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INFINITE QUEUEING SYSTEM WITH TREE STRUCTURE

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We focus on invariant measures of an interacting particle system in the case when the set of sites, on which the particles move, has a structure different from the usually considered set $\mathbb{Z}^d$. We have chosen the tree structure with the dynamics that leads to one of the classical particle systems, called the zero range process. The zero range process with the constant speed function corresponds to an infinite system of queues and the arrangement of servers in the tree structure is natural in a number of situations.

The main result of this work is a characterisation of invariant measures for some important cases of site-disordered zero range processes on a binary tree. We consider the single particle law to be a random walk on the binary tree. We distinguish four cases according to the trend of this random walk for which the sets of extremal invariant measures are completely different. Finally, we shall discuss the model with an external source of customers and, in this context, the case of totally asymmetric single particle law on a binary tree.

Keywords: invariant measures, zero range process, binary tree, queues

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1. INTRODUCTION OF MODEL

Let us consider infinitely many servers located one by one in the nodes of a full rooted binary tree T (an infinite countable graph whose each node $x \in T$, except the root $r \in T$ which has no ancestor, has exactly three neighbours: its *ancestor* (parent) x^-, its *left descendant* (child) x^+ and its *right descendant* (child) x_+). There is an arbitrary number of customers in this system which form arbitrary large but finite queues at these servers. If there is more than one customer in the queue at server $x \in T$ we assume that one of them is being served and the others are waiting. After an exponential service time with rate $\lambda_x > 0$ (it means with the expected value $1/\lambda_x$) the customer leaves server x and randomly chooses one of neighbouring servers to join the queue at it. The choice is made with respect to probabilities $p(x, y)$, $y \in T$. The service times in different queues are mutually independent and customers are assumed to be indistinguishable.

We are interested just in the length $\eta(x) \in \mathbb{N}$ of queue at each server $x \in T$. Let us denote by $\eta = (\eta(x) : x \in T)$ one particular configuration of the whole queueing system. Since the evolution of the system is given by individual jumps between servers, an actual configuration η can be only changed if one customer leaves server x for a queue at different server y. We denote this changed configuration by

$$\eta^{xy}(z) = \begin{cases} \eta(x) - 1 & \text{if } z = x \\ \eta(y) + 1 & \text{if } z = y \\ \eta(z) & \text{otherwise.} \end{cases}$$

The transition $\eta \longmapsto \eta^{xy}$ for some $x \neq y$ is the only possible transition in one jump.

We suppose that this system evolves in time and its description is given by the length $\eta_t(x)$ of queue at each server x at each time t. Since its dynamics is random and follows from exponentially distributed service times, we can expect the system to be a time-continuous stochastic process with the Markov property . We denote by $\theta(\eta, \zeta)$ the *transition rates* for every $\eta \neq \zeta$. According to the previous description we put

$$\begin{aligned} \theta(\eta, \zeta) &= \theta(\eta, \eta^{xy}) && \text{if } \zeta = \eta^{xy} \text{ for some } x \neq y \\ &= 0 && \text{otherwise,} \end{aligned}$$

for every $\eta \neq \zeta$, where

$$\theta(\eta, \eta^{xy}) = \mathrm{I}_{[\eta(x)>0]} \lambda_x p(x, y)$$

for every $x \neq y$.

This system of queues coincides with a classical particle system, the so-called *zero range process* which was introduced in [7]. We shall use the notation typical for particle systems. Note that the zero range process is usually defined more generally in the following sense:

$$\theta(\eta, \eta^{xy}) = g(\eta(x)) \lambda_x p(x, y)$$

where $g : \mathbb{N} \to [0, \infty)$, $g(0) = 0$, $g(k) > 0$ otherwise, is the so-called *speed function.* Our queueing system is anyway the special case of the zero range process with the constant speed function $g(k) = \mathrm{I}_{[k>0]}$. In the context of particle systems we call family $(p(x, \cdot) : x \in T)$ of probabilities on T such that

$$p(x, x) = 0, \ \sum_y p(x, y) = 1 \text{ for every } x \in T \text{ and}$$

$$\forall\, x \neq y \quad \exists x = x_0, x_1, \ldots, x_n = y : \quad \prod_{i=1}^{n} p(x_{i-1}, x_i) + \prod_{i=1}^{n} p(x_i, x_{i-1}) > 0$$

the single particle law and we call service rates $(\lambda_x : x \in T)$ also *leaving rates* or an *environment* and we assume that

$$\text{there exists a constant } \Lambda \text{ such that } 0 < \lambda_x \leq \Lambda \text{ for every } x \in T. \tag{1.1}$$

In addition, we shall assume that the only possible movement of customers is between servers which are neighbours on the tree. Recall that each $x \in T$ different from the root r has exactly three neighbours which we denoted x^-, x^+ and x_+. The neighbours of the root r are just r^+, r_+. If we use notation $x \sim y$ for "x and y are neighbours" then it means

$$p(x, y) = 0 \quad \text{if} \quad x \not\sim y. \tag{1.2}$$

Notice that $\sim$ is a symmetric and antireflexive relation on T. We shall denote by $|x|$ the level of tree in which node x lies and the level of the root is assumed to be zero.

Definition 1.1. Let T be a countable set of sites and let us denote by

$$\mathfrak{X} = \mathbb{N}^T = \{\eta : T \to \mathbb{N}\}$$

the state space of configurations. Let us consider a canonical Markov process $\big(P^\eta : \eta \in \mathfrak{X}\big)$ on the set

$$\mathfrak{D} = \{\varphi : \mathbb{R}^+ \to \mathfrak{X} \text{ right continuous, having left-hand limit at each } s > 0\}.$$

If this Markov process is associated with infinitesimal generator

$$\mathcal{L}f(\eta) = \sum_{x\in T}\sum_{y\sim x} \mathrm{I}_{[\eta(x)>0]}\ \lambda_x\ p(x,y)\ [f(\eta^{xy}) - f(\eta)] \tag{1.3}$$

defined for every $\eta \in \mathfrak{X}$ and cylinder function f on $\mathfrak{X}$ then it is called *the zero range* (ZR) *process on T with the constant speed function, with nearest-neighbour single particle law $p(x,y)$ in non-homogeneous environment* $(\lambda_x : x\in T)$.

A function $f : \mathbb{N}^T \to \mathbb{R}$ is called *the cylinder function* if there exists a finite $K \subset T$ such that $f(\eta) = f(\zeta)$ holds for every $\eta, \zeta \in \mathfrak{X}$ such that $(\eta(x) = \zeta(x)\ \forall\ x \in K)$.

There is a need to guarantee the existence of the Markov process from the previous definition. One can follow approaches introduced in [1] or [3]. See [2] for a detailed proof of the existence of the zero range process with infinitesimal generator (1.3) under conditions (1.1) and (1.2).

To study a time asymptotic behaviour of particle systems means to investigate measures on $\mathfrak{X}$ which are invariant with respect to the given dynamics represented by infinitesimal generator $\mathcal{L}$. Recall that a measure μ on $\mathfrak{X}$ is invariant for the zero range process with generator $\mathcal{L}$ if and only if

$$\int \mathcal{L}f(\eta)\,\mathrm{d}\mu(\eta) = 0 \qquad \text{for every cylinder function } f \text{ on } \mathfrak{X}. \tag{1.4}$$

The following proposition gives sufficient conditions for the existence of an invariant measure and, moreover, it shows an algorithm how to construct such a measure. The constructed measures are well known in the theory of particle systems or Jackson networks and are mentioned already by Spitzer [7].

