KYBERNETIKA — VOLUME *42* (2006), NUMBER 5, PAGES 569 – 584

LASLETT'S TRANSFORM FOR THE BOOLEAN MODEL IN $\mathbb{R}^d$

Rostislav Černý

Consider a stationary Boolean model X with convex grains in $\mathbb{R}^d$ and let any exposed lower tangent point of X be shifted towards the hyperplane $N_0 = \{x \in \mathbb{R}^d : x_1 = 0\}$ by the length of the part of the segment between the point and its projection onto the N_0 covered by X. The resulting point process in the halfspace (the Laslett's transform of X) is known to be stationary Poisson and of the same intensity as the original Boolean model. This result was first formulated for the planar Boolean model (see N. Cressie [3]) although the proof based on discretization is partly heuristic and not complete. Starting from the same idea we present a rigorous proof in the d-dimensional case. As a technical tool equivalent characterization of vague convergence for locally finite integer valued measures is formulated. Another proof based on the martingale approach was presented by A. D. Barbour and V. Schmidt [1].

Keywords: Boolean model, Laslett's transform

AMS Subject Classification: 60D05

1. INTRODUCTION

Let X be a stationary Boolean model in $\mathbb{R}^d$ with convex compact grains and intensity $\lambda > 0$, i.e.

$$X = \bigcup_{i=1}^{\infty} (x_i + G_{x_i}),$$

where $\cup_{i=1}^{\infty} x_i$ is a stationary Poisson point process (of germs) with intensity λ and G_{x_i} are i.i.d. random convex compact sets (grains), independent of $\cup_{i=1}^{\infty} x_i$, with lexicographical minimum at the origin (we will denote by $<_{lex}$ the *lexicographical order*, we put $(a_1, \ldots, a_d) <_{lex} (b_1, \ldots, b_d)$ iff $a_d < b_d$ or $(a_d = b_d$ and $a_{d-1} < b_{d-1})$ or ... or $((a_2, \ldots, a_d) = (b_2, \ldots, b_d)$ and $a_1 < b_1))$. Denote the distribution of G_{x_i} as Λ_0, it is usually called the distribution of a typical grain of X. Furthermore, points $\{x_i\}_{i=1}^{\infty}$ will be called *tangent* points of X and those of them that are not in the interior of X will be called *exposed*.

The Laslett's transform is defined in the half-space $\mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x_1 \geq 0\}$. Its idea is to remove all interiors of grains of the model X and then to close up the left gaps by shifting all remaining points from $\mathbb{R}^d_+$ towards the hyperplane $N_0 = \{x \in$

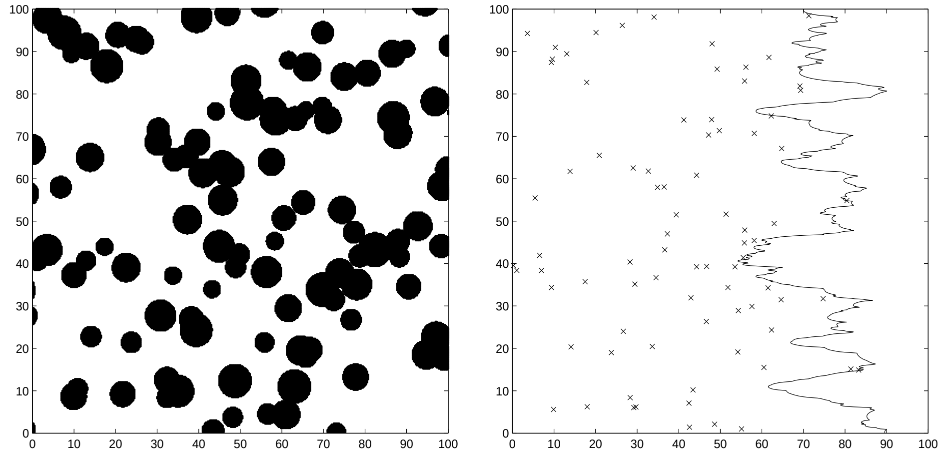

Fig. (a) A realization of planar Boolean model of discs. The intensity is 0,01; the distribution of disc radius is uniform U(2,4). (b) The corresponding point process of Laslett's transform and the shifted boundary of sample window.

$\mathbb{R}^d : x_1 = 0\}$ (see Figure). More precisely, assume L_X is the Laslett's transform and $x \in \mathbb{R}^d_+$. Then

$$L_X(x) = x - \lambda_1 \left(s(x) \cap X \right) \cdot e_1,$$

where $\lambda_1(\cdot)$ stands for the one-dimensional Lebesgue measure, e_1 is the unit vector $(1, 0, \ldots, 0) \in \mathbb{R}^d$ and $s(a)$ is the line segment connecting a point a with its orthogonal projection onto N_0. The point process of translated exposed germs of $X \cap \mathbb{R}^d_+$ is stationary Poisson with the same intensity λ as the original model X.

Remarkš. For a given realization X of the Boolean model we have defined Laslett's transform as the mapping $L_X : \mathbb{R}^d_+ \mapsto \mathbb{R}^d_+$. Moreover, we can consider the Laslett's transform acting on the space of particle processes (for the notation see Section 2):

$$L : \mathrm{N}(\mathcal{K}'(\mathbb{R}^d)) \mapsto \mathrm{N}(\mathbb{R}^d_+),$$

which assigns to a germ-grain model a point process in $\mathbb{R}^d_+$ of exposed tangent points shifted by L_X. Writing a subscript to the operator L we will distinguish between the first and second notion for the Laslett's transform.

Theorem 1. Let X be a stationary Boolean model in $\mathbb{R}^d$ with intensity $\lambda > 0$ and convex compact grains. Then the Laslett's transform of X forms the restriction of a stationary Poisson point process in $\mathbb{R}^d_+$ with the same intensity λ.

The theorem was originally formulated for a planar Boolean model by G. M. Laslett. Although the proof based on discretization and sequential conditioning (see [3], Section 9.5.3) is partly heuristic, the idea is correct and with some technical arguments it can be made complete. Another proof using martingale approach was given by A. D. Barbour and V. Schmidt (see [1]).

We start with the same idea of discretization of the model X and we give a rigorous proof of Theorem 1 in Section 3. Thus the Laslett's theorem can be generalized to the Euclidean space of arbitrary dimension $d \geq 2$.

Practical usage of Theorem 1 is straightforward. It is usual in practice that we do not observe particular grains of the model but only their union. One can therefore ask how to estimate the intensity λ then. Naturally, we can apply Laslett's transform. On the other hand the same estimator can be derived by working with the so-called *Tangent Point Process* (see [4] and [5]).

The second practical usage of Laslett's theorem lies in the fact that the resulting process is Poisson. Hence, using well-known approaches for testing Poisson point processes we can test that some observed random set is a part of a Boolean model X. Unfortunately, since the converse of Theorem 1 does not generally hold true, we can only reject that X is Boolean when the test rejects $L(X)$ to be poissonian. However, the opposite result of the test tells us nothing about X.

