WEAK LAW OF LARGE NUMBERS
FOR I.I.D. FUZZY RANDOM VARIABLES

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In this paper, weak laws of large numbers for sum of independent and identically distributed fuzzy random variables are obtained.

1. INTRODUCTION

The celebrated Feller weak law of large numbers provided a necessary and sufficient condition in the i.i.d. case in [1] as follows:

Theorem 1.1. (Feller) Let $\{X_n\}$ be i.i.d. random variables and $S_n = \sum_{i=1}^{n} X_i$. Suppose that $\mathrm{E}X_1 I_{[|X_1| \leq n]} \to u$ as $n \to \infty$. Then $S_n/n \to u$ in probability as $n \to \infty$ if and only if $n\Pr\{|X_1| > n\} \to 0$ as $n \to \infty$.

Let $a > 0$. A positive measurable function f on $[a, \infty)$ varies regularly at infinity with exponent ρ, $\infty < \rho < \infty$, denote $f \in RV(\rho)$, if and only if

$$\frac{f(tx)}{f(t)} \to x^\rho \quad \text{as} \quad t \to \infty \quad \text{for all } x > 0.$$

If $\rho = 0$ the function is slowly varying at infinity; $f \in SV$.

Recently, Gut [3] provided the following more general Feller law concerning regularly and slowly varying functions.

Theorem 1.2. (Gut) Let $\{X_n\}$ be i.i.d. random variables and $S_n = \sum_{i=1}^{n} X_i$. Further, let, for $x > 0$, $b \in RV(1/\rho)$, for some $\rho \in (0, 1]$, that is, let $b(x) = x^{1/\rho} l(x)$, where $l \in SV$. Set $b_n = b(n)$. Then $(1/b_n)(S_n - n\mathrm{E}X_1 I_{[|X_1| \leq b_n]}) \to 0$ in probability as $n \to \infty$ if and only if $n\Pr\{|X_1| > b_n\} \to 0$ as $n \to \infty$.

The author should simply think c (constant), $\log x$ or $\log \log x$ each time the function $l(x)$ appears.

The concept of a fuzzy random variable was introduced as a natural generalization of random sets in order to represent relationships between the outcomes of a random experiment and inexact data. By inexactness here we mean non-statistical inexactness due to the subjectivity and imprecision of human knowledge. Limit theorems for sums of independent fuzzy random variables have received much attentions because of its usefulness in several applied fields. This paper concerns with weak law of large numbers which is the one of limit theorems. Many authors have studied the strong laws of large numbers for sums of independent fuzzy random variables. For example, see Hong and Kim [4], Hong [5], Inoue [6], Joo and Kim [8], Kim [10], Klement et al. [11], Kruse [12], Miyakoshi and Shimbo [13], Uemura [16] and so on [7, 14]. On the other hand, weak laws of large numbers for fuzzy random variables have been studied only by Joo [9] and Taylor et al. [15]. The purpose of this paper is to obtain Feller, Gut [3] weak law of large numbers for sums of independent and identically distributed (i.i.d.) fuzzy random variables.

2. PRELIMINARIES

In this section, we describe some basic concepts of fuzzy numbers. Let R denote the real line. A fuzzy number is a fuzzy set $\tilde{u} : R \longrightarrow [0, 1]$ with the following properties;

1. $\tilde{u}$ is normal, i. e., there exists $x \in R$ such that $\tilde{u}(x) = 1$.

2. $\tilde{u}$ is upper semicontinuous.

3. $\operatorname{supp} \tilde{u} = cl\{x \in R | \tilde{u}(x) > 0\}$ is compact.

4. $\tilde{u}$ is a convex fuzzy set, i. e., $\tilde{u}(\lambda x + (1 - \lambda)y) \geq \min(\tilde{u}(x), \tilde{u}(y))$ for $x, y \in R$ and $\lambda \in [0, 1]$.

We denote the family of all fuzzy numbers by $F(R)$. For a fuzzy set $\tilde{u}$, the α-level set of $\tilde{u}$ is defined by

$$L_\alpha \tilde{u} = \begin{cases} \{x | \tilde{u}(x) \geq \alpha\}, & 0 < \alpha \leq 1, \\ \operatorname{supp} \tilde{u}, \alpha = 0. \end{cases} \tag{1}$$

Then it follows that $\tilde{u}$ is a fuzzy number if and only if $L_1 \tilde{u} \neq \phi$ and $L_\alpha \tilde{u}$ is a closed bounded interval for each $\alpha \in [0, 1]$. Form this characterization of fuzzy numbers, a fuzzy number, $\tilde{u}$ is completely determined by the endpoints of the intervals $L_\alpha \tilde{u} = [u_\alpha^1, u_\alpha^2]$.

