KYBERNETIKA — VOLUME 42 (2006), NUMBER 6, PAGES 665 – 672

NONQUADRATIC STABILIZATION OF CONTINUOUS–TIME SYSTEMS IN THE TAKAGI–SUGENO FORM

Miguel Bernal, Petr Hušek and Vladimír Kučera

This paper presents a relaxed scheme for controller synthesis of continuous-time systems in the Takagi-Sugeno form, based on non-quadratic Lyapunov functions and a non-PDC control law. The relaxations here provided allow state and input dependence of the membership functions' derivatives, as well as independence on initial conditions when input constraints are needed. Moreover, the controller synthesis is attainable via linear matrix inequalities, which are efficiently solved by commercially available software.

Keywords: fuzzy models, nonquadratic stabilization, nonlinear control, Lyapunov function, linear matrix inequality (LMI)

AMS Subject Classification: 93E12, 62A10, 62F15

1. INTRODUCTION

Fuzzy control systems have experienced a big growth of industrial applications in the recent years, because of their reliability and efficiency in dealing with highly nonlinear behavior, combining accuracy and simplicity [2].

Fuzzy systems in the Takagi–Sugeno form (TSFS) [12] are one of the most successful tools for modeling nonlinear systems. They are included in the more general class of quasi-LPV systems [1]. Their structure permits stability analysis via common quadratic Lyapunov functions [13] and controller synthesis via parallel distributed compensation (PDC). Controller design under this approach can perform decay rate specification, input and output constraints, robustness and optimality [14, 16]. Furthermore, results can be stated as linear matrix inequalities (LMIs) which are computationally solvable.

Nonetheless, common quadratic Lyapunov functions lack flexibility since there are many stable systems which do not admit them to prove stability. Analysis and synthesis of TSFS under this approach, specially when a large number of subsystems are involved, can be very conservative. Several approaches have been developed to overcome these drawbacks. Dropping the first condition (common functions) led to piecewise quadratic Lyapunov functions, which have been employed to enrich the set of candidates used to prove stability as well as to include rules' antecedent information in the searching [3, 8, 10, 11]. Very recent results on controller synthesis under this approach have appeared [4, 7], but they are still limited to continuous-time TSFS.

Dropping the second condition (quadratic functions) led to a more general approach based on non-quadratic Lyapunov functions where non-PDC control laws can be used [9, 15]. In contrast to the quadratic piecewise approach, the non-quadratic one can deal with non-linear premise variables, so TSFS' approximation capabilities can be fully exploited. Though this approach has been thoroughly developed for discrete-time TSFS [5], just few results are available for the continuous-time counterpart, mainly due to the difficulty in handling membership functions' derivatives.

This work intends to eliminate the previous lack by offering a non-quadratic controller synthesis on the basis of a non-PDC control law. It is shown how to allow state and input dependence of the membership functions' derivatives, as well as independence on initial conditions when input constraints are needed. Moreover, the results are expressed in terms of LMIs, which are numerically solvable with commercially available software.

This paper is organized as follows: Section 2 presents the dynamical fuzzy systems and the non-quadratic approach this work is based on; Section 3 develops the controller design which stabilizes a given continuous-time TSFS; Section 4 shows an illustrative example of the results and, finally, Section 5 gathers some concluding remarks.

2. FUZZY DYNAMIC MODEL AND NON–QUADRATIC APPROACH

Consider the following continuous-time Takagi–Sugeno fuzzy system:

$$\dot{x}(t) \quad = \quad A_z x(t) + B_z u(t) \tag{1}$$

where

$$A_z = \sum_{i=1}^{r} h_i(z(x(t)))A_i, \quad B_z = \sum_{i=1}^{r} h_i(z(x(t)))B_i,$$

$$h_i(z(x(t))) = \frac{w_i(z(x(t)))}{\sum_{i=1}^{r} w_i(z(x(t)))}, \quad w_i(z(x(t))) = \prod_{j=1}^{p} M_{ij}(z_j(x(t))),$$

M_{ij} is the ijth membership function, r is the number of rules, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}$ is the control input vector, A_i, B_i are matrices of suitable dimensions that represent the ith local model of the fuzzy system and $z(x(t)) = [z_1(x(t)) \ldots z_p(x(t))]$ is the premise vector which depends on the state vector $x(t)$.

