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G_δ–SEPARATION AXIOMS IN ORDERED FUZZY TOPOLOGICAL SPACES

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G_δ-separation axioms are introduced in ordered fuzzy topological spaces and some of their basic properties are investigated besides establishing an analogue of Urysohn's lemma.

Keywords: fuzzy G_δ-neighbourhood, fuzzy G_δ–T_1-ordered spaces, fuzzy G_δ–T_2 ordered spaces

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1. INTRODUCTION

The fuzzy concept has invaded all branches of Mathematics ever since the introduction of fuzzy set by Zadeh [10]. Fuzzy sets have applications in many fields such as information [5] and control [8]. The theory of fuzzy topological spaces was introduced and developed by Chang [3] and since then various notions in classical topology have been extended to fuzzy topological spaces. Sostak [6] introduced the fuzzy topology as an extension of Chang's fuzzy topology. It has been developed in many directions. Sostak [7] also published a new survey article of the developed areas of fuzzy topological spaces. Katsaras [4] introduced and studied ordered fuzzy topological spaces. Motivated by the concepts of fuzzy G_δ-set [2] and ordered fuzzy topological spaces the concept of increasing (decreasing) fuzzy G_δ-sets, fuzzy G_δ–T_1 ordered spaces and fuzzy G_δ–T_2 ordered spaces are studied. In this paper we introduce some new separation axioms in the ordered fuzzy topological spaces and we establish an analogue of Urysohn's lemma.

2. PRELIMINARIES

Definition 1. Let (X,T) be a fuzzy topological space and λ be a fuzzy set in X. λ is called a fuzzy G_δ-set [2] if $\lambda = \lambda_i$ where each $\lambda_i \in T$ for $i \in I$.

Definition 2. Let $X,T)$ be a fuzzy topological space and λ be a fuzzy set in X. λ is called a fuzzy F_σ-set if $\lambda = \lambda_i$ where each $1-\lambda_i \in T$ for $i \in I$ (see [2]).

Definition 3. A fuzzy set μ is a fuzzy topological space (X, T) is called a fuzzy G_δ-neighbourhood of $x \in X$ if there exists a fuzzy G_δ-set μ_1 with $\mu_1 \leq \mu$ and $\mu_1(x) = \mu(x) > 0$.

It is easy to see that a fuzzy set is fuzzy G_δ- if and only if μ is a fuzzy G_δ-neighbourhood of each $x \in X$ for which $\mu(x) > 0$.

Definition 4. A family H of fuzzy G_δ-neighbourhoods of a point x is called a base for the system of all fuzzy G_δ-neighbourhood μ of x if the following condition is satisfied. For each fuzzy G_δ-neighbourhood μ of x and for each θ, with $0 < \theta < \mu(x)$ there exists $\mu_1 \in H$ with $\mu_1 \leq \mu$ and $\mu_1(x) > 0$.

Definition 5. A function f from a fuzzy topological space (X, T) to a fuzzy topological space (Y, S) is called fuzzy irresolute if $f^{-1}(\mu)$ is fuzzy G_δ- in X for each fuzzy G_δ-set μ in Y. The function f is said to be fuzzy irresolute at $x \in X$ if $f^{-1}(\mu)$ is a fuzzy G_δ-neighbourhood of x for each fuzzy G_δ-neighbourhood μ of $f(x)$. Following the idea of Warren [10] it is easy to see that f is fuzzy irresolute $\Leftrightarrow$ f is-fuzzy irresolute at each $x \in X$.

Definition 6. A fuzzy set λ in (X, T) is called increasing/decreasing if $\lambda(x) \leq \lambda(y)/\lambda(x) \geq \lambda(y)$ whenever $x \leq y$ in (X, T) and x, $y \in X$.

Definition 7. (Katsaras [4]) An ordered set on which there is given a fuzzy topology is called an ordered fuzzy topological space.