Theorem 2.1. (Goetschel and Voxman [2]) For $\tilde{u} \in F(R)$, denote $u^1(\alpha) = u_\alpha^1$ and $u^2(\alpha) = u_\alpha^2$ by considering them as functions of $\alpha \in [0, 1]$. Then the following hold:

1. u^1 is a bounded non-decreasing function on [0,1].

2. u^2 is a bounded non-increasing function on [0,1].

3. $u^1(1) \leq u^2(1)$.

4. u^1 and u^2 are left continuous on [0,1] and right continuous at 0.

5. If v^1 and v^2 satisfy above $(1)-(4)$, then there exists a unique $\tilde{v} \in F(R)$ such that $v_\alpha^1 = v^1(\alpha)$, $v_\alpha^2 = v^2(\alpha)$, for all $\alpha \in [0, 1]$.

The above theorem implies that we can identify a fuzzy number $\tilde{u}$ with the parameterized representation $\{(u_\alpha^1, u_\alpha^2) \mid 0 \le \alpha \le 1\}$, where u^1 and u^2 satisfy $(1) - (4)$ of Theorem 2.1. Suppose now that $\tilde{u}, \tilde{v} \in F(R)$ are fuzzy numbers whose representations are $\tilde{u} = \{(u_\alpha^1, u_\alpha^2) \mid 0 \le \alpha \le 1\}$ and $\tilde{v} = \{(v_\alpha^1, v_\alpha^2) \mid 0 \le \alpha \le 1\}$, respectively. If we define

$$(\tilde{u} + \tilde{v})(z) = \sup_{x+y=z} \min(\tilde{u}(x), \tilde{v}(y)),$$

$$(\lambda\tilde{u})(z) = \begin{cases} \tilde{u}(z/\lambda), & \lambda \ne 0, \\ \tilde{0}, & \lambda = 0, \end{cases}$$

where $\tilde{0} = I_{\{0\}}$ is the indicator function of $\{0\}$, then

$$\tilde{u} + \tilde{v} = \{(u_\alpha^1 + v_\alpha^1 \, , \, u_\alpha^2 + v_\alpha^2) \mid 0 \le \alpha \le 1\}$$
$$\lambda\tilde{u} = \{(\lambda u_\alpha^1, \lambda u_\alpha^2) \mid 0 \le \alpha \le 1\} \text{ for } \lambda > 0.$$

Now, we define the metric d_∞ on $F(R)$ by

$$d_\infty(\tilde{u}, \tilde{v}) = \sup_{0 \le \alpha \le 1} d_H(L_\alpha\tilde{u}, L_\alpha\tilde{v}),$$

where d_H denotes Hausdorff metric on closed subsets of R which admits in our particular case the form

$$d_H(L_\alpha\tilde{u}, L_\alpha\tilde{v}) = \max(|u_\alpha^1 - v_\alpha^1|, |u_\alpha^2 - v_\alpha^2|).$$

Also, the norm $\|\tilde{u}\|$ of fuzzy number $\tilde{u}$ will be defined as

$$\|\tilde{u}\| = d_\infty(\tilde{u}, \tilde{0}) = \max(|u_0^1|, |u_0^2|).$$

Let $(\Omega, \mathcal{A}, P)$ denotes a complete probability space. For a fuzzy number valued function $\tilde{X} : \Omega \to F(R)$ and a subset B of R, $\tilde{X}^{-1}(B)$ denotes the fuzzy subset of Ω defined by

$$\tilde{X}^{-1}(B)(\omega) = \sup_{x \in B} \tilde{X}(\omega)(x)$$

for every $\omega \in \Omega$. The function $\tilde{X} : \Omega \to F(R)$ is called a fuzzy random variables if for every closed subset B of R, the fuzzy set $\tilde{X}^{-1}(B)$ is measurable when considered as a function from Ω to $[0, 1]$. If we denote $\tilde{X}(\omega) = \{(X_\alpha^1(\omega), X_\alpha^2(\omega) | 0 \le \alpha \le 1\}$, then it is well known that $\tilde{X}$ is a fuzzy random variable if and only if for each $\alpha \in [0, 1]$, X_α^1 and X_α^2 are random variables in the usual sense.

