KYBERNETIKA — VOLUME 43 (2007), NUMBER 1, PAGES 75 – 85

A NEW FAMILY OF TRIVARIATE PROPER QUASI–COPULAS

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In this paper, we provide a new family of trivariate proper quasi-copulas. As an application, we show that W^3 – the best-possible lower bound for the set of trivariate quasi-copulas (and copulas) – is the limit member of this family, showing how the mass of W^3 is distributed on the plane $x + y + z = 2$ of $[0, 1]^3$ in an easy manner, and providing the generalization of this result to n dimensions.

Keywords: copula, mass distribution, quasi-copula

AMS Subject Classification: 62H05, 60E05

1. INTRODUCTION

Let n be a natural number such that $n \geq 2$. An n-dimensional *copula* (briefly, n-copula) is the restriction to $[0, 1]^n$ of a continuous n-variate distribution function whose univariate margins are uniform on $[0, 1]$. Equivalently, an n-copula is a function $C\colon [0, 1]^n \to [0, 1]$ which satisfies the following conditions:

(C1) *boundary conditions*: for any $(u_1, u_2, \ldots, u_n)$ in $[0, 1]^n$ it holds that $C(u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_n) = 0$ and $C(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i$ for all $i \in \{1, 2, \ldots, n\}$;

(C2) *the n-increasing property*: for every $(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n) \in [0, 1]^n$, and each *$n$-box* B in $[0, 1]^n$, i.e., $B = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$, we have that $V_C(B) = \sum \operatorname{sgn}(c_1, c_2, \ldots, c_n) \cdot C(c_1, c_2, \ldots, c_n) \geq 0$ – $V_C(B)$ is defined as the *C-volume* of B –, where the sum is taken over all the *vertices* $(c_1, c_2, \ldots, c_n)$ of B (i.e., each c_k is equal to either a_k or b_k) and $\operatorname{sgn}(c_1, c_2, \ldots, c_n)$ is 1 if $c_k = a_k$ for an even number of k's, and -1 if $c_k = a_k$ for an odd number of k's.

The importance of copulas as a tool for statistical analysis and modeling stems largely from the observation that the joint distribution H of a random vector $(X_1, X_2, \ldots, X_n)$ with respective one-dimensional margins $F_1, F_2, \ldots, F_n$ can be expressed by

$$H(x_1, x_2, \ldots, x_n) = C(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)), \quad (x_1, x_2, \ldots, x_n) \in [-\infty, \infty]^n,$$

where C is an n-copula that is uniquely determined on $\operatorname{Range} F_1 \times \operatorname{Range} F_2 \times \cdots \times \operatorname{Range} F_n$. For a complete survey about copulas, see [14, 23, 24].

Alsina et al. [1] introduced the notion of *quasi-copula* in order to show that a certain class of operations on univariate distribution functions can, or cannot, be derived from corresponding operations on random variables defined on the same probability space (see also [19]). Cuculescu and Theodorescu [4] have given the characterization of an n-dimensional quasi-copula (or n-quasi-copula) as a function $Q\colon [0,1]^n \to [0,1]$ which satisfies condition (C1) of n-copulas, but instead of condition (C2), the weaker conditions:

(Q1) *monotonicity*: Q is nondecreasing in each variable;

(Q2) *Lipschitz condition*: for any $(u_1, u_2, \dots, u_n)$ and $(v_1, v_2, \dots, v_n)$ in $[0,1]^n$, it holds that $|Q(u_1, u_2, \dots, u_n) - Q(v_1, v_2, \dots, v_n)| \le \sum_{i=1}^n |u_i - v_i|$.