Definition 8. If λ is a fuzzy set of X and μ is a fuzzy set of Y then $\lambda \times \mu$ is a fuzzy set of $X \times Y$, defined by $(\lambda \times \mu)(x, y) = \min(\lambda(x), \mu(y))$, for each $(x, y) \in X \times Y$ [1]. A fuzzy topological space X is product related [1] to another fuzzy topological space Y if for any fuzzy set γ of X and η of Y whenever $(1 - \lambda) \geq \gamma$ and $1 - \mu \geq \eta \Rightarrow ((1-\lambda) \times 1) \vee (1 \times (1-\mu)) \geq \gamma \times \eta$, where λ is a fuzzy open set in X and μ is a fuzzy open set in Y, there exist λ_1 a fuzzy open set in X and μ_1 a fuzzy open set in Y such that $1 - \lambda_1 \geq \gamma$ or $1 - \mu_1 \geq \eta$ and $((1-\lambda_1) \times 1) \vee (1 \times (1-\mu_1)) = ((1-\lambda) \times 1) \vee (1 \times (1-\mu))$.

Definition 9. (Katsaras [4]) An ordered fuzzy topological space $(X, T, \leq)$ is called normally ordered if the following condition is satisfied. Given a decreasing fuzzy closed set μ and a decreasing fuzzy open set γ such that $\mu \leq \gamma$, there are decreasing fuzzy open set γ_1 and a decreasing fuzzy closed set μ_1 such that $\mu \leq \gamma_1 \leq \mu_1 \leq \gamma$.

3. FUZZY G_δ–T_1–ORDERED SPACES

Let $(X, T, \leq)$ be an ordered fuzzy topological space and let λ be any fuzzy set in $(X, T, \leq)$, λ is called increasing fuzzy G_δ/F_σ if $\lambda = \bigwedge_{i=1}^{\infty} \lambda_i$ /if $\lambda = \bigvee_{i=1}^{\infty} \lambda_i$, where each λ_i is increasing fuzzy open/closed in $(X, T, \leq)$. The complement of fuzzy increasing G_δ/F_σ-set is decreasing fuzzy F_σ/G_δ.

Definition 10. Let λ be any fuzzy set in the ordered fuzzy topological space $(X, T, \leq)$. Then we define

$$\begin{aligned} I_\sigma(\lambda) &= \text{increasing fuzzy } \sigma\text{-closure of } \lambda \\ &= \text{the smallest increasing fuzzy } F_\sigma\text{-set containing } \lambda; \\ D_\sigma(\lambda) &= \text{decreasing fuzzy } \sigma\text{-closure of } \lambda \\ &= \text{the smallest decreasing fuzzy } F_\sigma\text{-set containing } \lambda; \\ I^0_\sigma(\lambda) &= \text{increasing fuzzy } \sigma\text{-interior of } \lambda \\ &= \text{the greatest increasing fuzzy } G_\delta\text{-set contained in } \lambda; \\ D^0_\sigma(\lambda) &= \text{decreasing fuzzy } \sigma\text{-interior of } \lambda \\ &= \text{the greatest decreasing fuzzy } \mathrm{G}_\delta\text{-set contained in } \lambda. \end{aligned}$$

Proposition 1. For any fuzzy set λ of an ordered fuzzy topological space $(X, T, \leq)$, the following are valid.

$$\begin{aligned} &\text{(a)} \qquad 1 - I_\sigma(\lambda) = D^0_\sigma(1-\lambda), \\ &\text{(b)} \qquad 1 - D_\sigma(\lambda) = I^0_\sigma(1-\lambda), \\ &\text{(c)} \qquad 1 - I^0_\sigma(\lambda) = D_\sigma(1-\lambda), \\ &\text{(d)} \qquad 1 - D^0_\sigma(\lambda) = I_\sigma(1-\lambda). \end{aligned}$$

Proof. We shall prove (a) only, (b), (c) and (d) can be proved in a similar manner.

Since $I_\sigma(\lambda)$ is a increasing fuzzy F_σ-set containing λ, $1 - I_\sigma(\lambda)$ is a decreasing fuzzy G_δ-set such that $1 - I_\sigma(\lambda) \leq 1 - \lambda$. Let μ be another decreasing fuzzy G_δ-set such that $\mu \leq 1 - \lambda$. Then $1 - \mu$ is a increasing fuzzy F_σ-set such that $1 - \mu \geq \lambda$. It follows that $I_\sigma(\lambda) \leq 1 - \mu$. That is, $\mu \leq 1 - I_\sigma(\lambda)$. Thus, $1 - I_\sigma(\lambda)$ is the largest decreasing fuzzy G_δ-set such that $1-I_\sigma(\lambda) \leq 1-\lambda$. That is, $1-I_\sigma(\lambda) = 1-D^0_\sigma(1-\lambda)$. □

Definition 11. An ordered fuzzy topological space $(X, \tau, \leq)$ is said to be lower/upper fuzzy $G_\delta - T_1$-ordered if for each pair of elements $a \not\leq b$ in X, there exists an increasing/decreasing fuzzy G_δ-neighbourhood λ such that $\lambda(a) > 0/\lambda(b) > 0$ and λ is not a fuzzy G_δ-neighbourhood of b/a. X is said to be fuzzy G_δ–T_1-ordered if it is both lower and upper G_δ–T_1-ordered.