A fuzzy random variable $\tilde{X}(\omega) = \{(X_\alpha^1(\omega), X_\alpha^2(\omega) | 0 \le \alpha \le 1\}$, is called integrable if for each $\alpha \in [0, 1]$, X_α^1 and X_α^2 are integrable, equivalently, $\int \|\tilde{X}\| \, dP < \infty$. In this case, the expectation of $\tilde{X}$ is the fuzzy number $E\tilde{X}$ defined by

$$E\tilde{X} = \int \tilde{X} \, dP = \left\{ \left(\int X_\alpha^1 \, dP, \int X_\alpha^2 \, dP \right) \Big| 0 \le \alpha \le 1 \right\}.$$

Let $\{\tilde{X}_n\}$ be a sequence of fuzzy random variables. For $\alpha \in [0, 1]$ $\{X_{n\alpha}^1\}$, $\{X_{n\alpha}^2\}$ are sequences of independent and identically distributed random variables. Then $\{\tilde{X}_n\}$ is called a sequence of independent and identically distributed fuzzy random variables.

3. MAIN RESULTS

We define $\tilde{X}I_{[\|\tilde{X}\|\leq t]}$ as

$$\tilde{X}I_{[\|\tilde{X}\|\leq t]}(\omega) = \begin{cases} \tilde{X}(\omega) & \text{if}\|\tilde{X}(\omega)\| \leq t, \\ I_{\{0\}} & \text{otherwise.} \end{cases}$$

Theorem 3.1. Let $\{\tilde{X}_n\}$ be a sequence of independent and identically distributed fuzzy random variables and $\tilde{S}_n = \sum_{i=1}^n \tilde{X}_i$. Suppose that for a fuzzy number $\tilde{u} \in F(R)$, $d_\infty(\mathrm{E}\tilde{X}_1 I_{[\|\tilde{X}_1\|\leq n]}, \tilde{u}) \to 0$, then

$$d_\infty\left(\frac{\tilde{S}_n}{n}, \tilde{u}\right) \to 0 \ \text{ in probability} \tag{2}$$

iff
$$nP\{\|\tilde{X}_1\| > n\} \to 0. \tag{3}$$

To prove our main result, we need some lemmas.

Lemma 3.2. (Kim [10]) For each $\tilde{u} \in F(R)$ and each $\varepsilon > 0$, there exists a partition $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_r = 1$ of $[0,1]$ such that

$$\max(u_{\alpha_k}^1 - u_{\alpha_{k-1}^+}^1, u_{\alpha_{k-1}^+}^2 - u_{\alpha_k}^2) < \varepsilon, \quad k = 1, 2, \ldots, r,$$

where $u_{\alpha^+}^i$ denotes the right-hand limit of u^1 at α, $i = 1, 2$.

Lemma 3.3. (Hong [5]) For a sequence $\{\tilde{u}_n\} \in F(R)$ and a continuous $\tilde{u} \in F(R)$, suppose that for each $\alpha \in [0,1]$, $d_H(L_\alpha \tilde{u}_n, L_\alpha \tilde{u}) \to 0$. Then we have $d_\infty(\tilde{u}_n, \tilde{u}) \to 0$.

Lemma 3.4. Let $\{\tilde{X}_n\}$ be a sequence of independent and identically distributed fuzzy random variables. Let $\tilde{X}'_{jn} = \tilde{X}_j I_{[\|\tilde{X}_j\|\leq n]}$ and let $S'_n = \sum_{j=1}^n X'_{jn}$. If $nP\{\|\tilde{X}_1\| > n\} \to 0$, then for each $\alpha \in [0,1]$, $(1/n)((\tilde{S}'_n)_\alpha^1 - (\mathrm{E}\tilde{S}'_n)_\alpha^1) \to 0$ in probability.

P r o o f. By the assumption, for $\alpha \in [0,1]$,

$$\mathrm{E}\left(\left(\frac{\tilde{S}'_n}{n}\right)_\alpha^1 - \left(\frac{\mathrm{E}\tilde{S}'_n}{n}\right)_\alpha^1\right)^2 = \frac{1}{n^2}\sum_{j=1}^n \mathrm{E}((X'_{jn})_\alpha^1 - \mathrm{E}(X'_{jn})_\alpha^1)^2$$

$$\leq \frac{1}{n^2}\sum_{j=1}^n \mathrm{E}((X'_{jn})_\alpha^1)^2$$

$$= \frac{1}{n}\mathrm{E}((X'_{1n})_\alpha^1)^2$$

$$= \frac{1}{n}\sum_{j=1}^n \int_{[j-1<\|\tilde{X}_1\|\leq j]} ((X'_{1n})_\alpha^1)^2$$