We will refer to $V_Q(B)$ – the Q-volume of B – as the mass accumulated by Q on B. Every n-quasi-copula Q satisfies the inequalities

$$\begin{aligned} W^n(u_1, u_2, \dots, u_n) &= \max\left(0, \sum_{i=1}^n u_i - n + 1\right) \le Q(u_1, u_2, \dots, u_n) \\ &\le \min(u_1, u_2, \dots, u_n) = M^n(u_1, u_2, \dots, u_n) \end{aligned}$$

for every $(u_1, u_2, \dots, u_n)$ in $[0,1]^n$. While every n-copula is an n-quasi-copula, there exist *proper* n-quasi-copulas, i. e., n-quasi-copulas which are not n-copulas. For any $n \ge 2$, M^n is an n-copula; but W^n is an n-copula if and only $n = 2$, and a proper n-quasi-copula for $n \ge 3$.

One of the most important applications of quasi-copulas in statistics is the following result ([15, 17, 21]): Every pointwise ordered set of copulas has a least upper bound and greatest lower bound in the set of quasi-copulas. Of interest are sets of copulas of random variables with a specific statistical property (see [10, 11, 17, 18]). Furthermore, since quasi-copulas are a special type of binary aggregation operators satisfying the Lipschitz condition (Q2) (see [3]), these functions are becoming popular in fuzzy set theory (for instance, see [2, 8, 9, 12]).

In the literature, we cannot find many families of proper n-quasi-copulas when $n \ge 3$ – for some examples (different from W^n), see [7, 16, 22]. Recently, the mass distribution associated with a 3-quasi-copula and the differences with respect to the bivariate case – we recall that the (positive) mass of W^2 is distributed uniformly in $[0,1]^2$ on the segment which joins the points $(0,1)$ to $(1,0)$, and the (infinite positive and infinite negative) mass of W^3 is distributed on the plane $x+y+z=2$ of $[0,1]^3$ – have been studied in [7, 13]. Our purpose is to provide a new family of proper 3-quasi-copulas whose bivariate margins are 2-copulas – moreover, we construct the least upper bound and the greatest lower bound in the set of quasi-copulas with those margins. As an application, we prove that W^3 is the limit member of this new family, showing how the mass of W^3 is distributed on the plane $x+y+z=2$ of $[0,1]^3$ in an easy manner. In the last section, we provide the generalization of this problem to n dimensions.

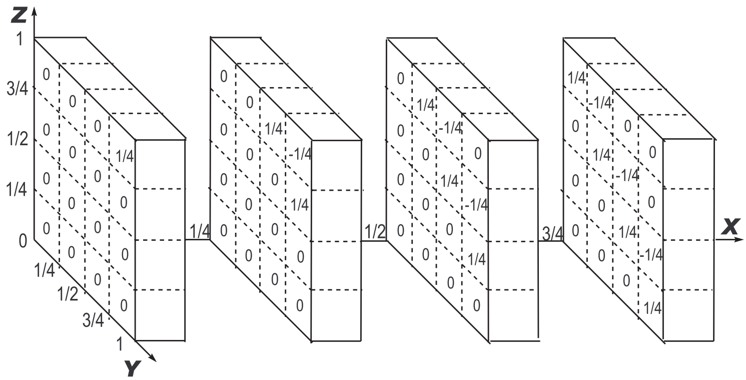

Fig. 1. Mass distribution of the trivariate function in Section 2 for $m = 4$.

2. A NEW FAMILY OF PROPER 3-QUASI-COPULAS

Let m be a natural number such that $m \geq 2$. We divide $[0,1]^3$ into m^3 3-boxes (or cubes, in this case), namely:

$$B_{i_1 i_2 i_3} = \left[\frac{i_1 - 1}{m}, \frac{i_1}{m}\right] \times \left[\frac{i_2 - 1}{m}, \frac{i_2}{m}\right] \times \left[\frac{i_3 - 1}{m}, \frac{i_3}{m}\right],$$

for all $i_1, i_2, i_3 = 1, 2, \ldots, m$. Now, we distribute $1/m$ of (positive) mass uniformly on each cube $B_{i_1 i_2 i_3}$ such that $i_1 + i_2 + i_3 = 2m + 1$; $-1/m$ of (negative) mass uniformly on each cube $B_{i_1 i_2 i_3}$ such that $i_1 + i_2 + i_3 = 2m + 2$; and 0 on the remaining cubes. It can be easily computed that there are $m(m+1)/2$ cubes with positive mass, and $m(m-1)/2$ cubes with negative mass; and the sum of positive mass is $(m+1)/2$, and the sum of negative mass is $-(m-1)/2$. Therefore, we have the amount of 1 of positive mass on $[0,1]^3$ (see Figure 1 for this construction in the case $m = 4$).