Proposition 2. For an ordered fuzzy topological space (X, τ, $\leq$) the following are equivalent.

1. $(X, \tau, \leq)$ is lower/upper fuzzy $G_\delta - T_1$-ordered.
2. For each a, $b \in X$ such that $a \not\leq b$, there exists an increasing/decreasing fuzzy G_δ-set λ such that $\lambda(a) > 0/\lambda(b) > 0$ and λ is not a fuzzy G_δ-neighbourhood of b/a.

3. For all $x \in X$, $\chi_{[\leftarrow,x]}/\chi_{[x,\rightarrow]}$ is fuzzy F_σ/G_δ – where $[\leftarrow, x] = \{y \in X | y \leq x\}$ and $[x, \rightarrow] = \{y \in X | y \geq x\}$.

Proof. (1) ⇒ (2) Let $(X, \tau, \leq)$ be lower fuzzy G_δ–T_1-ordered. Let a, $b \in X$ be such that $a \leq b$. There exists an increasing fuzzy G_δ-neighbourhood λ of a such that λ is not a fuzzy G_δ-neigbourhood of b. It follows that there exists a fuzzy G_δ-set μ_1 with $\mu_1 \leq \lambda$ and $\mu_1(a) = \lambda(a) > 0$. As λ is increasing, $\lambda(a) > \lambda(b)$ and since λ is not a fuzzy G_δ-neighbourhood of b, $\mu_1(b) < \lambda(b) \Rightarrow \mu_1(a) = \lambda(a) > \lambda(b) > \mu_1(b)$. This shows μ_1 is increasing and μ_1 is not a fuzzy G_δ-neighbourhood of b since λ is not a fuzzy G_δ-neighbourhood of b.

(2) ⇒ (3) consider $1 - \chi_{[\leftarrow,x]}$. Let y be such that $1 - \chi_{[\leftarrow,x]}(y) > 0$. This means $y \leq x$. Therefore by (2) there exists increasing fuzzy G_δ-set λ such that $\lambda(y) > 0$ and λ is not a fuzzy G_δ-neighbourhood of x and $\lambda \leq 1 - \chi_{[\leftarrow,x]}$. This means $1 - \chi_{[\leftarrow,x]}$ is fuzzy G_δ- and so $X_{(\leftarrow,x]}$ is fuzzy F_σ.

(3) ⇒ (1) This is obvious. □

Corollary 1. If $(X, \tau, \leq)$ is lower/upper fuzzy G_δ–T_1-ordered and $\tau \leq \tau^*$, then $(X, \tau^*, \leq)$ is also lower/upper fuzzy $G_\delta - T_1$-ordered.

Proposition 3. Let f be order preserving (that is $x \leq y$ in X if and only if $f(x) \leq *f(y)$ in X^*), fuzzy irresolute mapping from an ordered fuzzy topological space $(X, \tau, \leq)$ to an ordered fuzzy topological space $(X^*, \tau^*, \leq^*)$. If $(X^*, \tau^*, \leq^*)$ is fuzzy G_δ–T_1-ordered, then $(X, \tau, \leq)$ is fuzzy G_δ–T_1-ordered.

Proof. Let $a \leq b$ in X. As f is order preserving, $f(a) \leq^* f(b)$ in X^*. Hence there exists an increasing/decreasing fuzzy G_δ-set λ^* in X such that $\lambda^*(f(a)) > 0/\lambda^*(f(b)) > 0$ and λ^* is not a fuzzy G_δ-neighbourhood of $f(b)/f(a)$. Let $\lambda = f^{-1}(\lambda^*)$. As f is order preserving and fuzzy irresolute λ is an increasing/decreasing fuzzy G_δ-set in X. Also $\lambda(a) > 0/\lambda(b) > 0$ and λ is not a fuzzy G_δ-neighbourhood of b/a. Thus we have shown that X is lower/upper fuzzy G_δ–T_1-ordered. That is $(X, \tau, \leq)$ is fuzzy G_δ–T_1-ordered.