2. CONTINUITY OF THE LASLETT'S TRANSFORM

In this section we formulate first the equivalent characterization for the vague convergence of locally finite integer valued measures which will be later used to show the continuity of Laslett's transform (on sufficiently large spaces of measures).

Let V be a complete separable metric space with a metric ϱ and denote by $\mathcal{K}(V)$ all compact sets in V. Let $\mathcal{C}_c(V)$ be the space of all continuous functions on V with compact support.

Denote by $\mathrm{N}(V)$ the space of locally finite integer valued measures on V. Its elements can be considered as locally finite sets as well. Hence for $\phi \in \mathrm{N}(V)$ we will use the notation $x \in \phi$ which is equivalent to $\phi(x) > 0$ and $\cup\phi = \cup\{x : x \in \phi\}$. On $\mathrm{N}(V)$ we assume the topology given by vague convergence:

$$\phi_n \xrightarrow{v} \phi \quad \text{iff} \quad \forall\, f \in \mathcal{C}_c(V) : \int f \, \mathrm{d}\phi_n \to \int f \, \mathrm{d}\phi.$$

We will use the notation $d(x, A)$ for the distance of a point x from a set A, i. e. $d(x, A) = \inf\{\varrho(x, y) : \ y \in A\}$. Further set $B_x(r)$ the ball with center at x and radius r and let $(\cdot)^+$ denote the positive part, i. e. $(\cdot)^+ = \max(0, \cdot)$.

Lemma 1. Let $\phi_n, \phi \in \mathrm{N}(V)$. Then the following statements are equivalent:

(1) $\phi_n \xrightarrow{v} \phi$ for $n \to \infty$,

(2) $\forall\, K \in \mathcal{K}(V)\ \exists\, \varepsilon_0 > 0$ such that $\forall\, \varepsilon :\ 0 < \varepsilon < \varepsilon_0\ \exists\, n_0;\ \forall\, n > n_0$: there exists an injective mapping $\xi_n : \phi \cap K_\varepsilon \to \phi_n$ such that

(a) $\forall\, x \in \phi \cap K_\varepsilon : \quad \varrho(x, \xi_n(x)) < \varepsilon$,

(b) $(\phi_n \setminus \operatorname{Im} \xi_n) \cap K = \emptyset$,

where $K_\varepsilon = \{z \in X : \ d(z, K) \leq \varepsilon\}$ and $\operatorname{Im} \xi_n = \xi_n(\phi \cap K_\varepsilon)$.

P r o o f . (1)⇒(2): Let $\phi_n \xrightarrow{v} \phi$, $K \in \mathcal{K}(V)$ and $\varepsilon_1 > 0$. Put $\{x_1, \dots, x_l\} = \phi \cap K_{\varepsilon_1}$ and choose $\varepsilon_0 > 0$, $\varepsilon_0 < \varepsilon_1$ such that $x_j \notin \overline{B_{x_i}(\varepsilon_0)}$ for all $i \neq j$, $i, j = 1, \dots, l$. Let ε be given, $\varepsilon_0 > \varepsilon > 0$. For $i = 1, \dots, l$ set

$$f_{x_i}(x) = \left(1 - \frac{2 \cdot \varrho(x, x_i)}{\varepsilon}\right)^+ .$$

Then $f_{x_i} \in \mathcal{C}_c(V)$, $\operatorname{spt} f_{x_i} = \overline{B_{x_i}(\frac{\varepsilon}{2})}$ and it holds that

$$\int f_{x_i}\, \mathrm{d}\phi = 1 = \lim_{n \to \infty} \int f_{x_i}\, \mathrm{d}\phi_n .$$

Hence there exists n_1 such that for all $n > n_1$ there exists $y_i \in B_{x_i}(\frac{\varepsilon}{2}) \cap \phi_n$. Suppose there exists $\overline{y_i} \in B_{x_i}(\frac{\varepsilon}{2}) \cap \phi_n$, $\overline{y_i} \neq y_i$. Set

$$g_{x_i}(x) = \left(1 - \frac{2 \cdot d\left(x, B_{x_i}\left(\frac{\varepsilon}{2}\right)\right)}{\varepsilon}\right)^+ .$$

Again $g_{x_i} \in \mathcal{C}_c(V)$, $\operatorname{spt} g_{x_i} = \overline{B_{x_i}(\varepsilon)}$ and we derive a contradiction

$$\int g_{x_i}\, \mathrm{d}\phi = 1 = \lim_{n \to \infty} \int g_{x_i}\, \mathrm{d}\phi_n \geq 2.$$

Therefore it is possible to define uniquely $\xi_n(x_i) = y_i$ and it remains to prove assertion (b).

Set

$$h(x) = \left(1 - \frac{d(x, K)}{\varepsilon}\right)^+, \qquad h_{x_i}(x) = \begin{cases} 1 & x \notin B_{x_i}\left(\frac{\varepsilon}{2}\right), \\ \frac{2 \cdot \varrho(x, x_i)}{\varepsilon} & \text{otherwise} \end{cases}$$

and

$$k(x) = \min(h(x), h_{x_1}(x), \dots, h_{x_n}(x)) ,$$

$k \in \mathcal{C}_c(V)$, $\operatorname{spt} k = K_\varepsilon$. Then

$$\int k\, \mathrm{d}\phi = 0 = \lim_{n \to \infty} \int k\, \mathrm{d}\phi_n .$$

Consequently, there exists n_2 such that for all $n > n_2$ any point from $\phi_n \cap K$ belongs to some $B_{x_i}(\frac{\varepsilon}{2})$, $i = 1, \dots, l$. According to the first part of the proof this point must belong to $\operatorname{Im} \xi_n$. Now it suffices to set $n_0 = \max(n_1, n_2)$ and the assertion is proved.

(2)⇒(1): Let $f \in \mathcal{C}_c(V)$ be given, set $K = \operatorname{spt} f$. For this K choose ε_0 such that (2) holds. Denote $l = \operatorname{card}(\phi \cap K_{\varepsilon_0})$. Let $\varepsilon > 0$ be arbitrary. Since f is continuous there exists $0 < \delta < \varepsilon_0$, such that $\varrho(x, y) < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \frac{\varepsilon}{l}$. According to (2), for this δ there exists n_0 such that ξ_n is an injective mapping with properties (a), (b), for $n > n_0$. We have

$$\begin{aligned} \left| \int f \, d\phi - \int f \, \mathrm{d}\phi_n \right| &= \left| \sum_{x \in \phi \cap K_\delta} f(x) - \sum_{x \in \phi_n \cap K} f(x) \right| \\ &= \left| \sum_{x \in \phi \cap K_\delta} f(x) - f(\xi_n(x)) \right| \\ &\leq \sum_{x \in \phi \cap K_\delta} |f(x) - f(\xi_n(x))| < \varepsilon, \end{aligned}$$

using spt $f = K$, property (b) and property (a) for δ. □

In the rest of the section we will show the continuity of Laslett's transform. More precisely, we will show that if $\Phi_n \overset{n\to\infty}{\longrightarrow} \Phi$ in distribution, where Φ is a Poisson process of convex particles, then also $L(\Phi_n) \overset{n\to\infty}{\longrightarrow} L(\Phi)$ in distribution. The latter holds true if L is continuous on some region $\mathcal{U}_r$ and $\Pr(\Phi \in \mathcal{U}_r) = 1$, see [2], Theorem 2.7.