Note that if we project this construction on the planes $x = 1$, $y = 1$ and $z = 1$, we obtain a construction (similar on the three planes) with $1/m$ of (positive) mass distributed uniformly on each square of the form $R_{i_1 (m - i_1 + 1)}$, for $i_1 = 1, 2, \ldots, m$, where

$$R_{i_1 i_2} = \left[\frac{i_1 - 1}{m}, \frac{i_1}{m}\right] \times \left[\frac{i_2 - 1}{m}, \frac{i_2}{m}\right],$$

for all $i_1, i_2 = 1, 2, \ldots, m$; and 0 on $[0,1]^2 \setminus R_{i_1 (m - i_1 + 1)}$.

If (u_1, u_2, u_3) is a point in $[0,1]^3$, and $Q_m(u_1, u_2, u_3)$ is the mass spread on $[0, u_1] \times [0, u_2] \times [0, u_3]$, then Q_m is a proper 3-quasi-copula – whose three bivariate margins are 2-copulas –, as the following result shows.

Theorem 2.1. For each natural number $m \geq 2$, let $Q_m\colon [0,1]^3 \to [0,1]$ be the function defined by

$$Q_m(u_1,u_2,u_3) \tag{1}$$
$$= \begin{cases} 0, & (u_1,u_2,u_3) \in B_1, \\ m^2 \prod_{j=1}^{3}\left(u_j - \frac{i_j-1}{m}\right), & (u_1,u_2,u_3) \in B_2, \\ m\sum_{k=1}^{3}\prod_{\substack{j=1\\ j\neq k}}^{3}\left(u_j - \frac{i_j-1}{m}\right) - m^2\prod_{j=1}^{3}\left(u_j - \frac{i_j-1}{m}\right), & (u_1,u_2,u_3) \in B_3, \\ u_1+u_2+u_3-2, & \text{otherwise,} \end{cases}$$

where $B_1 = \{B_{i_1i_2i_3} : i_1+i_2+i_3 \leq 2m\}$, $B_2 = \{B_{i_1i_2i_3} : i_1+i_2+i_3 = 2m+1\}$, and $B_3 = \{B_{i_1i_2i_3} : i_1+i_2+i_3 = 2m+2\}$. Then, Q_m is a proper 3-quasi-copula for every $m \geq 2$ whose three bivariate margins (which are the same) are the 2-copula $C_m^{(2)}$ given by

$$C_m^{(2)}(v_1,v_2) = \begin{cases} 0, & (v_1,v_2) \in R_1, \\ m\prod_{j=1}^{2}\left(v_j - \frac{i_j-1}{m}\right), & (v_1,v_2) \in R_2, \\ v_1+v_2-1, & \text{otherwise,} \end{cases} \tag{2}$$

where $R_1 = \{R_{i_1i_2} : i_1+i_2 \leq m\}$ and $R_2 = \{R_{i_1i_2} : i_1+i_2 = m+1\}$.