Proposition 4. Suppose $(X_{t1}, \tau_{t1}, \leq_{t1})$ and $(X_{t2}, \tau_{t2}, \leq_{t2})$ be any two ordered fuzzy topological spaces such that X_{t1} and X_{t2} are product related (Zadeh [11]). Assume X_{t1} and X_{t2} are fuzzy G_δ–T_1-ordered. Let $(X, \tau, \leq)$ be the product ordered fuzzy topological space. Then $(X, \tau, \leq)$ is also fuzzy G_δ–T_1-ordered.

Proof. Let $a = (a_{t1}, a_{t2})$ and $b = (b_{t1}, b_{t2})$ be two elements of the product X such that $a \not\leq b$. Thus $a_{t1} \not\leq b_{t1}$ or $a_{t2} \not\leq b_{t2}$ or both. To be definite let us assume that $a_{t1} \not\leq b_{t1}$. Since $(X_{t1}, \tau_{t1}, \leq_{t1})$ is fuzzy $G_\delta - T_1$-ordered, there exists an increasing fuzzy G_δ-set θ_{t1} in $\tau_{t1,}$ such that $\theta_{t1}(a_{t1}) > 0$ and $\theta_{t1}(b_{t1}) = 0$. Define $\theta = \theta_{t1} \times 1_{Xt2}$. Then θ is an increasing fuzzy G_δ-set in X such that $\theta(a) > 0$ and $\theta(b) = 0$. (Since $\theta(b) = \theta(b_{t1}, b_{t2}) = \theta_{t1} \times 1_{xt2}\ (b_{t1}, b_{t2}) = \mathrm{Min}\{\theta_{t1}(b_{t1}), 1_{xt2}(b_{t2})\} = \mathrm{Min}\{0, 1\} = 0$).

Therefore $(X, \tau, \leq)$ is lower fuzzy $G_\delta - T_1$-ordered. Similarly we can prove it is also upper fuzzy G_δ–T_1-ordered. That is $(X, \tau, \leq)$ is fuzzy G_δ–T_1-ordered.

Definition 12. Let $\{(X_t, \tau_{t1}, \leq_t)\}_{t\in\Delta}$ be a collection of disjoint ordered fuzzy topological spaces. Let $X = \bigcup_{t\in\Delta} X_t$, $T = \{\lambda \in I^X | \lambda/X_t \in \tau_t\}$ and "$\leq$" be a partial order on X such that $x \leq y$ if and only if $x, y \in X_t$ for some $t \in \Delta$ and $x \leq_t y$. Then $(X, \tau, \leq)$ is called ordered fuzzy topological sum of $\{(X_t, \tau_t, \leq_t)\}_{t\in\Delta}$.

In this connection we prove the following proposition.

Proposition 5. $(X, \tau, \leq)$ is fuzzy G_δ–T_1-ordered $\Leftrightarrow$ $(X_t, \tau_t, \leq_t)$ is fuzzy G_δ–T_1-ordered for each $t \in \Delta$.

Proof. Let $(X, \tau, \leq)$ be fuzzy G_δ–T_1-ordered that $t \in \Delta$. Suppose $x, y \in X_t$ such that $x \not\leq_t y$. Then $x \not\leq y$. Hence there exists an increasing fuzzy G_δ-set λ in X such that $\lambda(x) > 0$ and $\lambda(y) = 0$. But λ/X_t is an increasing fuzzy G_δ- of X_t, such that $\lambda/X_t(x) > 0$ and $\lambda/X_t(y) = 0$. Therefore, $(X_t, \tau_t, \leq_t)$ is lower fuzzy $G_\delta - T_1$-ordered. Similarly, we can show that it is an upper fuzzy G_δ–T_1-ordered space.