Proposition 1.2. Consider a zero range process (1.3) on the full rooted binary tree T and assume that its single particle law satisfies: $p(x,y) \neq 0$ if and only if $x \sim y$. Let $(\pi(x) : x \in T)$ be a positive solution of equations:

$$\sum_{y\in T}\Big(p(y,x)\pi(y) - p(x,y)\pi(x)\Big) = 0 \qquad \forall\, x \in T. \tag{1.5}$$

Then product measures $\nu^{\varphi\pi}$ defined on the space $\mathbb{N}^T$ by their marginal distributions:

$$\nu^{\varphi\pi}(\eta : \eta(x) = k) = \Big(\frac{\varphi}{\lambda_x}\pi(x)\Big)^k\Big(1 - \frac{\varphi}{\lambda_x}\pi(x)\Big)\ \ \forall\, k \in \mathbb{N}, \tag{1.6}$$

$\forall\, x \in T$, are invariant for this zero range process for each nonnegative constant φ satisfying

$$\varphi < \frac{\lambda_x}{\pi(x)} \qquad \forall\, x\in T.$$

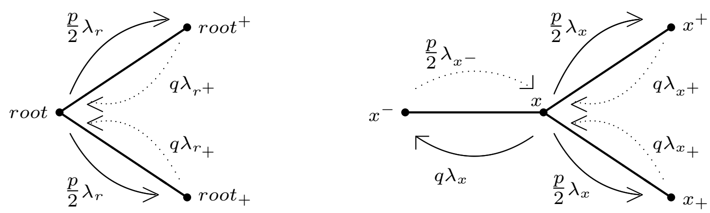
Proof. It is a straightforward verification of (1.4) for $\nu^{\varphi\pi}$. □

2. THE TIME ASYMPTOTIC BEHAVIOUR

The main result of this paper is a characterisation of invariant measures for the zero range process given by a generator (1.3) if the single particle law is a simple random walk on the binary tree. It means

$$\begin{array}{llcl} p(x,x^+) = \frac{p}{2} = p(x,x_+) & \forall\ x \in T \setminus \{r\} & \& & p(r,r^+) = \frac{1}{2} = p(r,r_+) \\ p(x,x^-) = 1-p =: q & \forall\ x \in T \setminus \{r\} & & \end{array} \tag{2.1}$$

where $p \in (0,1)$ is a parameter of this random walk on T. In order to avoid technical complications following from the different situation in the root let us slightly modify the service rate at the root and suppose that it is $p\lambda_r$ instead of λ_r.



Let us realise that in this special case balance equations (1.5) have a simple form:

$$\begin{array}{c} p\ \pi(r) = q\ (\pi(r^+) + \pi(r_+)) \\ \pi(x) = \frac{p}{2}\pi(x_-) + q(\pi(x^+) + \pi(x_+))\ \text{ for every } x \in T \setminus \{r\} \end{array} \tag{2.2}$$

and the function $\pi(x) = \left(\frac{p}{2q}\right)^{|x|}$ is a solution of them. Therefore the following consequence holds.

Corollary 2.1. Consider a zero range process (1.3) on the full rooted binary tree T with single particle law (2.1) for some $p \in (0,1)$. Then product measures ν^φ defined on the space $\mathbb{N}^T$ by their marginal distributions:

$$\nu^\varphi(\eta : \eta(x) = k) = \left(\frac{\varphi}{\lambda_x}\left(\frac{p}{2q}\right)^{|x|}\right)^k \left(1 - \frac{\varphi}{\lambda_x}\left(\frac{p}{2q}\right)^{|x|}\right)\ \ \forall\, k \in \mathbb{N}, \tag{2.3}$$

$\forall\, x \in T$, are invariant for this zero range process for each $\varphi \in \Phi_\lambda$, where

$$\Phi_\lambda = \{\varphi \geq 0 : \varphi < \lambda_x \left(\frac{2q}{p}\right)^{|x|}\ \ \forall\, x \in T\}.$$

Corollary 2.1 gives some representative set $\mathbf{V} := \{\nu^\varphi : \varphi \in \Phi_\lambda\} \subset \mathcal{I}$ of invariant measures and our aim is to find out set $\mathcal{I}$ of all invariant measures. Since $\mathcal{I}$ is a closed and convex subset of $\mathcal{P}$ (the set of all probability measures on $\mathfrak{X}$) we can employ a non compact version of the Krein–Milman theorem, see [8], which then says that $\mathcal{I}$ is closed convex hull of its extremal elements. We say that μ is extremal invariant iff $\mu = \alpha\mu_1 + (1-\alpha)\,\mu_2$ does not hold for any $0 < \alpha < 1$ and any other invariant measures μ_1, μ_2. So the set $\mathcal{I}_e$ of all extremal invariant measures characterises all invariant measures in the sense $\mathcal{I} = \overline{\mathrm{co}}\,\mathcal{I}_e$, where $\overline{\mathrm{co}}$ means the closed convex hull in $\mathcal{P}$ endowed with the topology of weak convergence. In what follows, we shall investigate extremal invariant measures for described zero range processes which is a standard approach how to characterise the set $\mathcal{I}$ of all invariant measures.

Remark on existence of an invariant measure

Since ν^φ should be a probability measure we are led to the assumption

$$0 \le \varphi < \lambda_x \left(\frac{2q}{p}\right)^{|x|} \qquad \forall\, x \in T.$$

Surely $\varphi = 0$ gives always one invariant measure ν^0 but it is degenerate (it is the Dirac measure on the zero configuration). Corollary 2.1 gives a nondegenerate invariant measure if

$$c := \inf_x \lambda_x \left(\frac{2q}{p}\right)^{|x|} > 0. \tag{2.4}$$

Then one of the following situations arises:

- c is attained at some $x \in T$ and therefore $\Phi_\lambda = [0, c)$
- c is not attained and $\Phi_\lambda = [0, c]$.

We can see immediately that if the parameter p of the random walk is greater than $2/3$ then $\Phi_\lambda = \{0\}$ without an influence of rates λ_x. Indeed, in this case $2q/p < 1$ and since we always assume $0 < \lambda_x \le \Lambda$ then c is forced to be zero. It means Corollary 2.1 gives no nondegenerate measure in this case, $\mathbf{V}$ is a singleton containing only measure ν^0. Intuitively one can say the random walk with $p > 2/3$ goes very quickly from the root to branches.

We can distinguish four cases for this random walk, with respect to the parameter $p \in (0, 1)$. In each case the situation concerning invariant measures of the associated zero range process is very different. In order to get an idea let us first assume, in addition, $\lambda_x \equiv 1$ for every x. Then these four cases can be expressed by the following:

1. the positive recurrent case $p < q$
2. the null recurrent case $p = q = 1/2$
3. the transient case $p > q$ & $p/2 \le q$
4. the transient case $p/2 > q$.

This classification follows some known result for general zero range processes (i. e. general countable set of sites with an irreducible single particle law $p(x, y)$ such that $\lim_{|x|\to\infty} p(x, y) < \infty$ for all y, homogeneous environment $\lambda_x \equiv 1$ and partially also more general speed function) which we shortly mention here and in more details below, in a proof of Theorem 2.3.

Zero range processes with a positive recurrent single particle law were studied from the equilibrium point of view by Waymire [9] and the set of invariant measures was specified there. Two years later, Andjel [1] was studying zero range processes with rather general single particle laws (also with a more general speed function) and an exact expression of the set of extremal invariant measures was given for zero range processes with a null recurrent single particle law. Note that a completely different approach was used to obtain the result in the latter case than it was in the former one.

Case 1. is thus covered by the first reference and case 2. comes under the second one. On the other hand there is no similar general result for transient single particle law which we can apply for the remaining cases. There exist only results which declare that the product invariant measures defined by (1.6) are extremal invariant. This result was proved by Saada [5] for a transient doubly stochastic single particle law $p(x, y)$ and recently also by Sethuraman [6] for an arbitrary single particle law satisfying only that $p(x, y) + p(y, x)$ is irreducible and $\lim_{|x|\to\infty} p(x, y) < \infty$ for all y, moreover a more general speed function is considered here.