Proof. Suppose m is a fixed natural number such that $m \geq 2$, and let (u_1,u_2,u_3) be a point in $[0,1]^3$. First, we show that Q_m is well-defined. Let $B_{i_1i_2i_3} \in B_2$ and $B_{j_1j_2j_3} \in B_3$ be two cubes in $[0,1]^3$ such that $i_2 = j_2$ and $i_3 = j_3$ (all the other cases can be proved in a similar manner). Then we have that $j_1 = 1 + i_1$. Since

$$\begin{aligned} Q_m(u_1,u_2,u_3) &= m^2\left(u_1 - \frac{i_1-1}{m}\right)\prod_{k=2}^{3}\left(u_k - \frac{j_k-1}{m}\right), \\ &\qquad (u_1,u_2,u_3) \in \left[\frac{i_1-1}{m},\frac{i_1}{m}\right]\times\left[\frac{j_2-1}{m},\frac{j_2}{m}\right]\times\left[\frac{j_3-1}{m},\frac{j_3}{m}\right], \end{aligned}$$

in particular, we obtain that

$$Q_m\left(\frac{i_1}{m},u_2,u_3\right) = m^2\left(\frac{i_1}{m} - \frac{i_1-1}{m}\right)\prod_{k=2}^{3}\left(u_k - \frac{j_k-1}{m}\right) = m\prod_{k=2}^{3}\left(u_k - \frac{j_k-1}{m}\right);$$

and since

$$Q_m(u_1,u_2,u_3) = m\sum_{k=1}^{3}\prod_{\substack{j=1\\ j\neq k}}^{3}\left(u_j-\frac{i_j-1}{m}\right)-m^2\prod_{j=1}^{3}\left(u_j-\frac{i_j-1}{m}\right),$$

$$(u_1,u_2,u_3)\in\prod_{k=1}^{3}\left[\frac{j_k-1}{m},\frac{j_k}{m}\right],$$

in particular, we obtain that

$$Q_m\left(\frac{i_1}{m},u_2,u_3\right)=Q_m\left(\frac{j_1-1}{m},u_2,u_3\right)=m\prod_{k=2}^{3}\left(u_k-\frac{j_k-1}{m}\right).$$

To prove the boundary conditions, suppose $u_2=u_3=1$ (the cases $u_1=u_2=1$ and $u_1=u_3=1$ use similar arguments) in a cube $B_{i_1i_2i_3}\in B_3$ (all the remaining cases can be proved in a similar manner). Thus $i_2=i_3=m$, and hence $i_1=2$. Then, we obtain that

$$\begin{aligned}Q_m(u_1,1,1) &= m\left[2\left(u_1-\frac{1}{m}\right)\left(1-\frac{m-1}{m}\right)+\left(1-\frac{m-1}{m}\right)^2\right]\\ &\quad -m^2\left(u_1-\frac{1}{m}\right)\left(1-\frac{m-1}{m}\right)^2=u_1.\end{aligned}$$

In what follows, let (u_1',u_2,u_3) and (u_1,u_2,u_3) be two points in a cube $B_{i_1i_2i_3}$ such that $u_1'>u_1$ (the case $u_1'=u_1$ is trivial in the following). We now check that Q_m is nondecreasing in the first variable and satisfies the Lipschitz condition (Q2) in the same variable (the cases for the other two variables can be proved in a similar manner) and in each cube $B_{i_1i_2i_3}$. We consider two cases (the remaining cases are trivial).

(i) Suppose $B_{i_1i_2i_3}\in B_2$. Then we have

$$Q_m(u_1',u_2,u_3)-Q_m(u_1,u_2,u_3)=m^2(u_1'-u_1)\prod_{j=2}^{3}\left(u_j-\frac{i_j-1}{m}\right).$$

It is trivial that $Q_m(u_1',u_2,u_3)-Q_m(u_1,u_2,u_3)\geq 0$. On the other hand, we have that $Q_m(u_1',u_2,u_3)-Q_m(u_1,u_2,u_3)\leq u_1'-u_1$ if, and only if, $m^2\cdot\prod_{j=2}^{3}(u_j-(i_j-1)/m)\leq 1$. Since $0\leq u_j-(i_j-1)/m\leq 1/m$, for $j=2,3$, the result follows.