Set $\mathcal{K}' = \mathcal{K}'(\mathbb{R}^d)$ the space of non-empty compact sets in $\mathbb{R}^d$ and $\mathcal{K}'_0$ those sets from $\mathcal{K}'$ with lexicographical minimum (denoted by lexmin) at the origin. $\mathcal{K}'$ is equipped with the Hausdorff metric d_H, defined as

$$d_H(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\} .$$

We will gradually define three mappings F_1, F_2, F_3 whose composition forms the Laslett's transform:

$$L = F_3 \circ F_2 \circ F_1 .$$

The mapping F_1 assigns marks to the process of grains. The marks have the interpretation of the shift length for the corresponding tangent points. The mapping F_2 preserves only germs and only those of them which are not overlapped by any other grain. Finally the mapping F_3 shifts the remaining points towards the hyperplane N_0 by the length given by the associated marks.

We say that two sets $A, B \subset \mathbb{R}^d$ *touch each other (touch e.a.)* if $A \cap B \neq \emptyset$ and $\lambda_d(A \cap B) = 0$.

Consider the following properties of two convex sets C_1, C_2:

(i) C_1, C_2 do not touch e.a.,

(ii) $\lambda_1 \left(s(\mathrm{lexmin}\, C_1) \cap \partial C_2 \right) = 0$.

Set

$$\mathcal{U}_{r_1} = \left\{ \phi \in \mathrm{N}(\mathcal{K}'(\mathbb{R}^d)) : \begin{array}{l} C_1, C_2 \in \phi, \\ C_1 \neq C_2 \end{array} \Rightarrow C_1, C_2 \text{ fulfill (i) and (ii).} \right\} .$$

Lemma 2. Let Φ be a stationary Poisson process on $\mathcal{K}'(\mathbb{R}^d)$ with convex grains. Then

$$\Pr(\Phi \in \mathcal{U}_{r_1}) = 1\,.$$

Proof. Assume without loss of generality that the intensity α of the process Φ equals 1. Let $M_2^!$ denote the second order factorial moment measure and Λ the intensity measure of Φ. Since Φ is Poisson process, it holds (see [7], p. 47)

$$M_2^! = \Lambda^2$$

and we have

$$\begin{aligned}
&\Pr(\exists\, C_1, C_2 \in \Phi,\ C_1 \neq C_2 :\ C_1, C_2 \text{ touch e.a.}) \\
&\qquad \le \mathrm{E} \sum_{C_1, C_2 \in \Phi, C_1 \neq C_2} 1_{\{C_1, C_2 \text{ touch e.a.}\}} \\
&\qquad = M_2^!(\{C_1, C_2 \text{ touch e.a.}\}) = \Lambda^2(\{C_1, C_2 \text{ touch e.a.}\}) \\
&\qquad = \int\int\int 1_{\{x + C_0, C_2 \text{ touch e.a.}\}}\, dx \Lambda_0(\mathrm{d}C_0)\Lambda(\mathrm{d}C_2) \\
&\qquad = \int\int \lambda_{\mathrm{d}}(\partial(C_2 \oplus (-C_0)))\Lambda_0(\mathrm{d}C_0)\Lambda(\mathrm{d}C_2) = 0
\end{aligned}$$

since $(C_2 \oplus (-C_0))$ is bounded and convex.

Since C_2 is convex its projection onto N_0 is convex as well. Denote this projection by $P_{N_0}(C_2)$. A necessary condition for the segment $s(\mathrm{lexmin}(C_1))$ to have intersection of positive measure with boundary ∂C_2 is that whole line covering $s(\mathrm{lexmin}(C_1))$ has non-empty intersection with boundary $\partial P_{N_0}(C_2)$. Then we have

$$\begin{aligned}
&\Pr\left(\exists\, C_1, C_2 \in \Phi,\ C_1 \neq C_2 :\ \lambda_1(s(\mathrm{lexmin}\, C_1) \cap \partial C_2) > 0\right) \\
&\qquad \le \mathrm{E} \sum_{C_1, C_2 \in \Phi, C_1 \neq C_2} 1_{\{\lambda_1(s(\mathrm{lexmin}\, C_1) \cap \partial C_2) > 0\}} \\
&\qquad = M_2^!\left(\left\{\lambda_1(s(\mathrm{lexmin}\, C_1) \cap \partial C_2) > 0\right\}\right) \\
&\qquad = \Lambda^2\left(\left\{\lambda_1(s(\mathrm{lexmin}\, C_1) \cap \partial C_2) > 0\right\}\right) \\
&\qquad \le \int\int\int \lambda_{\mathrm{d}-1}(\partial P_{N_0}(C_2))\, \lambda_1(\mathrm{d}x)\, \Lambda_0(\mathrm{d}C_0)\Lambda(\mathrm{d}C_2) = 0,
\end{aligned}$$

using Fubini theorem and the fact that the boundary of a convex set in $\mathbb{R}^{d-1}$ has measure 0. $\square$

Lemma 3. Define $F_1 : \mathrm{N}(\mathcal{K}'(\mathbb{R}^d)) \to \mathrm{N}(\mathcal{K}'(\mathbb{R}^d) \times \mathbb{R}_+)$ by

$$F_1(\phi) = \{(K, z) : \; K \in \phi, \; z = z(K, \phi)\},$$

where $z : \; \mathcal{K}'(\mathbb{R}^d) \times \mathrm{N}(\mathcal{K}'(\mathbb{R}^d)) \to [0, \infty)$ is defined as

$$z : \; (K, \phi) \mapsto \lambda_1\big(s(\operatorname{lexmin} K) \cap (\cup\phi)\big).$$

Suppose that Φ_n, Φ are point processes on $\mathcal{K}'(\mathbb{R}^d)$, $\Phi_n \xrightarrow{\mathcal{D}} \Phi$ for $n \to \infty$ and Φ is stationary Poisson process with convex grains. Then

$$F_1(\Phi_n) \xrightarrow{\mathcal{D}} F_1(\Phi) \qquad \text{for } n \to \infty.$$