For arbitrary service rates $0 < \lambda_x \le \Lambda$ we obtain analogous classification as above by employing the mean value of marginal distributions of measure ν^φ. Let us denote by $R_{\varphi,\lambda}(x) = \frac{\varphi}{\lambda_x}\left(\frac{p}{2q}\right)^{|x|}\left(1 - \frac{\varphi}{\lambda_x}\left(\frac{p}{2q}\right)^{|x|}\right)^{-1}$ the mean value of xth marginal distribution of ν^φ. Hence the expected number of particles at level n of the binary tree and the expected total number of particles on the binary tree are

$$RL_{\varphi,\lambda}(n) = \left(\frac{p}{2q}\right)^n \sum_{|x|=n} \frac{\frac{\varphi}{\lambda_x}}{1 - \frac{\varphi}{\lambda_x}\left(\frac{p}{2q}\right)^{|x|}} \quad \& \quad RT_{\varphi,\lambda} = \sum_{n=0}^{\infty} \left(\frac{p}{2q}\right)^n \sum_{|x|=n} \frac{\frac{\varphi}{\lambda_x}}{1 - \frac{\varphi}{\lambda_x}\left(\frac{p}{2q}\right)^{|x|}}$$

A classification which generalises the mentioned cases 1.−4. is then:

(i) $p \le q$ & $RT_{\varphi,\lambda} < \infty$ for every $\varphi \in (0, c)$ & $c \ne 0$

(ii) $p \le q$ & $RT_{\varphi,\lambda} = \infty$ for some $\varphi \in (0, c)$

(iii) $p > q$ & $p/2 \le q$

(iv) $p/2 > q$ or $(p \le q$ & $c = 0)$.

This formulation allows to see that individual classes are disjoint and also describes all possibilities. But some of the conditions are redundant. Let us realise relationships among them and then simplify the matter.

Lemma 2.2.

a) If $RT_{\varphi,\lambda} < \infty$ for some $\varphi \in (0, c)$, then $c > 0$ must be attained at some x,

b) if $RT_{\varphi,\lambda} < \infty$ for some $\varphi \in (0, c)$, then $RT_{\varphi,\lambda} < \infty$ for every $\varphi \in (0, c)$,

c) if $p \geq q$, then $RT_{\varphi,\lambda} = +\infty$,

d) if $p/2 > q$, then $c = 0$.

Proof.

a) Let us assume for a contradiction that $\lambda_x(\frac{2q}{p})^{|x|} > c$ for every x. There must exist a sequence (x_n) such that $|x_n| \to \infty$ and $\lambda_{x_n}(\frac{2q}{p})^{|x_n|} \searrow c$. Then $RL_{\varphi,\lambda}(|x_n|) \geq \frac{\varphi}{\lambda_{x_n}}(\frac{p}{2q})^{|x_n|} \longrightarrow \frac{\varphi}{c} > 0$ for each n. Hence $RL_{\varphi,\lambda}(|x_n|) \not\to 0$ and so $RT_{\varphi,\lambda} = \sum_n RL_{\varphi,\lambda}(n) < \infty$ does not hold.

b) Fix $\varphi \in (0, c)$. We can express $RT_{\varphi,\lambda} = \varphi \sum\limits_{n=0}^{\infty} \sum\limits_{|x|=n} \frac{1}{\lambda_x(\frac{2q}{p})^{|x|} - \varphi}$ and find out that

$$\varphi \sum_{n=0}^{\infty} \sum_{|x|=n} \frac{1}{\lambda_x\left(\frac{2q}{p}\right)^{|x|}} \leq RT_{\varphi,\lambda} \leq \varphi \frac{\varphi+\varepsilon}{\varepsilon} \sum_{n=0}^{\infty} \sum_{|x|=n} \frac{1}{\lambda_x\left(\frac{2q}{p}\right)^{|x|}}$$

for every $0 < \varepsilon < c - \varphi$ because function $\frac{1}{x-\varphi}$ is bounded from above by $\frac{\varphi+\varepsilon}{\varepsilon}\frac{1}{x}$ for every $x \geq \varphi + \varepsilon$, $\varepsilon > 0$. If $RT_{\psi,\lambda} < \infty$ for some $\psi \in (0, c)$ then also sum $\sum_{n=0}^{\infty} \sum_{|x|=n} \frac{1}{\lambda_x(\frac{2q}{p})^{|x|}}$ converges. Since for every $\varphi \in (0, c)$, there exists some $0 < \varepsilon < c - \varphi$, then $RT_{\varphi,\lambda}$ has to converge because of a convergent majorant.

c) Since $\sup_x \lambda_x \leq \Lambda$ then $\frac{\varphi}{\Lambda} \sum_n (\frac{p}{q})^n$ bounds $RT_{\varphi,\lambda}$ from below.

d) It is a direct consequence of (2.4) and the assumption $\sup_x \lambda_x \leq \Lambda$. □

We can therefore simplify the classes in this way:

(i) $RT_{\varphi,\lambda} < \infty$ for some $\varphi \in (0, c)$

(ii) $p \leq q$ & $RT_{\varphi,\lambda} = \infty$ for some $\varphi \in (0, c)$

(iii) $p > q$ & $p/2 \leq q$

(iv) $c = 0$ & $(p \leq q$ or $p > 2q)$

and investigate invariant measures for these cases in the following theorem.

Theorem 2.3. Consider a zero range process (1.3) on the full rooted binary tree T with simple random walk (2.1) for some $p \in (0,1)$ as the single particle law. According to parameter p and environment $(\lambda_x : x \in T)$ we distinguish four cases (i) – (iv) described above. For each of them we obtain the following result with corresponding numbering:

(iv) $\mathcal{I} = \{\nu^0\}$, i. e. measure ν^0 sitting on the null configuration is the only invariant measure,

(i) $\mathcal{I}_e = \{\nu_K : K = 0, 1, 2, \ldots\}$, where each ν_K defined on $\mathbb{N}^T$ by

$$\nu_K = \nu^\varphi(\cdot \mid \textstyle\sum_{x\in T} \eta(x) = K) \text{ for some } \varphi > 0$$

concentrates on configurations with finite number K of particles and independent of φ,

(ii) $\mathcal{I}_e = \{\nu^\varphi : \varphi \in \Phi_\lambda\}$,

(iii) $\mathcal{I}_e \supsetneq \{\nu^\varphi : \varphi \in \Phi_\lambda\}$ if $c > 0$, furthermore, there exists an infinitely dimensional set of product measures which are extremal invariant.

3. PROOF OF THE MAIN THEOREM 2.3

Case (iv)

Within item (iv), we are interested in two cases: the recurrent random walk $p \le q$ when $c = 0$ and the transient random walk when $p > 2q$ (here $c = 0$ holds always). Anyway, we have $c = 0$ and therefore Corollary 2.1 gives one invariant measure ν^0, degenerated to the null configuration. We are going to prove that for both cases it is the only invariant measure. Notice that we no longer obtain the same result for the case $q < p \le 2q$ and for this reason we shall describe the case $c = 0$ & $q < p \le 2q$ as a part of (iii).

Proof of Theorem 2.3 (iv). Let μ be an arbitrary invariant measure. If we denote $a(x) = \lambda_x \mu(\eta(x) > 0)$ for every $x \in T$ then a satisfies the set of balance equations (2.2). This fact follows from equality $\int \mathcal{L} f_x \mathrm{d}\mu = 0$ used for cylinder functions f_x defined by $f_x(\eta) = \eta(x)$.