$$\leq \frac{1}{n} \sum_{j=1}^{n} \int_{[j-1 < \|\tilde{X}_1\| \leq j]} \|\tilde{X}_1\|^2$$

$$\leq \frac{1}{n} \sum_{j=1}^{n} j^2 \Big[P\{\|\tilde{X}_1\| > j - 1\} - P\{\|\tilde{X}_1\| > j\} \Big]$$

$$= \frac{1}{n} \Big[P\{\|\tilde{X}_1\| > 0\} - n^2 P\{\|\tilde{X}_1\| > n\}$$

$$+ \sum_{j=1}^{n-1} ((j+1)^2 - j^2) P\{\|\tilde{X}_1\| > j\}]$$

$$\leq \frac{3}{n} \left[1 + \sum_{j=1}^{n-1} j P\{\|\tilde{X}_1\| > j\} \right]$$

$$= o(1),$$

which completes the proof. $\qquad\square$

Proof of Theorem 3.1. ($\Longleftarrow$) Set $\tilde{X}'_{jn} = \tilde{X}_j I_{[\|\tilde{X}_j\| \leq n]}$ for $1 \leq j \leq n$ and $\tilde{S}'_n = \sum_{j=1}^{n} \tilde{X}'_{jn}$. Then, for each $n \geq 2$, $\{\tilde{X}'_{jn}, 1 \leq j \leq n\}$ are i.i.d. and for $\varepsilon > 0$,

$$P\{d_\infty(\tilde{S}_n/n, \tilde{S}'_n/n) > \varepsilon\} \leq P\{\tilde{S}_n \neq \tilde{S}'_n\} = P\{\cup_{j=1}^{n} [\tilde{X}_j \neq \tilde{X}'_{jn}]\} \leq n P\{\|X_1\| > n\},$$

so that (2) entails

$$d_\infty(\tilde{S}_n/n, \tilde{S}'_n/n) \to 0 \quad \text{in probability.} \tag{4}$$

Next, we verify that $d_\infty(\tilde{S}'_n/n, \mathrm{E}\tilde{S}'_n/n) = d_\infty(\tilde{S}'_n/n, \mathrm{E}\tilde{X}_1 I_{[\|\tilde{X}_1\| \leq n]}) \to 0$ in probability. By Lemma 3.2, there exists a partition $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_k = 1$ satisfying $|u^1_{\alpha_i} - u^1_{\alpha^+_{i-1}}| < \varepsilon/4$ for all i, and since $(\mathrm{E}\tilde{X}_1 I_{[\|\tilde{X}_1\| \leq n]})^1_\alpha$ converges to $u^1(\alpha)$ uniformly with respect to α, there exist $N_1 > 0$ such that for any $n \geq N_1$, $|(\mathrm{E}\tilde{X}_1 I_{[\|\tilde{X}_1\| \leq n]})^1_\alpha - u^1_\alpha| < \varepsilon/4$ for all α. Now, since $n P\{|X^1_{1\alpha}| > n\} \leq n P\{\|X^1_1\| > n\} \to 0$, by Lemma 3.4 there exist $N_2 > 0$ such that for any $n \geq N_2$, $P\{|(\frac{\tilde{S}'_n}{n})^1_{\alpha_i} - (\frac{\mathrm{E}\tilde{S}'_n}{n})^1_{\alpha_i}| > \frac{\varepsilon}{4}\} \leq \varepsilon/k$ for all $i = 1, 2, \ldots, k$. Here, we note that, for $n \geq \max\{N_1, N_2\} = N$, if $\sup_i |(\tilde{S}'_n/n)^1_{\alpha_i} - (\mathrm{E}\tilde{X}_1 I_{[\|\tilde{X}_1\| \leq n]})^1_{\alpha_i}| \leq \varepsilon/4$, then $\sup_\alpha |(\tilde{S}'_n/n)^1_\alpha - (\mathrm{E}\tilde{X}_1 I_{[\|\tilde{X}_1\| \leq n]})^1_\alpha| \leq \varepsilon$. It is because for $\alpha \in (\alpha_{i-1}, \alpha_i]$

$$(\tilde{S}'_n/n)^1_\alpha - \Big(\mathrm{E}\tilde{X}_1 I_{[\|\tilde{X}_1\| \leq n]}\Big)^1_\alpha \leq (\tilde{S}'_n/n)^1_{\alpha_i} - \Big(\mathrm{E}\tilde{X}_1 I_{[\|\tilde{X}_1\| \leq n]}\Big)^1_{\alpha^+_{i-1}}$$