(ii) Suppose now $B_{i_1i_2i_3}\in B_3$. Then, we have that

$$\begin{aligned}&Q_m(u_1',u_2,u_3)-Q_m(u_1,u_2,u_3)\\ &= m(u_1'-u_1)\left[\sum_{j=2}^{3}\left(u_j-\frac{i_j-1}{m}\right)-m\prod_{j=2}^{3}\left(u_j-\frac{i_j-1}{m}\right)\right].\end{aligned}$$

Thus, $Q_m(u_1',u_2,u_3)-Q_m(u_1,u_2,u_3)\geq 0$ if, and only if,

$$m\prod_{j=2}^{3}\left(u_j-\frac{i_j-1}{m}\right)\le\sum_{j=2}^{3}\left(u_j-\frac{i_j-1}{m}\right). \tag{3}$$

Suppose $u_2-(i_2-1)/m>0$ and $u_3-(i_3-1)/m>0$ (the cases with the equality are trivial), then inequality (3) is equivalent to $m\le\sum_{j=2}^{3}(u_j-(i_j-1)/m)^{-1}$. Since $u_2\in((i_2-1)/m,i_2/m]$, we have that $u_2\le i_2/m=(i_2-1)/m+1/m$, thus $u_2-(i_2-1)/m\le 1/m$ (and similarly for u_3); whence the result follows.

On the other hand, we have that $Q_m(u_1',u_2,u_3)-Q_m(u_1,u_2,u_3)\le u_1'-u_1$ holds if, and only if, $m\prod_{j=2}^{3}(u_j-(i_j-1)/m)\ge\sum_{j=2}^{3}(u_j-(i_j-1)/m)+1/m$. Since $\prod_{j=2}^{3}(u_j-(i_j-1)/m)\ge 0$, i. e., $u_2u_3-u_2i_3/m-u_3i_2/m+i_2i_3/m^2\ge 0$, we have that $u_2u_3-u_2(i_3-1)/m-u_3(i_2-1)/m+(i_2-1)(i_3-1)/m^2\ge u_2/m+u_3/m-i_2/m^2-i_3/m^2+1/m^2$; whence the result follows.

Thus, we have proved that Q_m is a 3-quasi-copula. Now, since

$$\begin{aligned}
&V_{Q_m}\left(\left[\frac{1}{m},\frac{2}{m}\right]\times\left[\frac{m-1}{m},1\right]\times\left[\frac{m-1}{m},1\right]\right)\\
=\ & Q_m\left(\frac{2}{m},1,1\right)-Q_m\left(\frac{1}{m},1,1\right)-Q_m\left(\frac{2}{m},1,\frac{m-1}{m}\right)\\
&-Q_m\left(\frac{2}{m},\frac{m-1}{m},1\right)+Q_m\left(\frac{2}{m},\frac{m-1}{m},\frac{m-1}{m}\right)\\
&+Q_m\left(\frac{1}{m},1,\frac{m-1}{m}\right)+Q_m\left(\frac{1}{m},\frac{m-1}{m},1\right)\\
&-Q_m\left(\frac{1}{m},\frac{m-1}{m},\frac{m-1}{m}\right)=\frac{2}{m}-\frac{3}{m}=-\frac{1}{m},
\end{aligned}$$

we conclude that Q_m is a proper 3-quasi-copula.

Finally, since (as it is easy to check) the bivariate margins – or of higher dimension – of any n-quasi-copula are quasi-copulas, the three bivariate margins of Q_m – i. e., $Q_m(u_1,u_2,1)$, $Q_m(u_1,1,u_3)$ and $Q_m(1,u_2,u_3)$ – given by (2) are 2-copulas since the mass (only positive) of $C_m^{(2)}$ is distributed uniformly on $[0,1]^2$, which completes the proof. □

From Theorem 2.1, we first note that the 2-copulas given by (2) are a special type of orthogonal grid constructions of copulas studied in [6] with W^2 as background copula, and Π^2 – the copula of independent random variables, i. e., $\Pi^2(u,v)=uv$ for all (u,v) in $[0,1]^2$ – as foreground copula.

