formula (46) we have

$$\begin{aligned} F(s_1, \ldots, s_n) * F(t_1, \ldots, t_n) &= F(R_1(u,v), \ldots, R_n(u,v)) * F(R_1(v,w), \ldots, R_n(v,w)) \\ &\leqslant F(R_1(u,w), \ldots, R_n(u,w)) = F(s_1 * t_1, \ldots, s_n * t_n), \end{aligned}$$

which means that operation F fulfils (44). □

Example 12. (Saminger and Mesiar [20], p. 30) Each quasi-linear mean from Example 2 fulfils (44) with the operation $* = T_D$. Moreover, for $n = 2$ arbitrary t-norm $F = T$ fulfils (44) with the operation $* = T_D$.

Theorem 9. Let $*\colon [0,1]^2 \to [0,1]$. If an operation $F\colon [0,1]^n \to [0,1]$, which is increasing in each of its arguments fulfils (44), then it preserves $*$-transitivity.

Proof. Let an increasing operation F fulfil (44), $x, y, z \in X$. If relations R_k are $*$-transitive for $k = 1, \ldots, n$, then $R_k(x,y) * R_k(y,z) \leqslant R_k(x,z)$. We will prove that the relation R_F is $*$-transitive. Applying the notations $R_k(x,y) = s_k$, $R_k(y,z) = t_k$, for $k = 1, \ldots, n$ we obtain

$$\begin{aligned} R_F(x,y) * R_F(y,z) &= F(R_1(x,y), \ldots, R_n(x,y)) * F(R_1(y,z), \ldots, R_n(y,z)) \\ &= F(s_1, \ldots, s_n) * F(t_1, \ldots, t_n) \leqslant F(s_1 * t_1, \ldots, s_n * t_n) \\ &= F(R_1(x,y) * R_1(y,z), \ldots, R_n(x,y) * R_n(y,z)) \\ &\leqslant F(R_1(x,z), \ldots, R_n(x,z)) = R_F(x,z). \end{aligned}$$

As a result operation F preserves $*$-transitivity. □

Directly from Theorems 8 and 9 one obtains the following result

Theorem 10. Let $\operatorname{card} X \geqslant 3$, $*\colon [0,1]^2 \to [0,1]$ be an operation with zero element $z = 0$. An increasing in each of its arguments operation $F\colon [0,1]^n \to [0,1]$ preserves $*$-transitivity iff it fulfils condition (44).

By Theorem 10 one obtains the following result

Corollary 1. (Saminger et al. [20], Theorem 3.1) Let $\operatorname{card} X \geqslant 3$, operation $*$ be a t-norm (cf. Definition 2). An aggregation function $F\colon [0,1]^n \to [0,1]$ (cf. Definition 1) preserves $*$-transitivity iff $F \gg *$ (cf. (44)).

By Theorems 1, 2 and Examples 4, 5 we see that the minimum, the weighted minimum (2) and the projections fulfil condition (44) for $* = \min$. The weighted geometric mean preserves T_P-transitivity, the weighted arithmetic mean preserves T_L-transitivity, the minimum preserves T-transitivity with arbitrary t-norm T. The operation F described by the formula (15) preserves T_L-transitivity. In particular, one obtains the following result

Corollary 2. Minimum, the weighted minimum and each function of the form (14) preserve transitivity.

For given $F\colon [0,1]^n \to [0,1]$ and increasing bijection $\varphi\colon [0,1] \to [0,1]$ we define an operation F_φ,

$$F_\varphi(t_1, \ldots, t_n) = \varphi^{-1} F(\varphi(t_1), \ldots, \varphi(t_n)),\ t_1, \ldots, t_n \in [0,1] \tag{47}$$

and say that F and F_φ are isomorphic to each other (an order isomorphism). Using Lemma 2. for such functions F_φ, we get new functions fulfilling the above conditions. The conditions (37), (38), (40), (42) are invariant with respect to all increasing bijections, i. e. with any function F fulfilling one of these conditions, also each function of the form (47) fulfils the respective condition (cf. [9]). Moreover, the condition (44) is invariant with respect to any increasing bijection $\varphi\colon [0,1] \to [0,1]$, for $* = \min$ (cf. [20], Proposition 4.2). However, the condition (44) may not be invariant with respect to arbitrary $\varphi\colon [0,1] \to [0,1]$ (cf. [9], Example 10).