P r o o f. First we will show the continuity of the mapping z on $\mathcal{U}_{r_1}$. Let $\phi \in \mathcal{U}_{r_1}$, $K \in \phi$ and $\varepsilon_0 > 0$. Set $\{C_1, \dots, C_l\} = \{C' \in \phi : \; C' \cap S \oplus B_{\varepsilon_0}(o) \neq \emptyset\}$, $S = s(\operatorname{lexmin} K)$. We can write

$$\begin{aligned} z(K, \phi) &= \sum_{i=1}^{l} \lambda_1(C_i \cap S) \\ &\quad - \sum_{\substack{i < j \\ C_i \cap C_j \neq \emptyset}} \lambda_1(C_i \cap C_j \cap S) \\ &\quad + \sum_{\substack{i<j<k \\ C_i \cap C_j \cap C_k \neq \emptyset}} \lambda_1(C_i \cap C_j \cap C_k \cap S) - \dots \\ &= \sum_{k=1}^{l} (-1)^{k+1} \sum_{\substack{i_1 < \dots < i_k \\ C_{i_1} \cap \dots \cap C_{i_k} \neq \emptyset}} \lambda_1(C_{i_1} \cap \dots \cap C_{i_k} \cap S). \end{aligned}$$

Using property (i) it can be easily shown that the mapping $(C_1, C_2) \mapsto C_1 \cap C_2$ and (with a help of property (ii)) the mapping $(C_1, C_2) \mapsto \lambda_1(s(\operatorname{lexmin} C_1) \cap C_2)$ are both continuous in C_1, C_2 convex with properties (i) and (ii). It is then clear that $z(K, \phi)$ is continuous on processes with corresponding properties.

Assume now that $\phi_n, \phi \in \mathrm{N}(\mathcal{K}'(\mathbb{R}^d))$, $\phi \in \mathcal{U}_{r_1}$ and $\phi_n \xrightarrow{v} \phi$. We will show

$$F_1(\phi_n) \xrightarrow{v} F_1(\phi).$$

Using Lemma 1, for arbitrary $U \in \mathcal{K}\big(\mathcal{K}'(\mathbb{R}^d)\big)$ and sufficiently small $\varepsilon > 0$ we can find n_0 such that for all $n > n_0$ there exists an injective mapping $\xi_n : \; \phi \cap U_\varepsilon \to \phi_n$ with properties (a), (b) from Lemma 1.

Let $\overline{U} \in \mathcal{K}\big(\mathcal{K}'(\mathbb{R}^d) \times \mathbb{R}_+\big)$ be given. Set $U = \overline{U}|_{\mathcal{K}'(\mathbb{R}^d)}$ and define

$$\overline{\xi_n} : \; F_1(\phi) \cap \overline{U}_\varepsilon \to F_1(\phi_n), \quad (K, z) \mapsto (\xi_n(K), z(K, \phi_n)) \, .$$

Obviously this is an injective mapping which fulfills (b) from Lemma 1. It suffices to prove

$$\varrho((K,z),\overline{\xi_n}(K,z)) < \varepsilon\,. \tag{1}$$

This follows from the continuity of z and existence of m_0 such that for $n > m_0$, $\varrho(z(K,\phi_n), z(K,\phi)) < \varepsilon$. For $n > \max(n_0, m_0)$ we then get (1), so F_1 is continuous on $\mathcal{U}_{r_1}$. Together with Lemma 2 we derived

$$F_1(\Phi_n) \xrightarrow{\mathcal{D}} F_1(\Phi). \qquad \square$$

Let Π_k denote the projection onto the kth axis and consider further two properties of two convex sets C_1, C_2:

(iii) $\Pi_1(\mathrm{lexmin}(C_1)) \neq 0$,

(iv) $\mathrm{lexmin}(C_1) \notin \partial C_2$.

Set

$$\mathcal{U}_{r_2} = \left\{\phi \in \mathrm{N}(\mathcal{K}'(\mathbb{R}^d)) :\ \begin{array}{l} C_1, C_2 \in \phi, \\ C_1 \neq C_2 \end{array} \ \Rightarrow C_1, C_2 \text{ fulfill (iii) and (iv)}\right\}.$$

Lemma 4. Let Φ be a stationary Poisson process on $\mathcal{K}'(\mathbb{R}^d)$ with convex grains. Then

$$\Pr(\Phi \in \mathcal{U}_{r_2}) = 1\,.$$

Proof. Assume without loss of generality that the intensity α of the process Φ equals 1. Property (iii) follows easily from the definition of Poisson process. Similarly to Lemma 2 we have

$$\begin{aligned}
&\Pr(\exists\, C_1, C_2 \in \Phi,\ C_1 \neq C_2 :\ \mathrm{lexmin}\, C_1 \in \partial C_2) \\
&\qquad\qquad \leq \quad M_2^!(\{\mathrm{lexmin}\, C_1 \in \partial C_2\}) \\
&\qquad\qquad = \quad \int\!\!\int\!\!\int 1_{\{\mathrm{lexmin}(x + C_0) \in \partial C_2\}}\, \mathrm{d}x \Lambda_0(\mathrm{d}C_0)\Lambda(\mathrm{d}C_2) \\
&\qquad\qquad = \quad \int\!\!\int \lambda_{\mathrm{d}}(\partial C_2)\Lambda_0(\mathrm{d}C_0)\Lambda(\mathrm{d}C_2) = 0,
\end{aligned}$$

since C_2 is convex and compact. $\square$

Lemma 5. Define $F_2 : \mathrm{N}(\mathcal{K}'(\mathbb{R}^d) \times \mathbb{R}_+) \to \mathrm{N}(\mathbb{R}^d_+ \times \mathbb{R}_+)$ by

$$F_2(\phi) = \left\{(\mathrm{lexmin}\, K, z) :\ (K,z) \in \phi,\ \mathrm{lexmin}\, K \in \mathbb{R}^d_+ \cap \overline{\left(\bigcup_{(K,z)\in\phi} K\right)^C}\right\}.$$

Assume that Φ_n, Φ are point processes on $\mathcal{K}'(\mathbb{R}^d)$, $\Phi_n \xrightarrow{\mathcal{D}} \Phi$ for $n \to \infty$ and Φ is stationary Poisson process with convex grains. Then

$$F_2(F_1(\Phi_n)) \xrightarrow{\mathcal{D}} F_2(F_1(\Phi)) ,$$

where the mapping F_1 is defined in Lemma 3.

Proof. Set $\overline{\mathcal{U}_{r_2}} = \big\{\{(K_n, z_n)\}_{n=1}^{\infty} : \{K_n\}_{n=1}^{\infty} \in \mathcal{U}_{r_2}\big\}$. Lemma 4 can be easily applied to derive

$$\Pr(F_1(\Phi) \in \overline{\mathcal{U}_{r_2}}) = 1 .$$

We shall show the continuity of F_2 in every point of $\overline{\mathcal{U}_{r_2}}$.