$$\leq \Big(\mathrm{E}\tilde{X}_1 I_{[\|\tilde{X}_1\| \leq n]}\Big)^1_{\alpha_i} - \Big(\mathrm{E}\tilde{X}_1 I_{[\|\tilde{X}_1\| \leq n]}\Big)^1_{\alpha^+_{i-1}} + \frac{\varepsilon}{4}$$

$$\leq u^1(\alpha_i) - u^1(\alpha^+_{i-1}) + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4}$$

$$= \varepsilon,$$

and similarly we have

$$\Big(\mathrm{E}\tilde{X}_1 I_{[\|\tilde{X}_1\| \leq n]}\Big)^1_\alpha - \Big(\frac{\tilde{S}'_n}{n}\Big)^1_\alpha \leq \varepsilon,$$

and hence it follows. Now, for $n \geq N$

$$P \left\{ \sup_{\alpha} \left| \left(\frac{\tilde{S}'_n}{n} \right)^1_{\alpha} - \left(\frac{\mathrm{E}\tilde{S}'_n}{n} \right)^1_{\alpha} \right| > \varepsilon \right\}$$

$$\leq \ P \left\{ \sup_{i} \left| \left(\frac{\tilde{S}'_n}{n} \right)^1_{\alpha_i} - \left(\frac{\mathrm{E}\tilde{S}'_n}{n} \right)^1_{\alpha_i} \right| > \frac{\varepsilon}{4} \right\}$$

$$\leq \ \sum_{i=1}^{k} P \left\{ \left| \left(\frac{\tilde{S}'_n}{n} \right)^1_{\alpha_i} - \left(\frac{\mathrm{E}\tilde{S}'_n}{n} \right)^1_{\alpha_i} \right| > \frac{\varepsilon}{4} \right\}$$

$$\leq \ k\frac{\varepsilon}{k} = \varepsilon.$$

Hence, we proved $\sup_{\alpha} |(\tilde{S}'_n/n)^1_{\alpha} - (\mathrm{E}\tilde{X}_1 I_{[\|\tilde{X}_1\| \leq n]})^1_{\alpha}| \to 0$ in probability. Similarly, it can be proved $\sup_{\alpha} |(\tilde{S}'_n/n)^2_{\alpha} - (\mathrm{E}X_1 I_{[\|X_1\| \leq n]})^2_{\alpha}| \to 0$ in probability. Therefore, we obtain

$$d_{\infty}(\tilde{S}'_n/n, \mathrm{E}\tilde{S}'_n/n) \to 0 \ \text{ in probability.} \tag{5}$$

Now by (3) and (4), we have that

$$P \left\{ d_{\infty} \left(\frac{\tilde{S}_n}{n}, \tilde{u} \right) > \varepsilon \right\}$$

$$\leq P \left\{ d_{\infty} \left(\frac{\tilde{S}_n}{n}, \frac{\tilde{S}'_n}{n} \right) + d_{\infty} \left(\frac{\tilde{S}'_n}{n}, \frac{\mathrm{E}\tilde{S}'_n}{n} \right) + d_{\infty} \left(\frac{\mathrm{E}\tilde{S}'_n}{n}, \tilde{u} \right) > \varepsilon \right\}$$

$$\leq P \left\{ d_{\infty} \left(\frac{\tilde{S}_n}{n}, \frac{\tilde{S}'_n}{n} \right) > \frac{\varepsilon}{3} \right\} + P \left\{ d_{\infty} \left(\frac{\tilde{S}'_n}{n}, \frac{\mathrm{E}\tilde{S}'_n}{n} \right) > \frac{\varepsilon}{3} \right\} + P \left\{ d_{\infty} \left(\frac{\mathrm{E}\tilde{S}'_n}{n}, \tilde{u} \right) > \frac{\varepsilon}{3} \right\} \to 0$$

which completes the proof. $\square$

($\Longrightarrow$) If (1) holds, then $S^1_{n0}/n \to u^1_0$ in probability and $S^2_{n0}/n \to u^2_0$ in probability. Then the classical WLLN (see [1]) implies

$$nP\{|X^1_{10}| > n\} \to 0 \ \text{ and } \ nP\{|X^2_{10}| > n\} \to 0.$$

Therefore,

$$nP\{\|\tilde{X}_1\| > n\} = nP\{\max(|X^1_{10}|, |X^2_{10}|) > n\}$$

$$\leq \ n(P\{|X^1_{10}| > n\} + P\{|X^2_{10}| > n\}) \to 0.$$

Theorem 3.5. Let $\{\tilde{X}_n\}$ be a sequence of independent and identically distributed fuzzy random variables and $\tilde{S}_n = \sum_{i=1}^{n} \tilde{X}_i$. Let $b(x) = x^{1/\rho}l(x)$ for some $\rho \in (0, 1]$, where $l \in SV$ and $\lim_{x \to \infty} l(x) = \infty$ for $\rho = 1$. Set $b_n = b(n)$. Suppose that $(1/b_n) d_{\infty}(n\mathrm{E}\tilde{X}_1 I_{[\|\tilde{X}_1\| \leq b_n]}, \tilde{0}) \to 0$ as $n \to \infty$ then