5. PRESERVATION OF COMPOSED PROPERTIES

The properties from Definition 6 may be composed, which generates new classes of fuzzy relations. There are diverse kinds of such classifications (cf. [2], [6], [10], [13] and [15]) we will follow the classification of crisp relations (cf. [19]).

Definition 8. (Drewniak [6], p. 77) Let $*\colon [0,1]^2 \to [0,1]$. A relation $R \in FR(X)$ is called:

- *a tolerance*, if it is reflexive and symmetric,
- *a tournament*, if it is asymmetric and connected,

- *a $*$-equivalence*, if it is reflexive, symmetric and $*$-transitive,
- *a quasi-$*$-order*, if it is reflexive and $*$-transitive,
- *a partial $*$-order*, if it is reflexive, antisymmetric and $*$-transitive,
- *a linear quasi-$*$-order*, if it is a connected quasi-$*$-order,
- *a strict $*$-order*, if it is irreflexive and $*$-transitive,
- *a linear strict $*$-order*, if it is asymmetric, $*$-transitive and connected,
- *a linear $*$-order*, if it is a connected partial $*$-order.

In case of some composed properties we will deal with the characteristic functions of crisp relations only, which is motivated by the following statement (cf. [10], Theorem 4.15 in the case of linear order).

Theorem 11. If $R \in FR(X)$ fulfils one of the pairs of the listed conditions:

- antisymmetry, total connectedness,
- asymmetry, total connectedness,
- asymmetry, connecetedness

then $R(x, y) \in \{0, 1\}$ for $x, y \in X$.

P r o o f . Let $x, y \in X$. If a relation R is antisymmetric and totally connected, then if $x = y$ we see that $R(x, x) = 1$ (it follows from the fact that total connectedness (32) implies reflexivity (27)), if $x \neq y$ then by assumption of antisymmetry we have $\min(R(x, y), R(y, x)) = 0$ and by assumption of total connectedness we get $\max(R(x, y), R(y, x)) = 1$. As a result we see that $R(x, y) \in \{0, 1\}$. For the remain pairs of conditions the proof is similar. □

By Definition 8 and Theorem 11 one obtains the following result:

Corollary 3. Let $*\colon [0, 1]^2 \to [0, 1]$. If a relation $R \in FR(X)$ is a linear strict $*$-order, a linear $*$-order or a tournament, then $R(x, y) \in \{0, 1\}$ for $x, y \in X$.

During aggregation of characteristic functions of crisp relations we usually receive fuzzy relations. However, by Corollary 3 we have

Remark 2. If an operation $F\colon [0, 1]^n \to [0, 1]$ preserves a linear strict $*$-order, a linear $*$-order or a tournament, then the relation R_F is a characteristic function of a crisp relation.

Let $F\colon [0, 1]^n \to [0, 1]$ and $*\colon [0, 1]^2 \to [0, 1]$. Using theorems of the previous section we will give characterizations of operations preserving properties from Definition 8. Moreover, the respective examples will be recalled.

Corollary 4. Operation F preserves fuzzy tolerances iff $F(1,\dots,1) = 1$.

P r o o f . Operation F preserves a tolerance if it preserves reflexivity and symmetry. So by Theorems 4, 5 operation F preserves a tolerance if and only if $F(1,\dots,1) = 1$. □

Example 13. Every aggregation function, every idempotent operation, every operation with idempotent element 1, every t-norm and t-conorm preserve tolerance relations.

Corollary 5. Let $\operatorname{card} X \geqslant 2$. Operation F preserves fuzzy tournaments iff it fulfils conditions (40) and (42).

P r o o f . Operation F preserves a tournament if it preserves asymmetry and connectedness. So by Theorems 6 and 7 operation F preserves a tournament if and only if it fulfils conditions (40) and (42). □

Example 14. The median value and each operation F fulfilling (41) and (43) preserve fuzzy tournaments (these operations fulfil conditions (40) and (42)).

Corollary 6. Let $\operatorname{card} X \geqslant 3$, operation $*$ have zero element $z = 0$. A function F, which is increasing in each of its arguments preserves fuzzy $*$-equivalences (fuzzy quasi-$*$-orders) iff $F(1,\dots,1) = 1$ and $F \gg *$ (cf. (44)).