Assume $\phi_n, \phi \in \mathrm{N}(\mathcal{K}'(\mathbb{R}^d) \times \mathbb{R}_+)$, $\phi \in \overline{\mathcal{U}_{r_2}}$ and $\phi_n \xrightarrow{v} \phi$. Applying Lemma 1, for every sufficiently small $\varepsilon > 0$, every $L \in \mathcal{K}(\mathcal{K}'(\mathbb{R}^d) \times \mathbb{R}_+)$ and every $n > n_0$ there exists an injective mapping

$$\xi_n : \phi \cap L_\varepsilon \to \phi_n$$

with properties (a) and (b).

Denote $P_2(K, z) = (\operatorname{lexmin} K, z)$ for $K \in \mathcal{K}'(\mathbb{R}^d)$ and $z \in \mathbb{R}_+$. For $(x, z) \in F_2(\phi) \cap P_2(L_\varepsilon)$ (ε arbitrary, sufficiently small), let $K \in \mathcal{K}'(\mathbb{R}^d)$ be such that $(K, z) \in \phi \cap L_\varepsilon$. Define

$$\overline{\xi_n} : (x, z) \mapsto P_2\Big(\xi_n(K, z)\Big) .$$

Since ξ_n is injective and ϕ cannot contain two different grains with the same tangent point (property (iv)), $\overline{\xi_n}$ is injective.

We will show that $\overline{\xi_n}(x, z) \in F_2(\phi_n)$ for n sufficiently large, i.e. for $\xi_n(K, z) = (K_n, z_n)$ we have $\operatorname{lexmin} K_n \in \mathbb{R}_+^d \cap \overline{\Big(\bigcup_{(B,c)\in\phi_n} B\Big)}^C$. Obviously, there exists n_0 such that $\operatorname{lexmin} K_n \in \mathbb{R}_+^d$ for $n > n_0$, since $\operatorname{lexmin} K \in \mathbb{R}_+^d$, $\Pi_1(\operatorname{lexmin} K) \neq 0$ (property (iii)) and $|\operatorname{lexmin} K - \operatorname{lexmin} K_n| < d(\operatorname{lexmin} K, (\mathbb{R}_+^d)^C)$ for $n > n_0$ ((a) in Lemma 1).

Let $(x, z) \in F_2(\phi) \cap P_2(L_\varepsilon)$, $(B, c) \in \phi \cap L_\varepsilon$ be arbitrary such that $\operatorname{lexmin} B \neq x$. Then using (iv) we derive $x \in B^C$ and so there exists $\delta > 0$; $B_x(2\delta) \cap B = \emptyset$.

Denote $(B_n, c_n) = \xi_n(B, c)$. Choose n_1 such that for any $u \in \phi \cap L_\varepsilon$ and any $n > n_1$, $\varrho(\xi_n(u), u) < \delta$ (ϱ meaning here the maximal metric on product space $\mathcal{K}'(\mathbb{R}^d) \times \mathbb{R}_+$). Then $d_H(B_n, B) < \delta$, $|x_n - x| < \delta$ and hence $x_n \notin B_n$. We have shown for $n > \max(n_0, n_1)$ that $\overline{\xi_n}$ is an injective mapping from $F_2(\phi) \cap P_2(L_\varepsilon)$ to $F_2(\phi_n)$.

Condition (a) for $\overline{\xi_n}$ from Lemma 1 follows easily from the properties of ξ_n. It remains to show the condition (b).

Let $(x, z) \notin F_2(\phi)$ but $\exists\, K \in \mathcal{K}'$ such that $(K, z) \in \phi \cap L_\varepsilon$ and $P_2(K, z) = (x, z)$. Then one of the following statements must be true:

- $\exists\,(B, c) \in \phi$, $\operatorname{lexmin} B \neq x$, that fulfills $x \in \operatorname{int} B$.
  Set $\delta > 0$ such that $B_x(4\delta) \subseteq B$. Let $(B_n, c_n) = \xi_n(B, c)$ and n_2 be such

that for all $n > n_2$ and arbitrary $u \in \phi \cap L_\delta$ holds $\varrho(u, \xi_n(u)) < \delta$. Then $d_H(B, B_n) < 2\delta$ and hence $B_x(\delta) \subseteq (B)_{-2\delta} \subseteq B_n$ ($(B)_{-2\delta} = \{x : B_x(2\delta) \subseteq B\}$).

We have shown that $x_n \in B_n$ and hence

$$(x_n, z_n) \notin F_2(\phi_n)\,.$$

- $x \notin \mathbb{R}^d_+$.
  Then $\exists\,\delta > 0$ such that $B_x(\delta) \subseteq (\mathbb{R}^d_+)^C$ and hence $x_n \notin \mathbb{R}^d_+$ for $n > n_2$. Again

$$(x_n, z_n) \notin F_2(\phi_n)\,.$$

Setting $m_0 = \max(n_0, n_1, n_2)$, we found for $n > m_0$ an injective mapping

$$\overline{\xi_n} : F_2(\phi) \cap P_2(L_\varepsilon) \to F_2(\phi_n)\,,$$

which fulfills conditions of Lemma 1. Hence $F_2(\phi_n) \xrightarrow{v} F_2(\phi)$ and together with results of Lemma 3 we finally obtain

$$F_2(F_1(\Phi_n)) \xrightarrow{\mathcal{D}} F_2(F_1(\Phi)) \quad \text{for } n \to \infty\,. \qquad \square$$

Lemma 6. Set $\mathcal{U}_{r_3} \subseteq \mathrm{N}\left(\mathbb{R}^d_+ \times \mathbb{R}_+\right)$:

$$\mathcal{U}_{r_3} = \left\{\phi :\ \phi\left(\left\{(x,z) :\ (x_1 - z, x_2, \dots, x_d) \in K\right\}\right) < \infty,\ \forall\, K \in \mathcal{K}'(\mathbb{R}^d_+)\right\}.$$

Let Φ be a stationary Poisson point process on $\mathcal{K}'(\mathbb{R}^d)$ with convex grains, intensity $\alpha > 0$ and the distribution of typical grain Λ_0. Then

$$\Pr\left[F_2\left(F_1(\Phi)\right) \in \mathcal{U}_{r_3}\right] = 1\,.$$

Proof. Let $K \in \mathcal{K}'(\mathbb{R}^d_+)$ be an arbitrary compact set. Choose a rectangle $R_K \supseteq K$ with edges parallel to axes and one of its faces lying in N_0. Denote this face by F_{N_0}.