Non-PDC control law

$$u(t) \quad = \quad -\left(\sum_{j=1}^{r} h_j(z(x(t)))F_j\right)\left(\sum_{k=1}^{r} h_k(z(x(t)))P_k\right)^{-1} x(t) = -F_z P_z^{-1} x(t) \tag{2}$$

with the Lyapunov function candidate

$$V(x(t)) = x^T(t)\left(\sum_{k=1}^{r} h_k(z(x(t)))P_k\right)^{-1} x(t) = x^T(t) P_z^{-1} x(t), \quad P_k = P_k^T > 0 \tag{3}$$

is considered, which leads to the following closed-loop continuous-time TSFS under the previous definitions:

$$\dot{x}(t) \quad = \quad (A_z - B_z F_z P_z^{-1})\, x(t). \tag{4}$$

As in [9], hereinafter if $Y_z = \sum_{k=1}^{r} h_k(z(x(t)))Y_k$ then $Y_z^{-1} = \left(\sum_{k=1}^{r} h_k(z(x(t)))Y_k\right)^{-1}$, $Y_z^{-T} = \left(Y_z^{-1}\right)^T$, $\dot{Y}_z^{-1} = \frac{\mathrm{d}}{\mathrm{d}t}\left(Y_z^{-1}\right)$ and $Y_{z_0} = Y(z(x(0)))$. Congruence with Y for expression $P < 0$ is defined as the expression $YPY^T < 0$.

3. NON–QUADRATIC STABILITY

Theorem. (Non-quadratic stability) Assume that the initial condition $x(0)$ of the TSFS (4) satisfies $\|x(0)\| < c$. Then, the system (4) is stable if there exist matrices F_j, $P_k = P_k^T > 0$ and constants ϕ_l and μ such that

$$A_i P_k + P_k A_i^T - B_i F_j - F_j^T B_i^T - \sum_{l=1}^{r} \phi_l P_l < 0 \tag{5}$$

$$P_k - c^2 I > 0, \quad \begin{bmatrix} P_k & F_j^T \\ F_j & \mu^2 I \end{bmatrix} > 0, \quad \phi_l < \dot{h}_l(x(t)) \tag{6}$$

$$i, j, k, l \in \{1, \ldots, r\}.$$

P r o o f. Consider the Lyapunov function candidate (3). Since $\forall\, k,\ P_k > 0$ and $h_k(z(x(t))) > 0$, then $P_z > 0, \forall\, z(x(t))$. Let $\underline{\lambda}[M]$ and $\overline{\lambda}[M]$ denote the smallest and highest eigenvalue of matrix M respectively. Since $\underline{\lambda}[P_z] = \overline{\lambda}[P_z^{-1}]$ and $\overline{\lambda}[P_z] = \underline{\lambda}[P_z^{-1}]$ then $P_z^{-1} > 0 \Rightarrow V(x(t)) > 0$. Then

$$\begin{aligned} \dot{V}(x(t)) &= \dot{x}^T(t)P_z^{-1}x(t) + x^T(t)P_z^{-1}\dot{x}(t) + x^T(t)\dot{P}_z^{-1}x(t) \\ &= x^T(t)(A_z - B_z F_z P_z^{-1})^T P_z^{-1} x(t) \\ &\quad + x^T(t)P_z^{-1}(A_z - B_z F_z P_z^{-1})\, x(t) \\ &\quad + x^T(t)\dot{P}_z^{-1}x(t) \\ &= x^T(t)\big[(A_z - B_z F_z P_z^{-1})^T P_z^{-1} \\ &\quad + P_z^{-1}(A_z - B_z F_z P_z^{-1}) + \dot{P}_z^{-1}\big]x(t). \end{aligned}$$

$\dot{V}(x(t)) < 0$ is implied by

$$(A_z - B_z F_z P_z^{-1})^T P_z^{-1} + P_z^{-1}(A_z - B_z F_z P_z^{-1}) + \dot{P}_z^{-1} < 0. \tag{7}$$

Congruence with P_z gives

$$P_z A_z^T - F_z^T B_z^T + A_z P_z - B_z F_z + P_z \dot{P}_z^{-1} P_z < 0. \tag{8}$$