P r o o f . Operation F preserves a $*$-equivalence if it preserves reflexivity, symmetry and $*$-transitivity. Thus by Theorems 4, 5 and 10 operation F preserves fuzzy $*$-equivalences if and only if $F(1,\dots,1) = 1$ and $F \gg *$.

Operation F preserves a quasi-$*$-order if it preserves $*$-transitivity and reflexivity. So by Theorems 4 and 10 operation F preserves fuzzy quasi-$*$-orders if and only if $F(1,\dots,1) = 1$ and $F \gg *$. □

Corollary 7. (De Baets and Mesiar [4]) Let $\operatorname{card} X \geqslant 3$, $n = 2$, T and T^* be arbitrary t-norms. T preserves fuzzy T^*-equivalences iff $T \gg T^*$.

Example 15. The weighted arithmetic mean preserves fuzzy T_L-equivalences and T_P-equivalences (quasi-T_L-orders and quasi-T_P-orders).

Example 16. Minimum and the weighted minimum preserve fuzzy T-equivalences (quasi-T-orders), where T is an arbitrary t-norm.

Corollary 8. Let card $X \geqslant 3$, operation $*$ have zero element $z = 0$. A function F, which is increasing in each of its arguments preserves fuzzy linear quasi-$*$-orders iff $F \gg *$ and F fulfils condition (42).

P r o o f. Operation F preserves a linear quasi-$*$-order if it preserves connectedness, reflexivity and $*$-transitivity. So by Theorems $7, 10$ and by the fact that total connectedness (32) implies reflexivity (27) we see that operation F preserves fuzzy linear quasi-$*$-orders if and only if F fulfils condition (42) and $F \gg *$. □

Corollary 9. Let card $X \geqslant 3$, operation $*$ have zero element $z = 0$. A function F, which is increasing in each of its arguments preserves fuzzy partial $*$-orders iff $F \gg *$, $F(1, \ldots, 1) = 1$ and F fulfils condition (40).

P r o o f. Operation F preserves a partial $*$-order if it preserves reflexivity, antisymmetry and $*$-transitivity. So by Theorems $4, 6, 10$ operation F preserves fuzzy partial $*$-orders if and only if F fulfils condition (40), $F(1, \ldots, 1) = 1$ and $F \gg *$.□

Example 17. The weighted geometric mean preserves fuzzy partial T_P-orders.

Corollary 10. Let card $X \geqslant 3$, operation $*$ have zero element $z = 0$. A function F, which is increasing in each of its arguments preserves fuzzy linear $*$-orders (linear strict $*$-orders) iff $F \gg *$ and F fulfils conditions (40) and (42).

P r o o f. Operation F preserves a linear $*$-order if it preserves connectedness, reflexivity, antisymmetry and $*$-transitivity. So by Theorems $6, 7, 10$ and by the fact that total connectedness (32) implies reflexivity (27) we see that operation F preserves fuzzy linear $*$-orders if and only if F fulfils conditions (40), (42) and $F \gg *$.

Operation F preserves a linear strict $*$-order if it preserves connectedness, asymmetry and $*$-transitivity. So by Theorems $6, 7, 10$ operation F preserves fuzzy linear strict $*$-orders if and only if operation F fulfils conditions (40), (42) and $F \gg *$. □

Corollary 11. Let card $X \geqslant 3$, operation $*$ have zero element $z = 0$. A function F, which is increasing in each of its arguments preserves fuzzy strict $*$-orders iff $F \gg *$ and $F(0, \ldots, 0) = 0$.

P r o o f. Operation F preserves a strict $*$-order if it preserves $*$-transitivity and irreflexivity. So by Theorems 4 and 10 operation F preserves fuzzy strict $*$-orders if and only if F fulfils condition $F(0, \ldots, 0) = 0$ and $F \gg *$. □

Example 18. The weighted arithmetic mean preserves fuzzy strict T_L-orders.

6. CONCLUSION

In our considerations we look for new characterizations of transformations preserving fuzzy relations properties. We see that some non-monotonic transformations may be used for the preservation of fuzzy relations properties. Such transformations may not be good aggregation functions for applications, but from the theoretical point of view non-monotonic transformations may be taken into account. Besides, our conditions are useful in a verification of concrete aggregation functions.