For $L \in \mathcal{K}'(\mathbb{R}^d_+)$ set $H(L) = \{(x,z) \in \mathbb{R}^d_+ \times \mathbb{R}_+ :\ (x_1 - z, x_2, \dots, x_d) \in L\}$. Then $F_2(F_1(\Phi))(H(K)) \le F_2(F_1(\Phi))(H(R_K))$ and hence

$$\Pr[F_2(F_1(\Phi))(H(K)) = \infty] \le \Pr[F_2(F_1(\Phi))(H(R_K)) = \infty]\,.$$

Let $V = F_{N_0} \times \mathbb{R}$. We will define a one-dimensional process $\widetilde{\Phi}$ of convex grains on the axis x_1, that arises as an orthogonal projection of the part of the process Φ that intersects V:

$$\widetilde{\Phi} = \sum_{G \in \Phi:\ G \cap V \neq \emptyset} \delta_{\Pi_d(G)}\,.$$

Denote $\widetilde{\Xi} = \bigcup_{G \in \widetilde{\Phi}} G$. It can be easily shown that $\widetilde{\Xi}$ is a Boolean model and hence it is ergodic. Let p be the volume fraction of $\widetilde{\Xi}$. Using the ergodicity of $\widetilde{\Xi}$ we derive

$$
\begin{aligned}
&\Pr\left[F_2(F_1(\Phi))(H(R_K)) = \infty\right] \\
&\qquad \leq \Pr[\forall\, r \geq 0 :\ \lambda_d([0,r] \times F_{N_0}) - \lambda_d((\Xi \cap ([0,r]) \times F_{N_0})) \leq \lambda_d(R_K)] \\
&\qquad \leq \Pr[\forall\, r \geq 0 :\ \lambda_1([0,r]) - \lambda_1(\widetilde{\Xi} \cap [0,r]) \leq \lambda_1(\Pi_1(R_K))] \\
&\qquad = \Pr\left[\forall\, r \geq 0 :\ \frac{\lambda_1(\widetilde{\Xi} \cap [0,r])}{\lambda_1([0,r])} \geq 1 - \frac{\lambda_1(\Pi_1(R_K))}{r}\right] \\
&\qquad = \Pr\left[\left(\forall\, r \geq 0 :\ \frac{\lambda_1(\widetilde{\Xi} \cap [0,r])}{\lambda_1([0,r])} \geq 1 - \frac{\lambda_1(\Pi_1(R_K))}{r}\right)\right. \\
&\qquad\qquad \left.\bigcap \left(\lim_{r \to \infty} \frac{\lambda_1(\widetilde{\Xi} \cap [0,r])}{\lambda_1([0,r])} = p\right)\right] \\
&\qquad = \Pr[p \geq 1] = 0\,.
\end{aligned}
$$

□

Lemma 7. Define $F_3 : \mathrm{N}(\mathbb{R}^d_+ \times \mathbb{R}_+) \to \mathrm{N}(\mathbb{R}^d_+)$ by

$$F_3(\phi) = \{((x_1 - z)^+, x_2, \ldots, x_d) :\ (x, z) \in \phi\}.$$

Assume that Φ_n, Φ are point processes on $\mathcal{K}'(\mathbb{R}^d)$, $\Phi_n \xrightarrow{\mathcal{D}} \Phi$ for $n \to \infty$ and Φ is stationary Poisson with convex grains. Then

$$F_3(F_2(F_1(\Phi_n))) \xrightarrow{\mathcal{D}} F_3(F_2(F_1(\Phi)))\,.$$

P r o o f. We will show the continuity of F_3 on $\mathcal{U}_{r_3}$. Let $\phi_n, \phi \in \mathrm{N}(\mathbb{R}^d_+ \times \mathbb{R}_+)$, $\phi_n \xrightarrow{v} \phi$ and $\phi \in \mathcal{U}_{r_3}$. Our aim is to show $F_3(\phi_n) \xrightarrow{v} F_3(\phi)$.

From the assumption it follows that for all $h \in \mathcal{C}_c(\mathbb{R}^d_+ \times \mathbb{R}_+)$,

$$\int h \, \mathrm{d}\phi_n = \sum_{(x_n, z_n) \in \phi_n} h(x_n, z_n) \overset{n \to \infty}{\longrightarrow} \sum_{(x,z) \in \phi} h(x, z) = \int h \, \mathrm{d}\phi\,. \tag{2}$$

Let $f \in \mathcal{C}_c(\mathbb{R}^d_+)$. Denote

$$u_{\phi,f} = \sup\left\{z :\ (x, z) \in \phi,\ ((x_1 - z)^+, x_2, \ldots, x_d) \in \operatorname{spt} f\right\}$$

and similarly $u_{\phi_n,f}$. Since $\phi \in \mathcal{U}_{r_3}$, $u_{\phi,f} < \infty$. Applying Lemma 1 we derive from convergence $\phi_n \xrightarrow{v} \phi$ the existence of an n_0 such that for all $n > n_0$ we have $u_{\phi_n,f} < u_{\phi,f} + 1$. Choose $g \in \mathcal{C}_C(\mathbb{R}_+)$ such that

$$g(z) = 1, \qquad \text{when } 0 \le z \le u_{\phi,f} + 1\,.$$

It is chosen to fulfill

$$\begin{aligned}\sum_{(x_n,z_n)\in\phi_n} f(x_n, z_n) &= \sum_{(x_n,z_n)\in\phi_n} f(x_n, z_n)\, g(z_n),\\ \sum_{(x,z)\in\phi} f(x, z) &= \sum_{(x,z)\in\phi} f(x, z)\, g(z).\end{aligned}$$

Since the positive part is a continuous function, $f((x_1 - z)^+, x_2, \ldots, x_d)\, g(z) \in \mathcal{C}_c$ and from (2) we derive

$$\begin{aligned}\lim_{n\to\infty} \int f \,\mathrm{d}F_3(\phi_n) &= \lim_{n\to\infty} \sum_{(x_n,z_n)\in\phi_n} f((x_{n1} - z_n)^+, x_{n2}, \ldots, x_{nd})\\ &= \lim_{n\to\infty} \sum_{(x_n,z_n)\in\phi_n} f((x_{n1} - z_n)^+, x_{n2}, \ldots, x_{nd})\, g(z_n)\\ &= \sum_{(x,z)\in\phi} f((x_1 - z)^+, x_2, \ldots, x_{\mathrm{d}})\, g(z)\\ &= \int f \,\mathrm{d}F_3(\phi).\end{aligned}$$

Hence it holds that $F_3(\phi_n) \xrightarrow{v} F_3(\phi)$ and F_3 is continuous on $\mathcal{U}_{r_3}$. Finally

$$F_3(F_2(F_1(\Phi_n))) \xrightarrow{\mathcal{D}} F_3(F_2(F_1(\Phi)))\,. \qquad \square$$

3. PROOF OF THEOREM 1

The following proof proceeds through several steps. The goal is to derive void probabilities for the resulting transformed process, which determine its distribution (see [7], p. 37).

First, the discrete version of the Boolean model is defined and its convergence (in distribution) to the original model is shown. Then we derive the distribution of the process corresponding to the Laslett's transform of the discrete model and compute its void probabilities. From the continuity of the Laslett's transform it follows that these probabilities converge to those of the transformed non-discrete process corresponding to Boolean model.