If $(a(x) : x \in T)$ satisfy (2.2) then $a_n := \sum_{|x|=n} a(x)$ for every $n \ge 0$ must satisfy

$$\begin{aligned} pa_0 &= qa_1 \\ a_n &= pa_{n-1} + qa_{n+1} \quad \text{for } n \ge 1. \end{aligned} \tag{3.1}$$

Nevertheless (3.1) has a solution:

$$a_n = \psi \left(\frac{p}{q}\right)^n \quad \text{for every } n \ge 0$$

unique up to a multiplicative constant $\psi \ge 0$. It means

$$\frac{1}{2^n} \sum_{|x|=n} \lambda_x \mu(\eta(x) > 0) = \psi \left(\frac{p}{2q}\right)^n \quad \text{for every } n \ge 0. \tag{3.2}$$

If $p > 2q$ then the right hand side of the last formula is approaching infinity as $n \to \infty$ and $\psi > 0$ but the left hand side is bounded by Λ. Thus necessarily $\psi = 0$ and hence also $\mu(\eta(x) > 0) = 0$ for every x. It implies $\mu = \nu^0$ and the result for case $p > 2q$ is proved.

What remains to show is the case when $p \leq q$. We are going to prove that under the assumption $p \leq q$ the set of equations (2.2) has only one nonnegative bounded solution and it is $a(x) = \psi(\frac{p}{2q})^{|x|}$ for every $x \in T$.

Let us assume for a contradiction that there exists a smallest $n \geq 1$ and $x : |x| = n$ which does not satisfy $a(x) = \psi(\frac{p}{2q})^n$. It means there exist $\delta > 0$ and $y : |y| = n$ such that $a(y) = \psi(\frac{p}{q})^n(\frac{1}{2^n} - \delta)$.

Using an induction along k we obtain from (2.2) for every $k \geq 1$

$$\sum_{x \in S(y),\ |x|=n+k} a(x) = \psi \left(\frac{p}{q}\right)^{n+k} \Big(\frac{1}{2^n} - \delta A_k\Big),$$

where $S(y)$ stands for the set of successors of node y and where for every $k \geq 1$

$$A_k = -\frac{p}{q-p} + \frac{q}{q-p}\left(\frac{q}{p}\right)^k \qquad \text{if } p \neq q$$

$$A_k = k+1 \qquad \text{if } p = q.$$

For $p \leq q$ it means $A_k \stackrel{k\to\infty}{\longrightarrow} +\infty$. Thus there exists k such that $\sum_{\substack{x \in S(y) \\ |x|=n+k}} a(x) < 0$ which leads to a contradiction because $a(x)$ are assumed to be nonnegative.

We obtained that whenever μ is invariant then under assumption $p \leq q$

$$\mu(\eta(x) > 0) = \frac{\psi}{\lambda_x}\left(\frac{p}{2q}\right)^{|x|} \qquad \forall\, x \in T \tag{3.3}$$

for some nonnegative parameter ψ. Since $\inf_x \lambda_x (2q/p)^{|x|} = 0$ is assumed and μ is probabilistic, then $\psi = 0$ must hold and therefore $\mu = \nu^0$. □

Case (i)

Let us analyse the product measures ν^φ from Corollary 2.1 for this case. Recall from Lemma 2.2 that $\Phi_\lambda = [0, c)$. The assumption that the expected total number of particles $RT_{\varphi,\lambda}$ is finite implies

$$\nu^\varphi\Big(\sum_{x \in T} \eta(x) = +\infty\Big) = 0, \tag{3.4}$$

for every $\varphi \in \Phi_\lambda$. Indeed, $\nu^\varphi(\sum_{x \in T} \eta(x) = +\infty)$ is for arbitrary $K > 0$ bounded from above by

$$\lim_{n\to\infty} \int_{\{\sum_{|x|\leq n} \eta(x) \geq K\}} \nu^\varphi(\mathrm{d}\eta) \leq \lim_{n\to\infty} \mathsf{E}_{\nu^\varphi} \frac{1}{K} \sum_{|x|\leq n} \eta(x) = \frac{1}{K}\mathsf{E}_{\nu^\varphi} \sum_{x \in T} \eta(x) = \frac{1}{K} RT_\varphi.$$

From (3.4) we immediately obtain

$$1 = \nu^\varphi\Big(\bigcup_K \Big\{\sum_{x\in T}\eta(x) = K\Big\}\Big) = \sum_K \nu^\varphi\Big(\sum_{x\in T}\eta(x) = K\Big)$$

and for each measurable $A \subseteq \mathbb{N}^T$

$$\nu^\varphi A = \sum_K \nu^\varphi\Big(A \,\Big|\, \sum_{x\in T}\eta(x) = K\Big)\nu^\varphi\Big(\sum_{x\in T}\eta(x) = K\Big) = \sum_K \alpha_K^\varphi \nu_K(A)$$

where we denote by $\alpha_K^\varphi = \nu^\varphi\Big(\sum_{x\in T}\eta(x) = K\Big)$. It means

$$\nu^\varphi \in \overline{co}\{\nu_K : K \in \mathbb{N}\}.$$

Zero range processes in homogeneous environment whose single particle law is a positive recurrent chain were solved in [9] under assumptions: speed function $g(k) = \mathrm{I}_{[k>0]}$, an arbitrary countable set of sites with an irreducible positive recurrent transition probability matrix $\big(p(x,y) : x,y \in T\big)$ satisfying $\sup_y \sum_x p(x,y) < \infty$. In our context, if we assume a homogeneous environment and case (i) then the result of [9] gives, first, that the clustering of particles will occur at the root if we start from a configuration η such that $\sum_x \eta(x) = \infty$, and secondly, that the measures ν_K carried on configurations with exactly a finite number K of particles are extremal invariant measures.

The key to prove analogous results also for a non-homogeneous environment is to fulfil condition $RT_{\varphi,\lambda} < \infty$. Note that the clustering now will occurs in each z such that $\lambda_z(\frac{2q}{p})^{|z|} = c$ if we start from a configuration with infinitely many particles.

P r o o f of T h e o r e m 2.3 (i). It is then the same as in [9, Th.2.15]. □

3.1. Case (ii)

Recall that we obtained set $\mathbf{V} \subset \mathcal{I}$ of special product measures from Corollary 2.1. In this paragraph we prove that $\mathbf{V} = \mathcal{I}_e$ under assumptions: $p \le q$, environment $(\lambda_x : x \in T)$ is such that $RT_{\varphi,\lambda} = \infty$ for some $\varphi \in (0, c)$ and therefore $c > 0$.

This statement in the special case when $\lambda_x \equiv 1$ is covered by [1, Theorem 1.10] which sets all extremal invariant measures for a zero range processes with a null-recurrent irreducible single particle law in homogeneous environment. It means just case $p = \frac{1}{2}$ in our context. To prove Theorem 2.3 (ii), we use the same approach as [1, Th.1.10] but simplified by the recent result of [6, Th.1.4], from which the validity of one inclusion $\mathbf{V} \subset \mathcal{I}_e$ immediately follows.

It means it is enough to prove the inclusion $\mathcal{I}_e \subset \mathbf{V}$. Let us outline the main idea of the proof. Firstly, we introduce a suitable partial ordering $\le$ on the set of all probability measures on $\mathfrak{X}$:

$$\mu_1 \le \mu_2 \quad \text{iff} \quad \int f \,\mathrm{d}\mu_1 \le \int f \,\mathrm{d}\mu_2 \quad \text{for every } f \in \mathcal{M} \tag{3.5}$$

where $\mathcal{M} := \{f \text{ bounded cylinder function on } \mathfrak{X} \text{ s.t.}: \ \eta \le \zeta \Rightarrow f(\eta) \le f(\zeta)\}$, with an aim to prove that each $\mu \in \mathcal{I}_e$ is comparable in this ordering with each $\nu \in \mathbf{V}$. As a tool to prove it we employ the so called *coupling process*.