Conversely, let $(X_t, \tau_t, \leq_t)$ be fuzzy G_δ–T_1-ordered for all $t \in \Delta$. Consider $x, y \in X$ such that $x \leq y$. Then there exists $t_0 \in \Delta$ such that $x, y \in X_{t_0}$, with $x \not\leq t_0 y$ or $x \in X_t,\ y \in X_s,\ t \neq s\ t, s \in \Delta$. If $x, y \in X_{t_0},\ t_0 \in \Delta$, then by hypothesis there exists an increasing fuzzy G_δ-set λ in X_{t_0} such that $\lambda(x) > 0,\ \lambda(y) = 0$. Then λ is the required increasing fuzzy G_δ-set of X. But if $x \in X_t,\ y \in X_s,\ t \neq s$, $t, s \in \Delta$ then 1_{Xt}, is the required increasing fuzzy G_δ-set of X. Hence in either cases $(X, \tau, \leq)$ is lower fuzzy G_δ–T_1-ordered. Similarly we can prove that $(X, \tau, \leq)$ is upper G_δ–T_1-ordered. □

4. FUZZY G_δ–T_2–ORDERED SPACES

Definition 13. $(X, \tau, \leq)$ is said to be fuzzy G_δ–T_2-ordered if for $a, b \in X$, with $a \not\leq b$, there exists fuzzy G_δ-sets λ and μ such that λ is an increasing fuzzy G_δ-neighbourhood of a, μ is a decreasing fuzzy G_δ-neighbourhood of a and $\lambda \wedge \mu = 0$.

Definition 14. Let $(X \leq)$ be any partially ordered set. Let $G = \{(x, y) \in X \times X | x \leq y\}$. Then G is called the graph of the partial order "$\leq$".

Proposition 6. For an ordered fuzzy topological space $(X, \tau, \leq)$ the following are equivalent.

(1) X is fuzzy G_δ–T_2-ordered.

(2) For each pair $a, b \in X$ such that $a \not\leq b$, there exists fuzzy G_δ-sets λ and μ such that $\lambda(a) > 0$, $\mu(b) > 0$ and $\lambda(x) > 0$ and $\mu(y) > 0$ together imply that $x \leq y$.

(3) The characteristic function χ_G where G is the graph of the partial order of G, is fuzzy F_σ- in $(X \times X, \tau \times \tau, \leq)$.

Proof. (1) $\Rightarrow$ (2) Suppose $\lambda(x) > 0$, and $\mu(y) > 0$ and suppose $x \leq y$. Since λ is increasing and μ is decreasing, $\lambda(x) \leq \lambda(y)$ and $\mu(x) \geq \mu(y)$. Therefore, $0 < \lambda(x) \wedge \mu(y) \leq \lambda(y) \wedge \mu(x)$, which is a contradiction to the fact that $\lambda \wedge \mu = 0$.Therefore $x \not\leq y$.

(2) $\Rightarrow$ (1) Let a, $b \in X$ with $a \not\leq b$. Then there exist fuzzy sets λ and μ satisfying the properties in (2). Consider $I^0_\sigma(\lambda)$ and $D^0_\sigma(\mu)$. Clearly $I^0_\sigma(\lambda)$ in increasing and $D^0_\sigma(\mu)$ is decreasing. So the proof is complete if we show that $I^0_\sigma(\lambda) \wedge D^0_\sigma(\mu) = 0$. Suppose $z \in X$ is such that $I^0_\sigma(\lambda)(z) \wedge D^0_\sigma(\mu)(z) > 0$. Then $I^0_\sigma(\lambda)(z) > 0$ and $D^0_\sigma(\mu)(z) > 0$. So if $y \leq z \leq x$, then $y \leq z \Rightarrow D^0_\sigma(\mu)(y) \geq D^0_\sigma(\mu)(z)$ and $z \leq x \Rightarrow I^0_\sigma(\lambda)(x) \geq I^0_\sigma(\lambda)(z) > 0$. Hence by (2) $x \not\leq y$; but then $x \leq y$ and this is a contradiction.

(1) $\Rightarrow$ (3) We want to show that χ_G is fuzzy F_σ- in $(X \times X, \tau \times \tau)$. So it is sufficient if we show that $1-\chi_G$ is a fuzzy G_δ-neighbourhood of $(x, y) \in X \times X$ such that $(1 - \chi_G)(x, y) > 0$. Suppose $(x, y) \in X \times X$ is such that $(1 - \chi_G)(x, y) > 0$. That is $\chi_G(x, y) < 1$. This means $\chi_G(x, y) = 0$. That is $(x, y) \not\leq G$. That is, $x \not\leq y$. Therefore by (1) there exists fuzzy G_δ-sets λ and μ such that λ is increasing fuzzy G_δ-neighbourhood of a, μ is a decreasing fuzzy G_δ-neighbourhood of b and $\lambda \wedge \mu = 0$. Clearly, $\lambda \times \mu$ is a fuzzy G_δ-neighbourhood of (x, y). It is easy to verify that $\lambda \times \mu < 1-\chi_G$. Thus we find that $1-\chi_G$ is fuzzy G_δ-. Hence (3) is established.