P r o o f o f T h e o r e m 1. Let $\{Z_m\}_{m\in\mathbb{N}}$ be a system of square grids of points in the space $\mathbb{R}^d$:

$$Z_m = \left\{ \frac{1}{\sqrt[d]{m}} \cdot z, \ z \in \mathbb{Z}^d \right\}.$$

We will use the notation Z_{m+} for $Z_m \cap \mathbb{R}^d_+$.

Let m be given and let $\{Y_z\}_{z\in Z_m}$ be a collection of independent and identically distributed Bernoulli random variables, for which

$$Y_z = \begin{cases} 1 & \text{with probability} \quad \frac{\lambda}{m}, \\ 0 & \text{with probability} \quad 1-\frac{\lambda}{m}. \end{cases}$$

Define a point process on Z_m by setting

$$\psi_m(z) = Y_z.$$

Further let $\{G_z\}_{z\in Z_m}$ be a given collection of i.i.d. grains distributed according to Λ_0 and independent of $\{Y_z\}_{z\in Z_m}$. Define

$$\begin{aligned} M^m_{G_z} &= \left\{ y \in Z_m : \ G_z \ \cap \ \left(y \oplus \left(0, \tfrac{1}{\sqrt[d]{m}}\right]^d \right) \neq \emptyset \right\}, \\ G^{m_0}_z &= \bigcup_{y\in M^m_{G_z}} y \oplus \left(0, \tfrac{1}{\sqrt[d]{m}}\right]^d, \\ G^m_z &= G^{m_0}_z - \operatorname{lexmin}(G^{m_0}_z). \end{aligned}$$

It is now possible to define the underlying process for the discrete version of the Boolean model:

$$\Phi_m = \sum_{z\in Z_m, \ \psi_m(z)>0} \delta_{(z,G^m_z)}.$$

Note that the Laslett's transform of Φ_m leaves its points in Z_m.

We shall show that the processes $\{\Phi_m\}_{m=1}^{\infty}$ converge in distribution to the Poisson process corresponding to X. This is equivalent to convergence of corresponding Laplace transforms (see [6], p. 27).

$$\begin{aligned} \mathcal{L}_{\Phi_m}(f) &= \mathrm{E}\, e^{-\Phi_m(f)} = \mathrm{E} \exp\left\{ -\sum_{z\in Z_m} f\left(z, G^m_z\right) \psi_m(z) \right\} \\ &= \mathrm{E}\left[\mathrm{E}\left[\exp\left\{ -\sum_{z\in Z_m} f\left(z, G^m_z\right) \psi_m(z) \right\} \middle| \{G_z\}_{z\in Z_m\cap \operatorname{spt} f|_{\mathbb{R}^d}} \right]\right] \\ &= \mathrm{E} \prod_{z\in Z_m\cap \operatorname{spt} f|_{\mathbb{R}^d}} \mathrm{E}\left[\exp\left\{ -f\left(z, G^m_z\right) \psi_m(z) \right\} \middle| \{G_z\}_{z\in Z_m\cap \operatorname{spt} f|_{\mathbb{R}^d}} \right] \\ &= \mathrm{E} \prod_{z\in Z_m\cap \operatorname{spt} f|_{\mathbb{R}^d}} \left(\frac{\lambda}{m} e^{-f(z,G^m_z)} + \left(1 - \frac{\lambda}{m}\right)\right) \end{aligned}$$

$$= \prod_{z \in Z_m \cap \operatorname{spt} f|_{\mathbb{R}^d}} \left(1 - \frac{\lambda}{m} \int \left(1 - e^{-f(z, G_z^m)}\right) \Lambda_0\left(\mathrm{d}G_z\right)\right)$$

$$= \exp\left\{\sum_{z \in Z_m} \log\left(1 - \frac{\lambda}{m} \int \left(1 - e^{-f(z, G_z^m)}\right) \Lambda_0\left(\mathrm{d}G_z\right)\right)\right\} .$$

Using the Taylor expansion we have the estimate

$$-x - \frac{x^2}{2} \frac{1}{(1-x)^2} \leq \log(1-x) \leq -x - \frac{x^2}{2}, \quad \text{where } x \in [0, 1) .$$

Since f is bounded we derive:

$$\lim_{m \to \infty} \sum_{z \in Z_m} \frac{\lambda^2}{2m^2} \left(\int \left(1 - e^{-f(z, G_z^m)}\right) \Lambda_0(\mathrm{d}G_z)\right)^2$$

$$\leq \lim_{m \to \infty} \sum_{z \in Z_m \cap \operatorname{spt} f|_{\mathbb{R}^d}} \frac{\lambda^2}{2m^2} \left(\int \left(1 - e^{-K}\right) \Lambda_0(\mathrm{d}G_z)\right)^2$$

$$= \lim_{m \to \infty} \lambda_{\mathrm{d}}(\operatorname{spt} f|_{\mathbb{R}^d})\, m \,\frac{\lambda^2}{2m^2}\, (1 - e^{-K})^2 = 0$$

and

$$\lim_{m \to \infty} \sum_{z \in Z_m} \frac{\lambda^2}{2m^2} \frac{\left(\int \left(1 - e^{-f(z, G_z^m)}\right) \Lambda_0(\mathrm{d}G_z)\right)^2}{\left(1 - \frac{\lambda}{m} \int \left(1 - e^{-f(z, G_z^m)}\right) \Lambda_0(\mathrm{d}G_z)\right)^2}$$

$$\geq \lim_{m \to \infty} \sum_{z \in Z_m \cap \operatorname{spt} f|_{\mathbb{R}^d}} \frac{\lambda^2}{2m^2} \frac{\left(\int \left(1 - e^{-L}\right) \Lambda_0(\mathrm{d}G_z)\right)^2}{\left(1 - \frac{\lambda}{m} \int \left(1 - e^{-K}\right) \Lambda_0(\mathrm{d}G_z)\right)^2}$$

$$= \lim_{m \to \infty} \lambda_{\mathrm{d}}(\operatorname{spt} f|_{\mathbb{R}^d})\, m \,\frac{\lambda^2}{2m^2} \,\frac{c_1^2}{(1 - \frac{\lambda}{m} c_2)^2} = 0 .$$

Hence $\lim\limits_{m \to \infty} \mathcal{L}_{\Phi_m}(f) = \lim\limits_{m \to \infty} \exp\left\{\sum\limits_{z \in Z_m} -\frac{\lambda}{m} \int \left(1 - e^{-f(z, G_z^m)}\right) \Lambda_0\left(\mathrm{d}G_z\right)\right\}$.
Further

$$d_H\left(G_z, G_z^m\right) \leq d^{\frac{1}{2}} m^{-\frac{1}{d}} + \lambda_1\Big(G_z^{m_0} \cap \{(x_1, 0, \ldots, 0) : x_1 \leq 0\}\Big) \to 0, \quad \text{for } m \to \infty.$$