ACKNOWLEDGEMENT

The authors are grateful to the referees for their valuable comments and suggestions, which helped to improve the final version of the paper.

(Received April 11, 2006.)

REFERENCES

[1] J. C. Bezdek and J. D. Harris: Fuzzy partitions and relations: an axiomatic basis for clustering. Fuzzy Sets and Systems *1* (1978), 111–127.
[2] U. Bodenhofer: A Similarity–Based Generalization of Fuzzy Orderings. PhD Thesis. Universitätsverlag Rudolf Trauner, Linz 1999.
[3] T. Calvo, A. Kolesárová, M. Komorníková, and R. Mesiar: Aggregation operators: properties, classes and construction methods. In: Aggregation Operators (T. Calvo et al., ed.), Physica–Verlag, Heildelberg 2002, pp. 3–104.
[4] B. De Baets and R. Mesiar: T-partitions. Fuzzy Sets and Systems *97* (1998), 211–223.
[5] F. Chiclana, F. Herrera, E. Herrera-Viedma, and L. Martínez: A note on the reciprocity in the aggregation of fuzzy preference relations using OWA oprators. Fuzzy Sets and Systems *137* (2003), 71–83.
[6] J. Drewniak: Fuzzy Relation Calculus. Silesian University, Katowice 1989.
[7] J. Drewniak and U. Dudziak: Safe transformations of fuzzy relations. In: Current Issues in Data and Knowledge Engineering (B. De Baets et al., ed.), EXIT, Warszawa 2004, pp. 195–203.
[8] J. Drewniak and U. Dudziak: Aggregations preserving classes of fuzzy relations. Kybernetika *41* (2005), 3, 265–284.
[9] J. Drewniak and U. Dudziak: Aggregations in classes of fuzzy relations. Ann. Acad. Paed. Cracoviensis 33, Studia Math. *5* (2006), 33–43.
[10] J. Fodor and M. Roubens: Fuzzy Preference Modelling and Multicriteria Decision Spport. Kluwer Academic Publishers, Dordrecht 1994.
[11] E. P. Klement, R. Mesiar, and E. Pap: Triangular Norms. Kluwer Academic Publishers, Dordrecht 2000.
[12] J. L. Marichal: On an axiomatization of the quasi-arithmetic mean values without the symmetry axiom. Aequationes Math. *59* (2000), 74–83.
[13] S. Ovchinnikov: Similarity relations, fuzzy partitions, and fuzzy orderings. Fuzzy Sets and Systems *40* (1991), 107–126.
[14] V. Peneva and I. Popchev: Aggregation of fuzzy relations. C. R. Acad. Bulgare Sci. *51* (1998), 9–10, 41–44.
[15] V. Peneva and I. Popchev: Properties of the aggregation operators related with fuzzy relations. Fuzzy Sets and Systems *139* (2003), 3, 615–633.

[16] V. Peneva and I. Popchev: Transformations by parameterized t-norms preserving the properties of fuzzy relations. C. R. Acad. Bulgare Sci. *57* (2004), 10, 9–18.
[17] V. Peneva and I. Popchev: Aggregation of fuzzy preference relations with different importance. C. R. Acad. Bulgare Sci. *58* (2005), 5, 499–506.
[18] V. Peneva and I. Popchev: Aggregation of fuzzy preference relations by composition. C. R. Acad. Bulgare Sci. *59* (2006), 4, 373–380.
[19] M. Roubens and P. Vincke: Preference Modelling. Springer–Verlag, Berlin 1985.
[20] S. Saminger, R. Mesiar, and U. Bodenhofer: Domination of aggregation operators and preservation of transitivity. Internat. J. Uncertainty, Fuzziness, Knowledge–Based Systems *10* (2002), 11–35.
[21] S. Saminger, K. Maes, and B. De Baets: Aggregation of T-transitive reciprocal relations. In: Proc. of AGOP'05, Lugano 2005, pp. 113–118.
[22] L. A. Zadeh: Similarity relations and fuzzy orderings. Inform. Sci. *3* (1971), 177–200.

Józef Drewniak and Urszula Dudziak, Institute of Mathematics, University of Rzeszów, Rejtana 16A PL-35–310 Rzeszów. Poland.
e-mails: jdrewnia@univ.rzeszow.pl, ududziak@univ.rzeszow.pl