(3) $\Rightarrow$ (1) Suppose $x \leq y$. Then $(x, y) \notin G$, where G is the graph of the partial order. Given that χ_G is fuzzy F_σ in $(X, \times X, \tau \times \tau)$, $1 - \chi_G$ is fuzzy G_δ- in $(X \times X, \tau \times \tau)$. Now, $(x, y) \notin G \Rightarrow (1-\chi_G)(x, y) = 1 > 0$. Therefore, $(1-\chi_G)$ is a fuzzy G_δ-neighbourhood of $(x, y) \in X \times X$. Hence we can find a fuzzy G_δ-set $\lambda \times \mu$ such that $\lambda \times \mu < (1 - \chi_G)$ and λ is fuzzy G_δ-set such that $\lambda(x) > 0$ and μ is a fuzzy G_δ-set such that $\mu(y) > 0$.

We now claim that $I^0_\sigma(\lambda) \wedge D^0_\sigma(\mu) = 0$. For if $z \in X$ is such that $(I^0_\sigma(\lambda) \wedge D^0_\sigma(\mu)(z) > 0$, then $I^0_\sigma(\lambda)(z) \wedge D^0_\sigma(\mu)(z) > 0$. This means $I^0_\sigma(\lambda)(z) > 0$ and $D^0_\sigma(\mu)(z) > 0$. And if $b \leq z \leq a$, then $z \leq a \Rightarrow I^0_\sigma(\lambda)(a) > I^0_\sigma(\lambda)(z) > 0$, and $b \leq z \Rightarrow D^0_\sigma(\mu)(b) \geq D^0_\sigma(\mu)(z) > 0$. Then $I^0_\sigma(\lambda)(a) > 0, D^0_\sigma(\mu)(b) > 0 \Rightarrow a \not\leq b$; but then $a \leq b$. This is a contradiction. Hence (1) is established. □

Definition 15. $(X, \tau, \leq)$ is said to be weakly fuzzy G_δ–T_2-ordered if given $b < a$ (i.e., $b \leq a$, and $b \neq a$) there exists fuzzy G_δ-sets λ and μ such that $\lambda(a) > 0$ and $\mu(b) > 0$ and such that if x, $y \in X$, $\lambda(x) > 0$, $\mu(y) > 0$ together imply that $y < x$.

Notation. The symbol $x\|y$ means that $x \not\leq y$ and $y \not\leq x$.

Definition 16. $(X, \tau, \leq)$ is said to be almost fuzzy G_δ–T_2-ordered if given $a\|b$ there exists fuzzy G_δ-sets λ and μ such that $\lambda(a) > 0$ and $\mu(b) > 0$ and such that if x, $y \in X$, $\lambda(x) > 0$ and $\mu(y) > 0$ together imply that $x\|y$.

Proposition 7. $(X, \tau, \leq)$ is fuzzy G_δ–T_2-ordered, $\Leftrightarrow$ $(X, \tau, \leq)$ is weakly fuzzy G_δ–T_2-ordered and almost fuzzy G_δ–T_2-ordered.

P r o o f. Clearly if X is a fuzzy G_δ–T_2-ordered, then it is weakly fuzzy G_δ–T_2-ordered. So now let $a\|b$. Then $a \not\leq b$ and $b \not\leq a$. Since $a \not\leq b$ and since X is fuzzy G_δ–T_2-ordered we have fuzzy G_δ-sets λ and μ such that $\lambda(a) > 0$, $\mu(b) > 0$, $\lambda(x) > 0$ and $\mu(y) > 0$ together imply that $x \leq y$. Also since $b \leq a$, there exists fuzzy G_δ-sets μ^* and λ^* such that $\lambda^*(a) > 0$, and $\mu^*(b) > 0$, and $\lambda^*(x) > 0$ and $\mu^*(y) > 0$ together $\Rightarrow y \not\leq x$. Thus $I^0_\sigma(\lambda\wedge\lambda^*)$ is a fuzzy G_δ-set such that $I^0_\sigma(\lambda\wedge\lambda^*)(a) > 0$ and $I^0_\sigma(\mu\wedge\mu^*)$ is such that $I^0_\sigma(\mu \wedge \mu^*)(b) > 0$ and $I^0_\sigma(\lambda \wedge \lambda^*)(x) > 0$ and $I^0_\sigma(\mu \wedge \mu^*)(y) > 0$ together imply that $x\|y$. Hence X is almost fuzzy G_δ–T_2-ordered.