$$(1/b_n) d_{\infty}(\tilde{S}_n, \tilde{0}) \to 0 \ \text{ in probability} \ \text{ as } \ n \to \infty \tag{6}$$

iff

$$nP\left\{\|\tilde{X}_1\| > b_n\right\} \to 0 \quad \text{as} \quad n \to \infty. \tag{7}$$

To prove above result, we need the following lemma.

Lemma 3.6. (Gut [3]) Let $\{X_n\}$ be a sequence of i.i.d. random variables. Let $X_j' = X_j I_{[|X_j| \leq b_n]}$ for $1 \leq j \leq n$ and let $S_n' = \sum_{j=1}^n X_j'$. Let $b(x) = x^{1/\rho} l(x)$ for some $\rho \in (0, 1]$, where $l \in SV$ and set $b_n = b(n)$. If $nP\{|X_1| > b_n\} \to 0$ as $n \to \infty$, then $\frac{1}{b_n}(S_n' - ES_n') \to 0$ in probability as $n \to \infty$.

Proof of Theorem 3.5. ($\Longleftarrow$) Set $\tilde{X}_j' = \tilde{X}_j I_{[\|\tilde{X}_j\| \leq b_n]}$ for $1 \leq j \leq n$ and $\tilde{S}_n' = \sum_{j=1}^n \tilde{X}_j'$. Then, for each $n \geq 2$, $\{\tilde{X}_j', 1 \leq j \leq n\}$ are i.i.d. and for $\varepsilon > 0$,

$$P\{d_\infty(\tilde{S}_n/n, \tilde{S}_n'/n) > \varepsilon\} \leq P\{\tilde{S}_n \neq \tilde{S}_n'\} = P\{\cup_{j=1}^n [\tilde{X}_j \neq \tilde{X}_j']\} \leq nP\{\|X_1\| > b_n\},$$

so that (6) entails

$$(1/b_n)\, d_\infty(\tilde{S}_n, \tilde{S}_n') \to 0 \quad \text{in probability as } n \to \infty. \tag{8}$$

Next, we verify that $(1/b_n)\, d_\infty(\tilde{S}_n', \tilde{0}) \to 0$ in probability as $n \to \infty$. We note that

$$
\begin{aligned}
d_\infty(\tilde{S}_n', \tilde{0}) &\leq |(\tilde{S}_n')_0^1| + |(\tilde{S}_n')_0^2| \\
&\leq \left[|(\tilde{S}_n')_0^1 - (nE\tilde{X}_1 I_{[\|\tilde{X}_1\| \leq b_n]})_0^1| + d_\infty(nE\tilde{X}_1 I_{[\|\tilde{X}_1\| \leq b_n]}, \tilde{0})\right] \\
&\quad + \left[|(\tilde{S}_n')_0^2 - (nE\tilde{X}_1 I_{[\|\tilde{X}_1\| \leq b_n]})_0^2| + d_\infty(nE\tilde{X}_1 I_{[\|\tilde{X}_1\| \leq b_n]}, \tilde{0})\right] \\
&\leq 2\left[d_H(L_0\tilde{S}_n', L_0(nE\tilde{X} I_{[\|\tilde{X}_1\| \leq b_n]})) + d_\infty(nE\tilde{X}_1 I_{[\|\tilde{X}_1\| \leq b_n]}, \tilde{0})\right].
\end{aligned}
$$

Therefore, by Lemma 3.6 and the assumption, we obtain

$$(1/b_n)\, d_\infty(\tilde{S}_n', \tilde{0}) \to 0 \quad \text{in probability as } n \to \infty. \tag{9}$$

Now by (7), and (8), we have that

$$
\begin{aligned}
&P\left\{(1/b_n)d_\infty(\tilde{S}_n, \tilde{0}) > \varepsilon\right\} \\
\leq\ &P\left\{(1/b_n)\, d_\infty(\tilde{S}_n, \tilde{S}_n') + (1/b_n)\, d_\infty(\tilde{S}_n', \tilde{0}) > \varepsilon\right\} \\
\leq\ &P\left\{(1/b_n)\, d_\infty(\tilde{S}_n, \tilde{S}_n') > \frac{\varepsilon}{2}\right\} + P\left\{(1/b_n)\, d_\infty(\tilde{S}_n', \tilde{0}) > \frac{\varepsilon}{2}\right\} \to 0.
\end{aligned}
$$