Definition 3.1. Let us consider a zero range process given by the infinitesimal generator (1.3). Then the *coupling process* with respect to this zero range process is a Markov process with the state space $\mathfrak{X} \times \mathfrak{X}$ defined by the infinitesimal generator

$$\begin{aligned} \tilde{\mathcal{L}}\tilde{f}(\eta,\zeta) = \sum_{x,y}\Big[\quad & \lambda_x p(x,y)\ 1_{[\eta(x)>0,\zeta(x)>0]}\ \big(\tilde{f}(\eta^{xy},\zeta^{xy}) - \tilde{f}(\eta,\zeta)\big) \\ + \quad & \lambda_x p(x,y)\ 1_{[\eta(x)>0,\zeta(x)=0]}\ \big(\tilde{f}(\eta^{xy},\zeta) - \tilde{f}(\eta,\zeta)\big) \\ + \quad & \lambda_x p(x,y)\ 1_{[\eta(x)=0,\zeta(x)>0]}\ \big(\tilde{f}(\eta,\zeta^{xy}) - \tilde{f}(\eta,\zeta)\big)\ \Big] \end{aligned} \tag{3.6}$$

for every cylinder function $\tilde{f}$ on $\mathfrak{X} \times \mathfrak{X}$.

It is a well known fact (cf. the Strassen theorem [4]) that the existence of a probability measure $\tilde{\nu}$ on $\mathfrak{X} \times \mathfrak{X}$ such that μ_1 is its first marginal, μ_2 is its second marginal and $\tilde{\nu}((\eta,\zeta) : \eta \leq \zeta) = 1$, implies $\mu_1 \leq \mu_2$, where $\mu_1,\ \mu_2$ are arbitrary probability measures on $\mathfrak{X}$. We also employ a property of zero range processes with a nondecreasing speed function which is called *attractiveness* and means:

$$\eta \leq \zeta \quad \Rightarrow \quad P^{(\eta,\zeta)}(\eta_t \leq \zeta_t) = 1, \forall\, t \geq 0.$$

Here $P^{(\eta,\zeta)}$ stands for the distribution of coupling process (η_t,ζ_t) started at $(\eta,\zeta) \in \mathfrak{X} \times \mathfrak{X}$. This result is a direct consequence of the zero range dynamics and the construction of a coupling process; by (3.6) both coordinates of coupling process (η_t,ζ_t) follow the same (random) schema of jumping, i. e. an opportunity to jump from a site x to another site y at a time t arises always in both coordinates. It means that the jump does not occur simultaneously in both coordinates only if site x is empty for one of the coordinates. Hence if coordinates are ordered at the start such that $\eta \leq \zeta$ then this relation remains true in every time $t > 0$.

Finally, since the set $\mathbf{V}$ is fully ordered by $\leq$ (i. e. $\varphi_1 \leq \varphi_2 \Rightarrow \nu^{\varphi_1} \leq \nu^{\varphi_2}$) then it is not difficult to prove that arbitrary $\mu \in \mathcal{I}_e$ must be exactly equal to some measure from $\mathbf{V}$.

P r o o f o f T h e o r e m 2.3 (ii). Let us fix an arbitrary $\mu \in \mathcal{I}_e$ and an arbitrary $\nu^\varphi \in \mathbf{V}$. We are looking for a measure $\tilde{\nu^\varphi}$ on $\mathfrak{X} \times \mathfrak{X}$ with μ as its first marginal, ν as its second marginal. In [1, Lemma 4.2] such a measure $\tilde{\nu^\varphi}$ is constructed; which is, in addition, invariant for the coupling process (3.6).

The second step is nontrivial and can be found again in [1, Propositions 6.2, 6.3]. Note that the rates λ_x played no role yet. Employing basic assumption that simple random walk $p(x,y)$ is recurrent (i. e. $p \leq q$) we can prove that the above defined measure $\tilde{\nu^\varphi}$ satisfies:

$$\tilde{\nu^\varphi}((\eta,\zeta) : \eta \leq \zeta \text{ or } \eta \geq \zeta) = 1. \tag{3.7}$$

In the third step, we already need assumption $RT_{\varphi,\lambda} = \infty$. Under this assumption [6, Theorem 1.4] affirms that each measure $\nu^\varphi \in \mathbf{V}$ is extremal invariant. Nevertheless for μ and ν^φ both extremal invariant, one can find a coupling measure $\tilde{\nu^\varphi}$ as in the first step which is, moreover, extremal invariant for the coupling process. Then from the extremity of $\tilde{\nu^{\varphi}}$ and due to attractiveness result (3.7) implies

$$\tilde{\nu^{\varphi}}((\eta,\zeta):\eta\le\zeta)=1 \quad \text{or} \quad \tilde{\nu^{\varphi}}((\eta,\zeta):\eta\ge\zeta)=1. \tag{3.8}$$

See [1, Lemmas 4.3, 4.5] for details. Let us summarise the result of the first three steps in the following lemma.

Lemma 3.2. Under assumptions $p\le q$ & $RT_{\varphi,\lambda}=\infty$ for every $0<\varphi\in\Phi_\lambda$ either $\mu\le\nu^{\varphi}$ or $\mu\ge\nu^{\varphi}$.

Proposition 3.3. Under the assumption of the previous lemma the following holds: if $\mu\in\mathcal{I}_e$ then there exists $\varphi\in\Phi_\lambda$ such that $\mu=\nu^{\varphi}$.

Recall $\Phi_\lambda=[0,c)$, if infimum c is attained, or $\Phi_\lambda=[0,c]$, if it is not. The following proof will show exactly the differences between these cases. Note that this step is not needed in case of homogeneous environment and in it our proof differs from Andjel's proof of Theorem [1, Th.1.10].

P r o o f o f P r o p o s i t i o n 3.3. Let us show that the statement of Lemma 3.2 implies that one of the following situations must arise: either

$$\text{there exists } 0\le\varphi<c \text{ such that } \mu=\nu^{\varphi} \tag{3.9}$$

or

$$\nu^{\varphi}\le\mu \quad \text{for all } 0\le\varphi<c. \tag{3.10}$$

Towards a contradiction, assume that neither (3.9) nor (3.10) is satisfied. It means

$$\exists\psi<c\ :\mu\le\nu^{\psi} \qquad \& \qquad \forall\, 0\le\varphi<c\ :\nu^{\varphi}\ne\mu.$$

Nevertheless we know that μ is comparable with each ν^{φ}, for $0<\varphi<c$. Denote $\psi^\star:=\inf\{\psi\in[0,c):\mu\le\nu^{\psi}\}$. If $\psi^\star$ is positive then $\nu^{\varphi_n}\le\mu$ must hold for any sequence $\varphi_n\nearrow\psi^\star$, $\varphi_n<\psi^\star$. Since $\nu^{\varphi_n}\xrightarrow{w}\nu^{\psi^\star}$, then $\nu^{\psi^\star}\le\mu$. On the other hand, there is a sequence $\phi_n\ge\psi^\star$ converging to $\psi^\star$ from the right such that $\nu^{\phi_n}\ge\mu$. So $\nu^{\psi^\star}\ge\mu$. Finally $\nu^{\psi^\star}=\mu$ (because set $\mathcal{M}$ separates measures on $\mathfrak{X}$). Since it was forbidden, $\psi^\star$ must be equal to 0. So $\mu\le\nu^0$ which implies $\mu=\nu^0$, a contradiction.

Now we know that either (3.9) or (3.10) arises. The proof is finished if we are in the situation described by (3.9). If situation (3.10) arises then Φ_λ must be just $[0,c]$. Why: let us assume that x_c is such that $c=\lambda_{x_c}(2q/p)^{|x_c|}$. If we use (3.5) for cylinder functions $f(\eta)=\mathrm{I}_{[\eta(x_c)>k]}$ then we obtain for every $k\ge0$:

$$\mu(\eta:\eta(x_c)>k)\ge\nu^{\varphi}(\eta:\eta(x_c)>k)=(\varphi/c)^{k+1}\xrightarrow{\varphi\to c}1\ .$$

It means $\mu(\eta:\eta(x_c)=k)=0$ holds for every $k\in\mathbb{N}$. It is a contradiction, since we assume μ to be a probability measure on $\mathfrak{X}$.