Conversely let X be weakly fuzzy G_δ–T_2-ordered and almost fuzzy G_δ–T_2-ordered. We want to show that $X$is fuzzy G_δ–T_2-ordered. So let $a \not\leq b$. Then either $b < a$ or $b \leq a$. If $b < a$, then X being weakly fuzzy G_δ–T_2-ordered there exists fuzzy G_δ-sets λ and μ such that $\lambda(a) > 0$ and $\mu(b) > 0$ and such that $\lambda(x) > 0$, $\mu(y) > 0$ together imply $y < x$. That is $x \not\leq y$. If $b \not\leq a$, then $a\|b$ and the result follows easily since X is almost fuzzy $G_\delta - T_2$-ordered. □

Definition 17. Let λ and μ be fuzzy sets in $(X, \tau, \leq)$. λ is called a fuzzy G_δ-neighbourhood of μ if $\mu \leq \lambda$ and there exists a fuzzy G_δ-set δ such that $\mu \leq \delta \leq \lambda$.

Proposition 8. An ordered fuzzy topological space $(X, \tau, \leq)$ is fuzzy G_δ–T_2-ordered $\Leftrightarrow$ For each pair of points $x \not\leq y$ in X, there exists a function f of $(X, \tau, \leq)$ into a fuzzy G_δ–T_2-ordered space $(X^*, \tau^*, \leq^*)$ such that (1) f is increasing/decreasing; (2) f is fuzzy irresolute; (3) $f(x) \leq^* f(y)/f(y) \leq^* f(x)$.

P r o o f. If $(X, \tau, \leq)$ is fuzzy G_δ–T_2-ordered space, then the identity mapping is the required function.

Conversely let $x \not\leq y$ in X. Hence by hypothesis, there exists a function f of $(X, \tau, \leq)$ into a fuzzy G_δ–T_2-ordered space $(X^*, \tau^*, \leq^*)$ satisfying the conditions (1), (2) and (3).

Since $f(x) \not\leq^* f(y)$ and $(X^*, \tau^*, \leq^*)$ is fuzzy G_δ–T_2-ordered there exists an increasing fuzzy G_δ-set λ and a decreasing fuzzy G_δ-set μ such that λ is a fuzzy G_δ-neighbourhood of $f(a)$ and μ is a fuzzy G_δ- neighbourhood of $f(b)$ such that $\lambda\wedge\mu = 0$. Since f is increasing and λ is increasing it follows by Proposition 3.8 of [4], $F^{-1}(\lambda)$ is increasing. Also since f is increasing and μ is decreasing again by Proposition 3.8 of [4], $f^{-1}(\mu)$ is decreasing. Also since f is fuzzy irresolute $f^{-1}(\lambda)$ and $f^{-1}(\mu)$ are fuzzy G_δ-sets in X and also $f^{-1}(\lambda) \wedge f^{-1}(\mu) = f^{-1}(\lambda \wedge \mu) = f^{-1}(0) = 0$.

Hence X is fuzzy G_δ–T_2-ordered. Analogously one can prove the proposition for decreasing function. □

Proposition 9. The product of a family of fuzzy G_δ–T_2-ordered spaces is also fuzzy G_δ–T_2-ordered.

P r o o f. Let $\{X_t, \tau_t, \leq_t)|t \in \Delta\}$ be a family of fuzzy G_δ–T_2-ordered spaces and $(X, \tau, \leq)$ be the product of ordered fuzzy topological spaces. If $(x(t), (y_t) \in X$ such that $(x_t) \not\leq (y_t)$, then there exists $t_0 \in \Delta$ such that $x_{t_0} \not\leq y_{t_0}$. Thus there exists fuzzy G_δ-sets λ_{t_0} and μ_{t_0} in X_{t_0}, where λ_{t_0} is increasing and μ_{t_0} is decreasing and λ_{t_0} is fuzzy G_δ-neighbourhood of x_{t_0}, μ_{t_0} is a fuzzy G_δ-neighbourhood of y_{t_0}, $\lambda_{t_0} \wedge \mu_{t_0} = 0$. Define

$$\lambda = \prod_{t \in \Delta} \lambda_t \quad \text{where} \quad \lambda_{t_0} = 1_{x_t} \quad \text{if} \quad t \neq t_0,$$

and

$$\mu = \prod_{t \in \Delta} \mu_t \quad \text{where} \quad \mu_{t_0} = 1_{x_t} \quad \text{if} \quad t \neq t_0.$$