Noticing that

$$-\dot{P}_z \quad = \quad \frac{\mathrm{d}}{\mathrm{d}t}(P_z P_z^{-1})P_z - \dot{P}_z = P_z \dot{P}_z^{-1} P_z + \dot{P}_z P_z^{-1} P_z - \dot{P}_z = P_z \dot{P}_z^{-1} P_z \tag{9}$$

the inequality (8) turns into

$$\begin{aligned} & P_z A_z^T - F_z^T B_z^T + A_z P_z - B_z F_z - \dot{P}_z \\ & \quad < P_z A_z^T - F_z^T B_z^T + A_z P_z - B_z F_z - P_\phi < 0 \end{aligned} \tag{10}$$

providing that

$$\dot{P}_z = \sum_{l=1}^{r} \dot{h}_l(z(t)) P_l > \sum_{l=1}^{r} \phi_l P_l = P_\phi, \tag{11}$$

where

$$\begin{aligned} \dot{h}_l &> \min_{x,u} \left[\frac{\partial h_l}{\partial z} \frac{\partial z}{\partial x} \frac{\mathrm{d}x}{\mathrm{d}t} \right] = \min_{x,u} \left[\frac{\partial h_l}{\partial z} \frac{\partial z}{\partial x} (A_z x + B_z u) \right] \\ &= \min_{x} \left[\frac{\partial h_l}{\partial z} \frac{\partial z}{\partial x} (A_z x \pm B_z \mu) \right] = \dot{h}_{\min} > \phi_l \end{aligned} \tag{12}$$

with

$$\|u(t)\| < \mu. \tag{13}$$

Inequality (10) can be expressed as (5) and it remains to guarantee inequality (13). To do so, recall that $\|x(0)\| < c$, which can be expressed as

$$\|x(0)\|^2 < c^2 \Leftrightarrow \frac{1}{c^2} x^T(0)\, x(0) < 1. \tag{14}$$

Since the Lyapunov function candidate (3) has been proved to be a positive function which monotonically decreases, we can assume without loss of generality that

$$V(x(t)) \quad < \quad V(x(0)) = x^T(0) P_{z_0}^{-1} x(0) < \frac{1}{c^2} x^T(0)\, x(0) < 1 \tag{15}$$

which is equivalent to $P_z - c^2 I > 0$ and can be rewritten as the first inequality in (6).

Condition $\|u(t)\| < \mu$ can be rewritten by means of (2) as follows:

$$\begin{aligned} u^T(t) u(t) \quad &= \quad x^T(t) P_z^{-T} F_z^T F_z P_z^{-1} x(t) < \mu^2 \\ \Leftrightarrow \quad & \frac{1}{\mu^2} x^T(t) P_z^{-T} F_z^T F_z P_z^{-1} x(t) < 1. \end{aligned} \tag{16}$$

Recalling (15), it is clear that the previous inequality holds if

$$\frac{1}{\mu^2} x^T(t) P_z^{-T} F_z^T F_z P_z^{-1} x(t) < x^T(t) P_z^{-1} x(t) = V(x(t)).$$

This condition is equivalent to

$$x^T(t) \left[\frac{1}{\mu^2} P_z^{-T} F_z^T F_z P_z^{-1} - P_z^{-1} \right] x(t) < 0. \tag{17}$$

Applying congruence with P_z and rearranging terms one gets that the previous inequality is implied by

$$P_z - \frac{1}{\mu^2} F_z^T F_z > 0 \tag{18}$$

and by Schur complement

$$\begin{bmatrix} P_z & F_z^T \\ F_z & \mu^2 I \end{bmatrix} > 0 \tag{19}$$

which is implied by the second inequality in (6). □

Remark 1. The previous stability analysis is semi-global, since it depends of the range considered by the membership functions. In other words, the wider the membership functions are, the more global the stability analysis is.

Remark 2. The previous controller design allows input dependence of the membership functions' derivatives, since input is constrained by constant μ. This makes easier the estimation of constants ϕ_l, $l = 1, \dots, r$.

Remark 3. Conditions (5) and (6) in Theorem 3 are LMIs if constants ϕ_l, $l = 1, \dots, r$ are given. Inequality (12) shows that constants ϕ_l can be found through the minimum of $\dot{h}_l$ in which μ is fixed.

Remark 4. Note that if $\dot{h}_l$, $l = 1, \dots, r$ do not depend on input $u(t)$, then conditions (5) guarantee stability of system (4) and conditions (6) can be dropped.

Remark 5. Note that if $P_z = P$, i. e. if the Lyapunov function is quadratic with constant matrix P, then conditions (5) reduce to stability conditions found in the common Lyapunov function approach [14]. Then, the present approach actually includes the common Lyapunov function one as a particular case, so it reduces conservativeness.