From the continuity of e^{-f}, for any $\varepsilon > 0$ there exists m_0 such that for all $m > m_0$,

$$\left|e^{-f(z, G_z^m)} - e^{-f(z, G_z)}\right| < \frac{\varepsilon}{\lambda \cdot \lambda_{\mathrm{d}}(\operatorname{spt} f|_{\mathbb{R}^d})} .$$

The definition of Riemann integral finally gives

$$\begin{aligned}\lim_{m\to\infty} \mathcal{L}_{\Phi_m}(f) &= \lim_{m\to\infty} \exp\left\{\sum_{z\in Z_m} -\tfrac{\lambda}{m}\int\left(1-e^{-f(z,G_z^m)}\right)\Lambda_0\left(\mathrm{d}G_z\right)\right\}\\ &= \lim_{m\to\infty} \exp\left\{\sum_{z\in Z_m} -\tfrac{\lambda}{m}\int\left(1-e^{-f(z,G_z)}\right)\Lambda_0\left(\mathrm{d}G_z\right)\right\}\\ &= \exp\left\{-\lambda\int\int\left(1-e^{-f(z,G_z)}\right)\Lambda_0(\mathrm{d}G_z)\,\mathrm{d}z\right\},\end{aligned}$$

which is the Laplace functional of the process Φ.

Now we will focus on the Laslett's transform for the discrete model Φ_m. Denote it by L. $L(\Phi_m)$ can be expressed in the following way:

$$L(\Phi_m)(z) = Y_{L^{-1}(z)},\ z\in Z_m.$$

According to Lemma 7 it follows that $L(\Phi_m) \xrightarrow{\mathcal{D}} L(\Phi)$.

Let $K\subseteq \mathbb{R}_+^d$ be an arbitrary compact set, $\mu = \mathrm{card}(K\cap Z_m^+)$. Set $A(z_0) = \{y\in Z_m : y <_{lex} z_0\}$ and $K(z) = K\cap A(z)$. Then

$$\Pr(L(\Phi_m)(K)=0) = \prod_{z\in K\cap Z_m^+} \Pr\left(L(\Phi_m)(z)=0\middle|L(\Phi_m)(K(z))=0\right).$$

This can be shown by sequential conditioning taking points from $K\cap Z_m^+$ from the largest to the smallest (according to lexicographical order). For $K=\emptyset$ the probability is taken to be unconditioned.

Set $g(z)=\{y:\ y_1\geq z_1,\ y_k=z_k,\ k=2,\dots,d\}$. We have

$$\begin{aligned}&\Pr(L(\Phi_m)(K)=0)\\ &= \prod_{z\in K\cap Z_m^+}\sum_{y\in g(z)} \Pr\left(Y_y=0 \mid L(\Phi_m)(K(z))=0,\ L^{-1}(z)=y\right)\\ &\qquad\qquad \times\Pr\left(L^{-1}(z)=y\mid L(\Phi_m)(K(z))=0\right)\\ &= \prod_{z\in K\cap Z_m^+}\left(1-\frac{\lambda}{m}\right)\sum_{y\in g(z)}\Pr\left(L^{-1}(z)=y\mid L(\Phi_m)(K(z))=0\right)\\ &= \prod_{z\in K\cap Z_m^+}\left(1-\frac{\lambda}{m}\right) = \left(1-\frac{\lambda}{m}\right)^{\mu}\longrightarrow e^{-\lambda\ \lambda_d(K)},\quad \text{for } m\to\infty.\end{aligned}$$

It remains to show that $\Pr(L(\Phi_m)(K)=0)\overset{m\to\infty}{\longrightarrow}\Pr(L(\Phi)(K)=0)$. Since we know that $L(\Phi_m)\xrightarrow{\mathcal{D}} L(\Phi)$ this follows from convergence in distribution for K stochastically continuous, i.e. $\Pr(L(\Phi)(\partial K)>0)=0$ (see [6], p. 26).

Since $\mathcal{K}$ is generated by rectangles with faces parallel to the coordinate system it will be sufficient to show that these rectangles are stochastically continuous sets.

Let F be a face of K that is not parallel to N_0. Since the Laslett's transform shifts points in direction orthogonal to N_0 it obviously follows from the properties of Boolean model X that $\Pr(L(\Phi)(F) > 0) = 0$.

Let $F_p \subseteq N_0$ be compact and assume for contradiction that there exists $x \in \mathbb{R}^d_+$ such that

$$\Pr\left(L(\Phi)(x + F_p) \geq 1\right) > 0.$$

The stationarity of X and property $\Pr(X \cap (F_p \times [0, r]) = \emptyset) > 0$ together imply that there exist many points with the same properties like x. Particulary

$$\Pr\left(L(\Phi)(x + F_p) \geq 1\right) > 0, \quad \text{for } x \in [a, b],\ b > a.$$

Hence $\Pr\left(L(\Phi)(F_p \times [a, b]) = \infty\right) \geq \Pr\left(L(\Phi)(F_p + a + \frac{b}{n}) \geq 1, n \geq 1\right) > 0$. This is a contradiction to the result of Lemma 6 and hence K is stochastically continuous.

Thus the derived limit implies that the process $L(\Phi)$ is stationary Poisson with intensity λ. Theorem 1 is proved. □

ACKNOWLEDGEMENT

This work was supported by the Grant Agency of the Czech Republic under Grants 201/05/H005 and 201/06/0302.

The author is thankful to his supervisor Doc. RNDr. Jan Rataj, CSc. who drew the attention to the problem of Laslett's transform, helped to show basic steps of the given proof and made helpful remarks on this text.

(Received December 27, 2005.)

REFERENCES

[1] A. D. Barbour and V. Schmidt: On Laslett's transform for the Boolean model. Adv. in Appl. Probab. *33* (2001), 1–5.
[2] P. Billingsley: Convergence of Probability Measures. Second edition. Wiley, New York 1999.
[3] N. A. C. Cressie: Statistics for Spatial Data. Second edition. Wiley, New York 1993.
[4] I. S. Molchanov: Statistics of the Boolean model: From the estimation of means to the estimation of distribution. Adv. Appl. Probab. *27* (1995), 63–86.
[5] I. S. Molchanov: Statistics of the Boolean Model for Practioners and Mathematicions. Wiley, Chichester 1997.
[6] J. Rataj: Point Processes (in Czech). Karolinum, Prague 2000.
[7] D. Stoyan, W. S. Kendall, and J. Mecke: Stochastic Geometry and Its Applications. Akademie–Verlag, Berlin 1987.

Rostislav Černý, Mathematical Institute of Charles University, Sokolovská 83, 186 75 Praha 8. Czech Republic.
e-mail: rostislav.cerny@karlin.mff.cuni.cz