It remains to show $\mu=\nu^c$ assuming (3.10). It is clear that $\nu^c\le\mu$ holds. It implies (due to the same argument as in the first step plus Lemma 3.2) the existence of a measure $\tilde{\nu^c}$ on $\mathfrak{T}\times\mathfrak{T}$ which is invariant for the coupling process, its first marginal is equal to μ, its second marginal is equal to ν^c and

$$\tilde{\nu^c}\big((\eta,\zeta):\eta\geq\zeta\big)=1.$$

Towards a contradiction, assume that $\mu=\nu^c$ does not hold, so there must exist $x\in T$ such that $\tilde{\nu}\Big((\eta,\zeta):\eta(x)>\zeta(x)\Big)>0$. Since $\{(\eta,\zeta):\eta(x)>\zeta(x)\}$ is a disjoint union of the sets $\{(\eta,\zeta):\eta(x)>\zeta(x)=k\}$ over $k=0,1,\dots$, there exists $k\in\mathbb{N}$ such that $\tilde{\nu}\big((\eta,\zeta):\eta(x)>\zeta(x)=k\big)>0$. We obtain by an induction

$$\begin{aligned}\tilde{\nu}\Big((\eta,\zeta):\eta(x)>\zeta(x)=0\Big)>0 \qquad &\text{and therefore also}\\ \mu(\eta:\eta(x)>0)\ >\ \nu^c(\zeta:\zeta(x)>0)=\frac{c}{\lambda_x}&\Big(\frac{p}{2q}\Big)^{|x|}.\end{aligned}\tag{3.11}$$

Since μ is invariant and we assume $p\leq q$, then we can employ (3.3) which states $\lambda_x\mu(\eta:\eta(x)>0)=\psi(p/2q)^{|x|}$ for every x, for some $0\leq\psi\leq c$. It is a contradiction with (3.11). Thus Proposition 3.3 is proved and Theorem 2.3 case (ii) as well. □

Case (iii)

We obtained by Corollary 2.1 that the product measures $\nu^\varphi\in\mathbf{V}$ are invariant. It is typical for this particular case of simple random walk $(1/2<p\leq 2/3)$ that the expected number of particles $RL_{\varphi,\lambda}(n)$ at nth level of binary tree, with respect to ν^φ, goes to infinity if $n\to\infty$.

Now unlike the previous cases set $\mathbf{V}$ is too small to describe all extremal invariant measures and therefore it is not possible in general to characterise $\mathcal{I}$ as a closed convex hull of $\mathbf{V}$. First of all notice that equations (2.2) has not the only solution $\pi_\varphi(x)=\varphi(p/2q)^{|x|}$ as it was in recurrent case $(p\leq q)$. Let us show an example of another solution.

Example 3.4. Assume $p\in(\frac{1}{2},\frac{2}{3}]$. Let us define a sequence

$$A_k=\frac{p}{2p-1}-\frac{q}{2p-1}\Big(\frac{q}{p}\Big)^k=\sum_{i=0}^{k}\Big(\frac{q}{p}\Big)^i\tag{3.12}$$

for every $k\geq 0$. Note that (3.12) is the unique solution of $A_k=qA_{k-1}+pA_{k+1}$ for every $k\geq 1$ with initial conditions $A_0=1, A_1=1/p$.

Let us consider some $0<\varepsilon\leq(2p-1)/(2p)$ and $0<\varphi\leq 1/2$.

$$\begin{aligned}&\pi_{\varphi,\varepsilon}(r)=\varphi\\ &\pi_{\varphi,\varepsilon}(r^+)=\varphi\tfrac{p}{2q}(1+2\varepsilon)\text{ and }\pi_{\varphi,\varepsilon}(r_+)=\varphi\tfrac{p}{2q}(1-2\varepsilon)\\ &\pi_{\varphi,\varepsilon}(x)=\varphi\left(\tfrac{p}{2q}\right)^{|x|}(1+2\varepsilon A_{|x|-1})\text{ for every }x\text{ which is a successor of }r^+\\ &\pi_{\varphi,\varepsilon}(x)=\varphi\left(\tfrac{p}{2q}\right)^{|x|}(1-2\varepsilon A_{|x|-1})\text{ for every }x\text{ which is a successor of }r_+,\end{aligned}$$

is a positive solution of equations (2.2), satisfying $\pi_{\varphi,\varepsilon}(x)<1$.

Note that one can modify solution π_φ as well at the successors of an arbitrary node $x \in T$ using a parameter $\varepsilon_x > 0$, in the same way as it has been done in Example 3.4 at the successors of the root with parameter $\varepsilon\,(=\varepsilon_r)$. Let $\pi_{\varphi,\varepsilon_x}$ denote solution of (2.2) obtained in this way. Since the set of all solutions of (2.2) is closed under addition and multiplication by constant then for a fixed choice of parameters $0 < \varepsilon_x \le (2p-1)/(2p)$ for each $x \in T$ the infinite countable set

$$\mathbb{B}_{(\varepsilon_x)} = \{\pi_1\} \cup \{\pi_{1,\varepsilon_x} : x \in T\}$$

of positive bounded solutions is a (Schauder) basis in the vector space of all solutions.

From Proposition 1.2 we know that the product measure ν^π defined by (1.6) is invariant for the zero range process (1.3) with an environment $(\lambda_x : x \in T)$ if

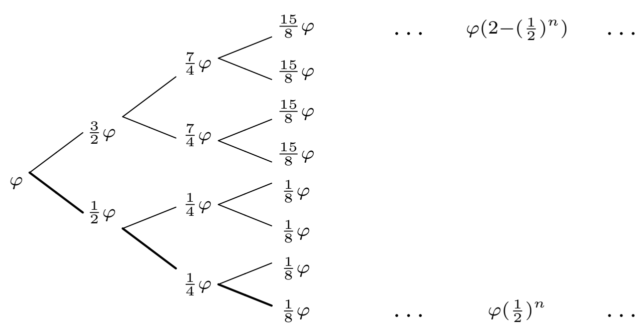
$$0 < \pi(x) < \lambda_x \quad \forall x \in T. \tag{3.13}$$

Let us investigate this condition for $\pi_{\varphi,\varepsilon}$ from Example 3.4 (one can consider $\pi_{\varphi,\varepsilon_x}$ for arbitrary $x \in T$ as well). Since

$$\varepsilon \le (2p-1)/(2p) \quad \text{and} \quad A_k \le p/(2p-1) \ \text{ for every } k$$

then condition $c = \inf_x \lambda_x (\frac{2q}{p})^{|x|} > 0$ is sufficient for (3.13) to hold for each $0 < \varphi \le c/2$. The next example illustrates that in case (iii), contrary of the remaining cases, the condition $c > 0$ is not necessary for the existence of nontrivial invariant measure. Nor for the existence of infinitely dimensional set of them.

Example 3.5. Let us consider specially $p = 2/3$ in the previous example, i.e. the symmetric random walk on the tree, and $\varepsilon = 1/4$. Then we get $\pi_{\varphi,\varepsilon}(x)$ as in the picture:



The expected number of particles with respect to ν^{π_φ} is $\mathsf{E}_{\nu^{\pi_\varphi}}(\eta(x)) = \frac{\pi_\varphi(x)/\lambda_x}{1-\pi_\varphi(x)/\lambda_x}$. So we can see that the density of particles with respect to ν^{π_φ} is no more constant at sites from the same level, even if $\lambda_x \equiv 1$. If we especially assume $\varphi = \frac{1}{2}$, we can find a sequence x_k, $|x_k| = k$, in the left branch of tree, such that $\mathsf{E}_{\nu^{\pi_{1/2}}}(\eta(x_k)) \overset{k\to\infty}{\longrightarrow} \infty$ and a sequence z_k, $|z_k| = k$, in the right branch, such that $\mathsf{E}_{\nu^{\pi_{1/2}}}(\eta(z_k)) \overset{k\to\infty}{\longrightarrow} 0$.