We also observe that there does not exist a 3-copula whose three bivariate margins are $C_m^{(2)}(u_1,u_2)$, $C_m^{(2)}(u_1,u_3)$ and $C_m^{(2)}(u_2,u_3)$ – this is related to the problem of the *compatibility* of three 2-copulas (for more details, see [5, 20]). The following result shows this fact.

Proposition 2.1. For any natural number $m\ge 2$, there does not exist a 3-copula whose three bivariate margins are the 2-copula $C_m^{(2)}$ given by (2).

P r o o f. Suppose C is a 3-copula whose three bivariate margins are $C_m^{(2)}$. Let B be the 3-box given by $B = [1/2, 1]^3$. Then we have that

$$\begin{aligned} V_C(B) &= C(1,1,1) - C\left(\frac{1}{2},1,1\right) - C\left(1,\frac{1}{2},1\right) - C\left(1,1,\frac{1}{2}\right) + C\left(\frac{1}{2},\frac{1}{2},1\right) \\ &\quad + C\left(\frac{1}{2},1,\frac{1}{2}\right) + C\left(1,\frac{1}{2},\frac{1}{2}\right) - C\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) \\ &= 1 - \frac{3}{2} + 3 \cdot C_m^{(2)}\left(\frac{1}{2},\frac{1}{2}\right) - C\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right). \end{aligned}$$

If m is even, it is easy to check that $V_C(B) = -1/2$ for every $m \geq 2$; and if m is odd, we have that $V_C(B) = (1-m)/(2m) < 0$ for every $m \geq 3$. In both cases we obtain a contradiction; therefore, C is not a 3-copula, which completes the proof. □

We also note that Q_m is not the unique proper 3-quasi-copula whose three bivariate margins are $C_m^{(2)}$ (for methods of constructing Lipschitz aggregation operators, see [2]). In fact, for any natural number $m \geq 2$, and given $C_m^{(2)}(u_1, u_2)$, $C_m^{(2)}(u_1, u_3)$ and $C_m^{(2)}(u_2, u_3)$, $(u_1, u_2, u_3) \in [0,1]^3$, we can construct an infinite number of proper 3-quasi-copulas whose three bivariate margins are $C_m^{(2)}$, as the following example shows.

Example 2.1. For every (u_1, u_2, u_3) in $[0,1]^3$, consider the function Q given by

$$Q(u_1,u_2,u_3) = \lambda \cdot Q_U(u_1,u_2,u_3) + (1-\lambda) \cdot Q_L(u_1,u_2,u_3),$$

where

$$Q_U(u_1,u_2,u_3) = \min(C_m^{(2)}(u_1,u_2), C_m^{(2)}(u_1,u_3), C_m^{(2)}(u_2,u_3))$$

and

$$\begin{aligned} & Q_L(u_1,u_2,u_3) \\ = \; & \max(0, C_m^{(2)}(u_1,u_2) + u_3 - 1, C_m^{(2)}(u_1,u_3) + u_2 - 1, C_m^{(2)}(u_2,u_3) + u_1 - 1), \end{aligned}$$

with $\lambda \in [0,1]$. Q_L and Q_U are two proper 3-quasi-copulas – whose three bivariate margins are $C_m^{(2)}$ – which satisfy the inequalities $Q_L(u_1,u_2,u_3) \leq Q_m(u_1,u_2,u_3) \leq Q_U(u_1,u_2,u_3)$ for every (u_1,u_2,u_3) in $[0,1]^3$ (see [22]). Observe that $Q_L(u_1,u_2,u_3) \neq Q_m(u_1,u_2,u_3) \neq Q_U(u_1,u_2,u_3)$ for some (u_1,u_2,u_3) in $[0,1]^3$ and for every $m \geq 2$. For instance, if i is a real number such that $3i = 2m+1$, after some elementary algebra we have that