($\Longrightarrow$) If (5) holds, then $S_{n0}^1/b_n \to 0$ in probability and $S_{n0}^2/b_n \to 0$ in probability. Then

$$(1/b_n)(S_{n1}^i - nEX_{10}^i I_{[|X_{10}^i| \leq b_n]}) \to 0, \quad i = 1, 2$$

in probability as $n \to \infty$, since

$$EX_{10}^i I_{[|X_{10}^i| \leq b_n]} \leq d_\infty E(\tilde{X}_1 I_{[\|\tilde{X}_1\| \leq b_n]}, \tilde{0}), \quad i = 1, 2.$$

Then Theorem 1.2 implies

$$nP\{|X_{10}^1| > b_n\} \to 0 \text{ and } nP\{|X_{10}^2| > b_n\} \to 0.$$

Therefore,

$$nP\left\{\|\tilde{X}_1\| > b_n\right\} = nP\left\{\max(|X_{10}^1|, |X_{10}^2|) > b_n\right\}$$

$$\leq n\left(P\left\{|X_{10}^1| > b_n\right\} + P\left\{|X_{10}^2| > b_n\right\}\right) \to 0$$

as $n \to \infty$, which completes the proof.   □

Remark 3.7. In Theorem 3.5, if $d_\infty(\mathrm{E}\tilde{X}_1 I_{[\|\tilde{X}_1\| \leq b_n]}, \tilde{u}) \to 0$ for a fuzzy number $\tilde{u} \in F(R)$ or $\|\mathrm{E}\tilde{X}_1 I_{[\|\tilde{X}_1\| \leq b_n]}\|$, $n = 1, 2, \ldots$, are bounded then the assumption $(1/b_n) d_\infty(n\mathrm{E}\tilde{X}_1 I_{[\|\tilde{X}_1\| \leq b_n]}, \tilde{0}) \to 0$ as $n \to \infty$ holds since $n/b_n \to 0$ as $n \to \infty$.

Example 3.8. Let $\tilde{u} \in F(R)$ be fixed and let $\{Y_n\}$ be a sequence of i.i.d. random variables with common density

$$f(x) = \begin{cases} \frac{c}{x^2 \log |x|} & \text{for} |x| \geq 2, \\ 0 & \text{otherwise,} \end{cases}$$

where c is a normalizing constants. We define $(\tilde{X}_n(\omega))(x) = \tilde{u}(x - Y_n(\omega))$, i.e., $\tilde{X}_n(\omega)$ is the translation of $\tilde{u}$ by $Y_n(\omega)$ in the real axis. Then

$$\tilde{X}_{n\alpha}^1(\omega) = u_\alpha^1 + Y_n(\omega) \text{ and } \tilde{X}_{n\alpha}^2(\omega) = u_\alpha^2 + Y_n(\omega).$$

We note that

$$nP\{\|\tilde{X}_1\| > n\} = nP\{\max(|X_{10}^1|, |X_{10}^2|) > n\}$$

$$\leq n(P\{|u_0^1 + Y_1| > n\} + P\{|u_0^2 + Y_1| > n\}) \to 0$$

and

$$d_H(L_\alpha \mathrm{E}\tilde{X}_1 I_{[\|\tilde{X}_1\| \leq n]}, L_\alpha \tilde{u})$$

$$= \max\left(\left|(\mathrm{E}\tilde{X}_1 I_{[\|\tilde{X}_1\| \leq n]})_\alpha^1 - u_\alpha^1\right|, \left|(\mathrm{E}\tilde{X}_1 I_{[\|\tilde{X}_1\| \leq n]})_\alpha^2 - u_\alpha^2\right|\right)$$

$$= \left|\int_{[\|\tilde{X}_1\| \leq n]} (u_\alpha^1 + Y_1) \, \mathrm{d}P - u_\alpha^1\right| + \left|\int_{[\|\tilde{X}_1\| \leq n]} (u_\alpha^2 + Y_1) \, \mathrm{d}P - u_\alpha^2\right|$$

$$\leq 2\left|\int_{[\|\tilde{X}_1\| \leq n]} Y_1 \, \mathrm{d}P\right| + \|\tilde{u}\| P\{\|\tilde{X}_1\| > n\}$$