Considering a non-homogeneous environment, the condition (3.13) means

$$\begin{aligned} &\varphi < \frac{\lambda_x}{2-\frac{1}{2^{|x|}}} && \text{for every left successor } x \text{ of the root,} \\ &\varphi < 2^{|x|}\lambda_x && \text{for every right successor } x \text{ of the root.} \end{aligned} \tag{3.14}$$

It explains why we exclude case (iii) under assumption $\inf_x \lambda_x(\frac{2q}{p})^{|x|} = 0$ from case (iv). Indeed, the choice:

$$\begin{aligned} &\lambda_x = |x|^{-1} && \text{for every } x : |x| \geq 1 \text{ from the branch emphasised at the picture,} \\ &\lambda_x = 1 && \text{otherwise,} \end{aligned}$$

leads to the fact that $\inf_x \lambda_x(2q/p)^{|x|} = \inf_x |x|^{-1} = 0$ but there are invariant measures ν^{π_φ} for every $\varphi \in [0, 1/2]$ because (3.14) is satisfied.

The example showed that $\lambda_x(2q/p)^{|x|}$ may approach zero for some branch of the tree. It is possible to find out a much weaker sufficient condition on rates λ_x than the condition $c > 0$ under which

$$\mathbf{W} := \{\nu^\pi : 0 < \pi(x) < \lambda_x \text{ solve equations (2.2)}\}$$

is an infinitely dimensional set of product measures.

Employing [6, Theorem 1.4] we can claim that

$$\mathbf{W} \subseteq \mathcal{I}_e.$$

Nevertheless, it is still an open problem whether the equality $\mathbf{W} = \mathcal{I}_e$ holds.

4. DISCUSSION ON OPEN MODEL

Let us consider in this paragraph the same queueing system on the tree as in the previous but open. We mean that there is an external (bottomless) source of customers and we shall assume that customers from outside of the system can only enter the root and no other node of the tree. We consider no departure of customers from the system either.

In the open model the external source is considered as a false node o and customers come from it to the root with a fixed rate $\lambda_o > 0$. It means the arrival of customers is a Poisson process with rate λ_o. So we investigate the same model as was described in Section 1 to whom false node o is added. The single particle law is changed just in the root so that $p(o, r) = 1$, in other nodes the situation remains without changes.

The open zero range process is defined by infinitesimal generator

$$\mathcal{L}_o f(\eta) = \sum_{x\in T} \sum_{y \in T, y\sim x} \mathrm{I}_{[\eta(x)>0]} \lambda_x p(x,y)[f(\eta^{xy}) - f(\eta)] \; + \; \lambda_o[f(\eta^{or}) - f(\eta)] \tag{4.1}$$

for every $\eta \in \mathbb{N}^T$ and cylinder function f on $\mathbb{N}^T$, where $\eta^{or}(x) = \begin{cases} \eta(x) & x \neq r \\ \eta(r)+1 & x = r. \end{cases}$

The following analogy of Proposition 1.2, i. e. an existence theorem for invariant measures of open zero range processes, can be proved in the same way.

Proposition 4.1. Consider an open zero range process (4.1) and assume that its single particle law satisfies: $y \in \{x_+, x^+\} \Rightarrow p(x,y) \neq 0$. Let $(\pi(x) : x \in T)$ be a nonnegative solution of equations:

$$\sum_{y \in T \cup \{o\}} \Big(p(y,x)\pi(y) - p(x,y)\pi(x) \Big) = 0 \qquad \forall\, x \in T$$
$$\pi(o) = 1 \tag{4.2}$$

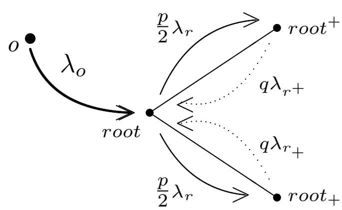
Then product measures ν^π defined on space $\mathbb{N}^T$ by their marginal distributions:

$$\nu^\pi(\eta : \eta(x) = k) = \Big(\frac{\lambda_o}{\lambda_x} \pi(x) \Big)^k \Big(1 - \frac{\lambda_o}{\lambda_x} \pi(x) \Big) \quad \forall\, k \in \mathbb{N} \tag{4.3}$$

$\forall\, x \in T$, are invariant for the open zero range process if

$$\lambda_o < \frac{\lambda_x}{\pi(x)} \qquad \forall\, x \in T.$$

Our aim is to describe invariant measures for the open model assuming again a simple random walk on tree T as a single particle law. So let us consider in what follows $p(x,y)$ defined by (2.1) changed in the root like the following picture shows (we again assume the correction of rate at the root).



Realise that if we put the outer source to the model it makes sense to assume in addition to $p \in (0,1)$ also $p = 1$, so called totally asymmetric random walk. We obtain the following result as a consequence of Proposition 4.1.

Theorem 4.2. Consider an open zero range process (4.1) with a simple random walk (2.1) as the single particle law for some $p \in (0,1]$, with arrival rate $\lambda_o > 0$ (see picture above). Then

(I) there is no invariant measure in cases

- $p \in (0, \frac{1}{2}]$
- $p \in (\frac{1}{2}, 1]$ and $\Big(\lambda_o \geq \lambda_x 2^{|x|}(p-q)$ for some $x \in T \Big)$

(II) there exists invariant measure ν which is product defined by marginals

$$\nu(\eta : \eta(x) = k) = \Big(\frac{\lambda_o}{\lambda_x} \frac{1}{2^{|x|}} \frac{1}{p-q} \Big)^k \Big(1 - \frac{\lambda_o}{\lambda_x} \frac{1}{2^{|x|}} \frac{1}{p-q} \Big) \quad \forall\, k \in \mathbb{N} \tag{4.4}$$

for every $x \in X$, in case

- $p \in (\frac{1}{2}, 1]$ and $\lambda_o < \lambda_x 2^{|x|}(p-q) \quad \forall\, x.$

Furthermore,

(α) ν is the unique invariant measure in case

- $p \in (\frac{2}{3}, 1]$,

(β) there exists an infinitely dimensional set of extremal invariant measures if

- $p \in (\frac{1}{2}, \frac{2}{3}]$ and $\lambda_o < \dfrac{\lambda_x}{\delta(x) + \frac{1}{2^{|x|}}\frac{1}{p-q}} \quad \forall\, x$, for some set of positive $(\delta_x)_{x \in T}$ such that $\inf_x \delta_x (\frac{2q}{p})^{|x|} > 0$.

This set is formed by product measures ν^π defined by (4.3) where

$$\pi(x) = \frac{1}{2^{|x|}} \frac{1}{p-q} + \alpha(x) \qquad \forall\, x \tag{4.5}$$

for any positive solution $\alpha(x)$ of (2.2) such that $\lambda_o < \dfrac{\lambda_x}{\alpha(x) + \frac{1}{2^{|x|}}\frac{1}{p-q}} \quad \forall\, x.$

P r o o f. Investigating equations (4.2) one can find out that there is

— no non-negative solution for case $p \in (0, \frac{1}{2}]$,

— the unique non-negative solution $\pi(x) = \frac{1}{2^{|x|}}\frac{1}{p-q}$ for case $p \in (\frac{2}{3}, 1]$ and

— a family of solutions (4.5) for case $p \in (\frac{1}{2}, \frac{2}{3}]$.

The uniqueness in the second item can be proved using the same approach as at the end of the proof of Theorem 2.3 (iv).

To obtain statement (I) we need in addition to realise that for each invariant measure μ function $a(x) = \lambda_x \mu(\eta(x) > 0)$, $a(o) = 1$, has to solve (4.2). Compare with the first part of the proof of Theorem 2.3 (iv).

The existence-part of (II) is an immediate consequence of Proposition 4.1. The sufficient condition for the existence of infinitely dimensional set of extremal invariant measures is adopted from Section 3, case (iii).