Then λ is an increasing fuzzy G_δ-set of X and μ is decreasing fuzzy G_δ-set of X such that λ is a fuzzy G_δ-neighbourhood of (x_t) and μ is a fuzzy G_δ-neighbourhood of (y_t) and $\lambda \wedge \mu = 0$. Hence $(X, \tau, \leq)$ is fuzzy G_δ–T_2-ordered. □

Proposition 10. Let $\{(X_t, \tau_t, \leq) | t \in \Delta\}$ be a family of disjoint ordered fuzzy topological spaces and let $(X, \tau, \leq)$ be the ordered fuzzy topological sum. Then $(X, \tau, \leq)$ is fuzzy G_δ–T_2-ordered $\Leftrightarrow$ $(X_t, \tau_t, \leq_t)$ is fuzzy G_δ–T_2-ordered for each $t \in \Delta$.

Proof. The proof is similar to Proposition 5. □

Definition 18. $(X, \tau, \leq)$ is said to be fuzzy G_δ-normally ordered if and only if the following condition is satisfied: Given decreasing fuzzy F_σ-set μ and decreasing fuzzy G_δ-set ρ such that $\mu \leq \rho$, there are decreasing fuzzy G_δ-set ρ_1 and a decreasing fuzzy F_σ-set μ_1 such that $\mu \leq \rho_1 \leq \mu_1 \leq \rho$.

Clearly every normally ordered space (see Katsaras [4]) is fuzzy G_δ-normally ordered.

Proposition 11. In an ordered fuzzy topological spaces $(X, \tau, \leq)$ the following are equivalent:
(1) $(X, \tau, \leq)$ is fuzzy G_δ-normally ordered;
(2) Given a decreasing fuzzy G_σ-set μ and a decreasing fuzzy G_δ-set ρ with $\mu \leq \rho$, there exists a decreasing fuzzy G_δ-set ρ_1 such that $\mu < \rho_1 < D_\sigma\ (\rho_1) \leq \rho$.

Proof. (1) $\Rightarrow$ (2) Let μ and ρ be as given in (2).

Hence by (1) we have fuzzy G_δ-decreasing set ρ_1 a decreasing fuzzy F_σ-set μ_1 such that $\mu \leq \rho_1 \leq \mu_1 \leq \rho$. Since μ_1 is a decreasing fuzzy F_σ-set such that $\rho_1 \leq \mu_1$, we have $\mu \leq \rho_1 \leq D_\sigma(\rho_1) \leq \mu_1 \leq \rho$. This proves (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1). Let μ be a decreasing fuzzy F_σ-set and ρ be a decreasing fuzzy G_δ-set such that $\mu \leq \rho$. Hence by (2) there exists a decreasing fuzzy G_δ-set ρ_1 such that $\mu \leq \rho_1 \leq D_\sigma\ (\rho_1) \leq \rho$.

Clearly $D_\sigma(\rho_1)$ is the smallest decreasing fuzzy F_σ-set containing ρ_1. Put $\mu_1 = D(\rho_1)$. Then $\mu \leq \rho_1 \leq \mu_1 \leq \rho$ shows that (2) $\Rightarrow$ (1) is proved. □

We have now the following result which is analogous to Urysohn's lemma.

Definition 19. A function f from a fuzzy topological space (X, T) to a fuzzy topological space (Y, S) is called fuzzy G_δ-continuous if $f^{-1}(\lambda)$ is fuzzy G_δ in (X, T) whenever λ is fuzzy open in (Y, S).

Theorem 12. $(X, \tau, \leq)$ is fuzzy G_δ-normally ordered $\Leftrightarrow$ Given a decreasing fuzzy F_σ-set μ in X and a decreasing fuzzy G_δ-set ρ with $\mu \leq \rho$, there exists an increasing function $f : X \to I(I)$ such that $\mu(x) < 1 - f(x)(0+) \leq 1 - f(x)(1-) \leq \rho(x)$ and f is fuzzy G_δ-continuous and $I(I)$ is fuzzy unit interval (see [4]).

Proof. The proof is similar to that of Theorem 5.3 in [4] with some slight suitable modifications.

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