$$Q_m\left(\frac{i}{m},\frac{i}{m},\frac{i}{m}\right) = \frac{1}{m} < \frac{m+2}{3m} = Q_U\left(\frac{i}{m},\frac{i}{m},\frac{i}{m}\right)$$

for any $m \geq 2$. Moreover, if we suppose that $i_1 = 1$ and $i_2 = i_3 = m$, then we have that

$$Q_L\left(\frac{1}{2m}, 1-\frac{1}{2m}, 1-\frac{1}{2m}\right) = 0 < \frac{1}{8m} = Q_m\left(\frac{1}{2m}, 1-\frac{1}{2m}, 1-\frac{1}{2m}\right)$$

for every $m \geq 2$.

3. APPROXIMATION OF W^3

In this section we show that W^3 is the limit member of the family of the proper 3-quasi-copulas defined by (1).

Theorem 3.1. Let $\varepsilon > 0$. For m sufficiently large, there exists a proper 3-quasi-copula Q_m given by (1) such that $|Q_m(u_1,u_2,u_3) - W^3(u_1,u_2,u_3)| < \varepsilon$ for all (u_1,u_2,u_3) in $[0,1]^3$.

Proof. Let m be a natural number such that $m \geq 6/\varepsilon$. We first prove that

$$Q_m\left(\frac{i_1}{m},\frac{i_2}{m},\frac{i_3}{m}\right) = W^3\left(\frac{i_1}{m},\frac{i_2}{m},\frac{i_3}{m}\right) = \max\left(0, \frac{i_1+i_2+i_3}{m} - 2\right),$$

for every $i_1, i_2, i_3 = 1, 2, \ldots, m$. For that, we consider the following four cases:

(i) If $i_1 + i_2 + i_3 < 2m$, then we have that

$$Q_m\left(\frac{i_1}{m},\frac{i_2}{m},\frac{i_3}{m}\right) = 0 \quad \text{and} \quad W^3\left(\frac{i_1}{m},\frac{i_2}{m},\frac{i_3}{m}\right) = \max\left(0, \frac{i_1+i_2+i_3}{m} - 2\right) = 0.$$

(ii) If $i_1 + i_2 + i_3 = 2m + 1$, then we have that

$$Q_m\left(\frac{i_1}{m},\frac{i_2}{m},\frac{i_3}{m}\right) = m^2 \prod_{j=1}^{3}\left(\frac{i_j}{m} - \frac{i_j - 1}{m}\right) = \frac{1}{m}$$

and

$$W^3\left(\frac{i_1}{m},\frac{i_2}{m},\frac{i_3}{m}\right) = \max\left(0, \frac{2m+1}{m} - 2\right) = \frac{1}{m}.$$

(iii) If $i_1 + i_2 + i_3 = 2m + 2$, then we have that

$$Q_m\left(\frac{i_1}{m},\frac{i_2}{m},\frac{i_3}{m}\right) = 3m\left(\frac{1}{m}\right)^2 - m^2\left(\frac{1}{m}\right)^3 = \frac{2}{m}$$

and

$$W^3\left(\frac{i_1}{m},\frac{i_2}{m},\frac{i_3}{m}\right) = \max\left(0, \frac{2m+2}{m} - 2\right) = \frac{2}{m}.$$

(iv) If $i_1 + i_2 + i_3 > 2m + 2$, then we have that

$$Q_m\left(\frac{i_1}{m},\frac{i_2}{m},\frac{i_3}{m}\right) = \frac{i_1+i_2+i_3}{m} - 2$$

and

$$W^3\left(\frac{i_1}{m},\frac{i_2}{m},\frac{i_3}{m}\right) = \max\left(0, \frac{i_1+i_2+i_3}{m} - 2\right) = \frac{i_1+i_2+i_3}{m} - 2.$$