$$\leq 2\int_{[n-u_0^1, n+u_0^1]} |Y_1| \, \mathrm{d}P + 2\int_{[n-u_0^2, n+u_0^2]} |Y_1| \, \mathrm{d}P + \|\tilde{u}\| P\{\|\tilde{X}_1\| > n\}$$

$$\leq 4\int_{[n-\|\tilde{u}\|, n+\|\tilde{u}\|]} |Y_1| \, \mathrm{d}P + \|\tilde{u}\| P\{\|\tilde{X}_1\| > n\}$$

$$\leq 4(n + \|\tilde{u}\|) P\{|Y_1| > n - \|\tilde{u}\|\}\} + \|\tilde{u}\| P\{\|\tilde{X}_1\| > n\}$$

$$\leq 4(n - \|\tilde{u}\|) P\{|Y_1| > n - \|\tilde{u}\|\} + 8\|\tilde{u}\| P\{\|\tilde{X}_1\| > n - \|\tilde{u}\|\} + \|\tilde{u}\| P\{\|\tilde{X}_1\| > n\}$$

$$\to 0.$$

Then we have

$$d_\infty\left(\frac{\tilde{S}_n}{n}, \tilde{u}\right) \to 0 \quad \text{in probability}$$

by the criteria in Theorem 3.1.

Example 3.9. Let $\rho \in (0,1]$, and suppose that $\{Y_n\}$ is a sequence of i.i.d. random variables with common density

$$f(x) = \begin{cases} \frac{\rho}{2|x|^{1+\rho}} & \text{for} |x| \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $nP\{|Y_1| > (n \log n)^{1/\rho}\} \to 0$ as $n \to 0$ (Example 1.3 [3]). We define $(\tilde{X}_n(\omega))(x) = \tilde{u}(x - Y_n(\omega))$, i.e., $\tilde{X}_n(\omega)$ is the translation of $\tilde{u}$ by $Y_n(\omega)$ in the real axis. Then

$$\tilde{X}_{n\alpha}^1(\omega) = u_\alpha^1 + Y_n(\omega) \quad \text{and} \quad \tilde{X}_{n\alpha}^2(\omega) = u_\alpha^2 + Y_n(\omega).$$

We can easily check that $nP\{\|\tilde{X}_1\| > (n \log n)^{1/\rho}\} \to 0$. Next we consider that

$$
\begin{aligned}
& d_H\left(L_\alpha \mathrm{E}\tilde{X}_1 I_{[\|\tilde{X}_1\| \leq (n \log n)^{1/\rho}]}, 0\right) \\
=\ & \max\left(\left|(\mathrm{E}\tilde{X}_1 I_{[\|\tilde{X}_1\| \leq (n \log n)^{1/\rho}]})_\alpha^1\right|, \left|(\mathrm{E}\tilde{X}_1 I_{[\|\tilde{X}_1\| \leq (n \log n)^{1/\rho}]})_\alpha^2\right|\right) \\
=\ & \left|\int_{[\|\tilde{X}_1\| \leq (n \log n)^{1/\rho}]} (u_\alpha^1 + Y_1)\,\mathrm{d}P\right| + \left|\int_{[\|\tilde{X}_1\| \leq (n \log n)^{1/\rho}]} (u_\alpha^2 + Y_1)\,\mathrm{d}P\right| \\
\leq\ & 2\left|\int_{[\|\tilde{X}_1\| \leq (n \log n)^{1/\rho}]} Y_1\,\mathrm{d}P\right| + 2\|\tilde{u}\| \\
\leq\ & 4\int_{[(n \log n)^{1/\rho}-\|\tilde{u}\|,(n \log n)^{1/\rho}+\|\tilde{u}\|]} |Y_1|\,\mathrm{d}P + 2\|\tilde{u}\| \\
\leq\ & 4((n \log n)^{1/\rho} + \|\tilde{u}\|)P\{|Y_1| > (n \log n)^{1/\rho} - \|\tilde{u}\|\}\} + 2\|\tilde{u}\|,
\end{aligned}
$$

and hence we have that $(1/(n \log n)^{1/\rho})\, d_\infty(n\mathrm{E}\tilde{X}_1 I_{[\|\tilde{X}_1\| \leq (n\log n)^{1/\rho}]}, \tilde{0}) \to 0$ as $n \to \infty$. Then we have

$$d_\infty\left(\frac{\tilde{S}_n}{(n \log n)^{1/\rho}}, \tilde{0}\right) \to 0 \quad \text{in probability}$$

by the criteria in Theorem 3.5.

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