4. EXAMPLE

Consider the following continuous-time Takagi–Sugeno fuzzy system [15]:

$$\dot{x}(t) = \sum_{i=1}^{2} h_i(z(x(t))) \left[A_i x(t) + B_i u(t) \right] \tag{20}$$

where

$$A_1 = \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 10 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix},$$

$$z = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \; h_1(z(x(t))) = \frac{1+\sin(z_1(t))}{2}, \; h_2(z(x(t))) = \frac{1-\sin(z_1(t))}{2}$$

Since $z(t) = x(t)$, $z(t)$ will be omitted in the sequel. Assuming $|x_i(t)| < \pi/4$, $i = 1, 2$, it can be found that:

$$\begin{aligned}
\dot{h}_1(x(t)) &= -\dot{h}_2(x(t)) = \frac{1}{2}\cos x_1(t)\dot{x}_1 \\
&= \frac{1}{2}\cos x_1(t)\left\{\frac{1+\sin(x_1(t))}{2}[-5 \;\; -4]\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}\right. \\
&\quad \left. +\frac{1-\sin(x_1(t))}{2}[-2 \;\; -4]\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \left(\frac{1+\sin(x_1(t))}{2}\right)u(t)\right\} \\
&= \cos x_1(t)\left(-\frac{7}{4}x_1(t) - 2x_2(t)\right) \qquad (21) \\
&\quad -\frac{3}{4}x_1(t)\sin x_1(t)\cos x_1(t) + \cos x_1(t)\left(\frac{1+\sin(x_1(t))}{4}\right)u(t) \\
&> \cos x_1(t)\left(-\frac{7}{4}x_1(t) - 2x_2(t)\right) \qquad (22) \\
&\quad -\frac{3}{4}x_1(t)\sin x_1(t)\cos x_1(t) \pm \cos x_1(t)\left(\frac{1+\sin(x_1(t))}{4}\right)\mu \\
&> -3.05 = \phi_1,
\end{aligned}$$

as long as $\|u(t)\| < \mu = 2$.

Similarly, it can be checked that $\dot{h}_2(x(t)) > \phi_2 = -3.05$.

Applying Theorem 1 with the previous constants ϕ_l, $l = 1, 2$ and $\mu = 2$, matrices P_k and F_j, $j, k \in 1, 2$ were found, so control law (2) stabilizes system (20) as shown in Figure 1. Note that the existing approaches [6, 15] are unable to stabilize this system through non-quadratic approach, since $\dot{h}_l(t)$ depends on input $u(t)$, so the presented scheme represents a significant extension.

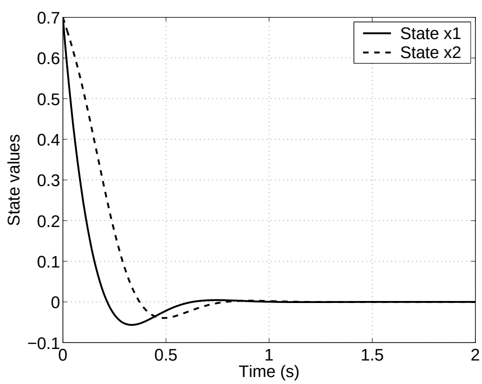

Fig. 1. Evolution of the states x_1 and x_2.

In Figure 2, the control input signal which stabilizes the system is showed. Note that constraint $\|u(t)\| < \mu = 2$ holds since this requirement was implicit in the proposed design, though it was employed to establish constants ϕ_l, $l = 1, 2$ rather than to reduce the magnitude of the control signal.

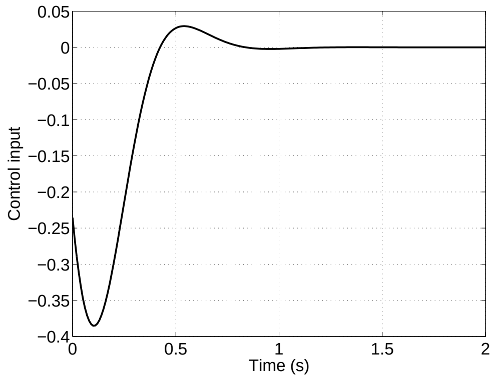

Fig. 2. Evolution of the control input signal $u(t)$.

Finally, Figure 3 shows the behavior of the found non-quadratic Lyapunov function.