It remains to prove the uniqueness-part of (II). Assume that μ is an arbitrary invariant measure in case $p \in (\frac{2}{3}, 1]$ & $\lambda_o < \lambda_x 2^{|x|}(p-q)$ for every x. Note that the uniqueness of π as a solution of (4.2) implies that

$$\lambda_x \mu(\eta(x) > 0) = \frac{1}{2^{|x|}} \frac{1}{p-q} \qquad \forall\, x \in T. \tag{4.6}$$

But this is not enough information about μ. To get more information let us employ the ordinary coupling process (associated with process (4.1)) given by generator $\tilde{\mathcal{L}}_o$ which differs from (3.6) only in additional term $\lambda_o(f(\eta^{or}) - f(\eta))$. One can construct a measure $\tilde{\nu}$ on $\mathbb{N}^T \times \mathbb{N}^T$ which is invariant for $\tilde{\mathcal{L}}_o$ and its first and second marginals are measures μ and ν, respectively. Cf. proof of Theorem 2.3 (ii).

We use equality $\int \tilde{\mathcal{L}}_o f \, \mathrm{d}\tilde{\nu} = 0$ for functions $f_z(\eta, \zeta) = (\eta(z) - \zeta(z))^+$, denote

$$h(z) = \lambda_z \, \tilde{\nu}\Big(\eta(z) > 0, \zeta(z) = 0\Big)$$

and then we immediately obtain the following inequalities

$$\begin{aligned} h(z) &\leq h(z_-)\frac{p}{2} + (h(z^+) + h(z_+))q \qquad \forall\, z \neq r, \\ h(r)\, p &\leq \qquad\qquad (h(r^+) + h(r_+))q. \end{aligned} \tag{4.7}$$

From (4.7), there exists a sequence $(x_i)_{i=0}^{\infty}$ of successors, $x_0 = r$, $|x_i| = i$, such that $h(x_i) \geq (\frac{p}{2q})\, h(x_{i-1})$ for every $i \geq 1$. But $\frac{p}{2q} > 1$. So either $h(x_i)$ is increasing for some sequence $(x_i)_{i=n}^{\infty}$ of successors or $h(x) \equiv 0 \;\forall\, x \in T$. Since from the definition of h and from (4.6)

$$h(x) \;\leq\; \lambda_x \, \tilde{\nu}\Big(\eta(x) > 0\Big) \;=\; \lambda_x \mu \Big(\eta(x) > 0\Big) \;=\; \frac{1}{2^{|x|}} \frac{1}{p - q}$$

for every x, only the latter variant can be considered. It means

$$\tilde{\nu}\,\Big(\eta(x) > 0, \zeta(x) = 0\Big) = 0 \qquad \forall\, x \in T.$$

Using an induction in k it can be proved that

$$\tilde{\nu}\,\Big(\eta(x) > \zeta(x), \zeta(x) = k\Big) = 0 \qquad \forall\, x \in T,\ \forall\, k \in \mathbb{N}. \tag{4.8}$$

It suffices to plug $f_{z,k}(\eta, \zeta) = \mathrm{I}_{[\eta(z) > \zeta(z) = k-1]}$ in $\int \tilde{\mathcal{L}}_o f \, \mathrm{d}\tilde{\nu} = 0$. Nevertheless, (4.8) implies that

$$\tilde{\nu}(\eta \leq \zeta) = 1$$

and hence $\mu \leq \nu$ follows from the Strassen theorem (the open model is again attractive particle system). But it is possible to repeat whole procedure also for $h(z) = \lambda_z \, \tilde{\nu}(\eta(z) = 0, \zeta(z) > 0)$ and therefore also $\nu \leq \mu$ holds. Since the relation $\leq$ is a partial order, $\mu = \nu$ must hold. □

We can see that the situation concerning a classification with respect to trend p is rather different than it was for closed model. Intuitively, a stationary distribution of the open model seems to put together a stationary distribution of closed model and a stationary distribution of totally asymmetric open model.

Remark 4.3. Note that the condition $\inf_x \delta_x (\frac{2q}{p})^{|x|} > 0$ from case (β) is not necessary at all. Compare with Example 3.5.

Remark 4.4. Let us observe that in the case of totally asymmetric random walk it is possible to prove the uniqueness of measure ν in a very direct way.

Let μ be an invariant measure with respect to generator (4.1) in the specified case. We use directly the fact that $\int \mathcal{L}_o f \, \mathrm{d}\mu = 0$ for any cylinder function f. Let us choose arbitrary finite tree T_n as the subset of tree T ended by nth level and arbitrary cylinder function f depending only on numbers of particles at nodes of T_n. In this very special case where $p(x, x_-) = 0$ for every x, function $\mathcal{L}_o f$ is again cylinder function and depends only on nodes of T_n. So equality $\int \mathcal{L}_o f \, \mathrm{d}\mu = 0$ can be rearranged as

$$0 = \sum_{\eta_{\restriction_n} \in \mathbb{N}^{T_n}} f(\eta_{\restriction_n}) \Bigg[\Big(\sum_{y \in T_n} \mathrm{I}_{[\eta_{\restriction_n}(y)>0]} \lambda_{y_-} p(y_-, y) \mu_{\restriction_n}(\eta_{\restriction_n}^{yy_-}) - \lambda_y \mu_{\restriction_n}(\eta_{\restriction_n}) \Big) + \Big(\sum_{y \in T_{n+1} \setminus T_n} \lambda_{y_-} p(y_-, y) \mu_{\restriction_n}(\eta_{\restriction_n}^{oy_-}) - \lambda_o \mu_{\restriction_n}(\eta_{\restriction_n}) \Big) \Bigg]. \tag{4.9}$$

We use notation: $\eta_{\restriction_n}$ is a "short" configuration from product space $\mathbb{N}^{T_n}$ and $\mu_{\restriction_n}$ stands for a measure on $\mathbb{N}^{T_n}$ defined by $\mu_{\restriction_n}(\zeta_{\restriction_n}) = \mu(\eta : \eta(x) = \zeta_{\restriction_n}(x) \ \forall x \in T_n)$.

What is important on equality (4.9) is its dependence on η only through nodes from T_n. From this reason we can choose stepwise the functions $f = \mathrm{I}_{\eta_{\restriction_n}}$ and then obtain that each square brackets in the last term are equal to zero for every n and for every $\eta_{\restriction_n} \in \mathbb{N}^{T_n}$. It means

$$0 = \sum_{\zeta_{\restriction_n} \in \mathbb{N}^{T_n}} L(\zeta_{\restriction_n}, \eta_{\restriction_n}) \ \mu_{\restriction_n}(\zeta_{\restriction_n}) - L(\eta_{\restriction_n}, \zeta_{\restriction_n}) \ \mu_{\restriction_n}(\eta_{\restriction_n})$$

for every n and for every $\eta_{\restriction_n} \in \mathbb{N}^{T_n}$, where $L(\eta_{\restriction_n}, \eta_{\restriction_n}^{z_- z}) = \mathrm{I}_{[\eta_{\restriction_n}(z_-)>0]} \lambda_{z_-} p(z_-, z)$.

Nevertheless, $\{L(\eta_{\restriction_n}, \zeta_{\restriction_n}) : \eta_{\restriction_n}, \zeta_{\restriction_n} \in \mathbb{N}^{T_n}\}$ are transition rates of an irreducible, countable state space Markov process which has the unique invariant measure

$$\upsilon_n(\zeta_{\restriction_n}) = \prod_{x \in T_n} \Big(\frac{\lambda_o}{\lambda_x} \frac{1}{2^{|x|}} \Big)^{\zeta_{\restriction_n}(x)} \Big(1 - \frac{\lambda_o}{\lambda_x} \frac{1}{2^{|x|}} \Big)$$

for every $\zeta_{\restriction_n} \in \mathbb{N}^{T_n}$ if we assume $\lambda_o < \lambda_x 2^{|x|} \quad \forall x$. It means $\mu_{\restriction_n}(\cdot) = \upsilon_n(\cdot)$ for every n, so we know every finite dimensional marginal of measure μ. Hence the uniqueness of the projective limit completes this proof.

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