Now, let (u_1,u_2,u_3) be a point in $[0,1]^3$. We have $|u_1-i_1/m|<1/m$, $|u_2-i_2/m|<1/m$, and $|u_3-i_3/m|<1/m$ for some (i_1,i_2,i_3). Then

$$\begin{aligned}|Q_m(u_1,u_2,u_3)-W^3(u_1,u_2,u_3)| \le\ & \left|Q_m(u_1,u_2,u_3)-Q_m\left(\frac{i_1}{m},\frac{i_2}{m},\frac{i_3}{m}\right)\right| \\ & +\left|Q_m\left(\frac{i_1}{m},\frac{i_2}{m},\frac{i_3}{m}\right)-W^3\left(\frac{i_1}{m},\frac{i_2}{m},\frac{i_3}{m}\right)\right| \\ & +\left|W^3\left(\frac{i_1}{m},\frac{i_2}{m},\frac{i_3}{m}\right)-W^3(u_1,u_2,u_3)\right| \\ \le\ & 2\left|u_1-\frac{i_1}{m}\right|+2\left|u_2-\frac{i_2}{m}\right|+2\left|u_3-\frac{i_3}{m}\right|<\frac{6}{m}\le\varepsilon,\end{aligned}$$

which completes the proof. □

As a consequence of Theorem 3.1, for m sufficiently large ($m\to\infty$), the mass of W^3 is distributed on the plane $x+y+z=2$ of $[0,1]^3$ with subsets with arbitrarily large W^3-volume and subsets with arbitrarily small W^3-volume (see also [13, 14]).

4. CONCLUSION

In this paper, we have defined a new family of proper 3-quasi-copulas for which W^3 is the limit member of that family. Although our study is restricted to the trivariate case – for the sake of simplicity –, similar results can be obtained in higher dimensions – with a tedious algebra – by defining families of proper n-quasi-copulas in a similar manner. Let m be a natural number such that $m\ge 2$, and suppose $n\ge 3$. We divide $[0,1]^n$ into m^n n-boxes, namely:

$$B_{i_1i_2\dots i_n}=\left[\frac{i_1-1}{m},\frac{i_1}{m}\right]\times\left[\frac{i_2-1}{m},\frac{i_2}{m}\right]\times\cdots\times\left[\frac{i_n-1}{m},\frac{i_n}{m}\right],$$

for all $i_1,i_2,\dots,i_n=1,2,\dots,m$. Now, we distribute $1/m$ of (positive) mass uniformly on each n-box $B_{i_1i_2\dots i_n}$ such that $i_1+i_2+\cdots+i_n=(n-1)m+1$; $-1/m$ of (negative) mass uniformly on each n-box $B_{i_1i_2\dots i_n}$ such that $i_1+i_2+\cdots+i_n=(n-1)m+2$; and 0 on the remaining n-boxes. For example, if $n=4$, the number of 4-boxes with positive mass is $\sum_{i=2}^{m+1}\binom{i}{2}$, and the number of 4-boxes with negative mass is $\sum_{i=2}^{m+1}\binom{i}{2}-m$; then, the amount of positive and negative mass can be easily computed. Therefore, W^n – whose (infinite positive and infinite negative) mass is distributed on the set $\{(x_1,x_2,\dots,x_n)\in[0,1]^n\,|\,x_1+x_2+\cdots+x_n=n-1\}$ – is the member limit of this family of proper n-quasi-copulas.

Finally, we note that the family introduced in this paper (and its generalization to n-dimensions) could be much interesting in applications, especially in the construction of aggregation operators to fitting a data set.

ACKNOWLEDGEMENT

This work was partially supported by the Ministerio de Ciencia y Tecnología (Spain) and FEDER, under research project BFM2003-06522. The author wishes to thank Roger B. Nelsen and the Department of Mathematical Sciences of the Lewis and Clark College (Portland, OR), where this work was carried out in part, and two anonymous referees for their helpful comments.

(Received May 29, 2006.)

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