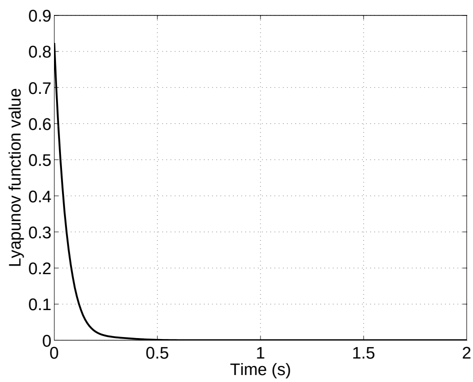

Fig. 3. Evolution of the non-quadratic Lyapunov function $V(x(t))$.

5. CONCLUSION

This paper fully develops a non-quadratic fuzzy design, which permits state and input dependence of the membership functions' derivatives, as well as independence on initial conditions when input constraints are needed. The proposed design employs a non-quadratic Lyapunov function with a non-PDC control law, which is proved to reduce conservativeness compared with common Lyapunov functions. A simulation example is provided to illustrate the design procedure and performance.

ACKNOWLEDGEMENT

This work has been supported by the program TALENT GACR 102/03/H116 from the Ministry of Education, Youth and Sports of the Czech Republic and by the Mexican National Council of Science and Technology (CONACYT) via scholarship 121109.

(Received May 17, 2005)

REFERENCES

[1] X. Apkarian and X. Gahinet: A convex characterization of gain scheduling Hinf controllers. IEEE Trans. Control Systems *40* (1995), 853–864.

[2] O. Begovich, E. N. Sanchez, and M. Maldonado: Takagi–Sugeno fuzzy scheme for real trajectory tracking of an underactuated robot. IEEE Trans. Control Systems Techn. *10* (2002), 14–20.

[3] M. Bernal and P. Hušek: Piecewise quadratic stability of affine Takagi–Sugeno fuzzy control systems. In: Proc. Advanced Fuzzy-Neural Control Conference, Oulu 2004, pp.157–162.

[4] M. Bernal and P. Hušek: Controller synthesis with input and output constraints for fuzzy systems. In: 16th IFAC World Congress DVD-edition, Prague 2005.

[5] M. Bernal: Non-quadratic discrete fuzzy controller design performing decay rate. In: FUZZ–IEEE Internat. Conference, Reno 2005, CD edition.

[6] M. Bernal, P. Hušek, and V. Kučera: Non-quadratic design for continuous-time systems in the Takagi–Sugeno form. Submitted to Automatica.

[7] G. Feng: Controller synthesis of fuzzy dynamic systems based on piecewise Lyapunov functions. IEEE Trans. Fuzzy Systems *11* (2003), 605–612.

[8] G. Feng: Stability analysis of discrete time fuzzy dynamic systems based on piecewise Lyapunov functions. IEEE Trans. Fuzzy Systems *12* (2004), 22–28.

[9] T. M. Guerra and L. Vermeiren: LMI-based relaxed non-quadratic stabilization conditions for nonlinear systems in Takagi–Sugeno's form. Automatica *40* (2004), 823–829.

[10] M. Johansson, A. Rantzer, and K. Arzen: Piecewise quadratic stability of fuzzy systems. IEEE Trans. Fuzzy Systems *7* (1999), 713–722.

[11] A. Rantzer and M. Johansson: Piecewise linear quadratic optimal control. IEEE Trans. Automat. Control *45* (2000), 629–637.

[12] T. Takagi and M. Sugeno: Fuzzy identification of systems and its applications to modeling and control. IEEE Trans. Systems, Man and Cybernet. *15* (1985), 116–132.

[13] K. Tanaka and M. Sugeno: Stability analysis of fuzzy systems using Lyapunov's direct method. In: Proc. NAFIPS'90, pp. 133–136.

[14] K. Tanaka, T. Ikeda, and H. O. Wang: Fuzzy regulators and fuzzy observers: Relaxed stability conditions and lmi-based designs. IEEE Trans. Fuzzy Systems *6* (1998), 250–264.

[15] K. Tanaka, T. Hori, and H. O. Wang: A multiple function approach to stabilization of fuzzy control systems. IEEE Trans. Fuzzy Systems *11* (2003), 582–589.

[16] H. O. Wang , K. Tanaka, and M. Griffin: An approach to fuzzy control of nonlinear systems: Stability and design issues. IEEE Trans. Fuzzy Systems *4* (1996), 14–23.

Miquel Bernal, Petr Hušek, and Vladimír Kučera, Faculty of Electrical Engineering, Czech Technical University in Prague, Technická 2, 166 27 Praha 6. Czech Republic.

e-mails: xbernal@control.felk.cvut.cz, husek@fel.cvut.cz, kucera@fel.cvut.cz