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POWER ANALYSIS OF VOTING BY COUNT AND ACCOUNT

Midori Hirokawa and Milan Vlach

Using players' Shapley–Shubik power indices, Peleg [4] proved that voting by count and account is more egalitarian than voting by account. In this paper, we show that a stronger shift in power takes place when the voting power of players is measured by their Shapley–Shubik indices. Moreover, we prove that analogous power shifts also occur with respect to the absolute Banzhaf and the absolute Johnston power indices.

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1. INTRODUCTION

We are concerned with a double majority voting rule composed of the common simple majority rule (voting by count) and the weighted majority rule (voting by account). This compound rule has a long history going back at least to elections in some Jewish communities in Europe in the seventeenth century, see Peleg [4]. It turns out that similar voting rules are today in use also.

For example, Taylor and Zwicker [5, page 19] notice that the creation of a new sewer district in certain municipalities requires the approval of half the people in the district, subject to the condition that those approving must own property that is worth more than fifty percent of the total assessed value of all property in the district.

Other examples are provided by decision making procedures used in stockholders' meetings or in conducting rehabilitation of financially troubled companies. For instance, as a response to the high increase of bankruptcies in Japan in the late 1990s, the Civil Rehabilitation Law was passed and took effect, April 1, 2000. According to this law, the passage of the rehabilitation plan requires affirmative votes only by unsecured creditors who are entitled to vote, have attended the creditors meeting, constitute the majority of attending persons entitled to vote, and also hold one half or more of the total amount of unsecured claims in face value; see Hirokawa and Xu [2] for details.

It is therefore of interest to study the relationship between game-theoretic solutions of a weighted majority rule (voting by account) and a compound rule (voting by count and account). One of the possible approaches is to examine power indices, which reflect voting power of players. From a rather general result, Peleg [4] derives that, in a certain well defined sense, the vector of players' Shapley–Shubik power indices for voting by count and account dominates the corresponding vector for voting by account. As a consequence, voting by count and account is more egalitarian than voting by account when the distribution of power among players is measured by their Shapley–Shubik indices.

In addition to the Shapley–Shubik power index, several other power indices for evaluating the power of a player have been proposed, see Brams, Lucas and Straffin [1]. Since it is far from clear which of the various proposed measures reflect better the realities of power relationships, it is of interest to know whether Peleg's result is valid also for other indices. Here we address this question for the case of the absolute Banzhaf, Johnston, and Deegan–Packel power indices.

Since some simple games are well suited to capture the essence of voting rules, we follow Peleg in using game theoretic tools for analysing the power of individual players in the mentioned voting systems. We provide an alternative proof of Peleg's result on egalitarianism, show that a stronger result hold, and prove that an analogous shift in power also occurs when the power distribution is measured by the absolute Banzhaf or absolute Johnston indices. On the other hand, we show this is not true for the case of the absolute Deegan–Packel index.

Throughout the paper the symbol $\sharp$ stands for "the number of elements in", and a $\Box$ indicates the end of proof.

2. SIMPLE GAMES

Let n be a positive integer. By a *simple game* of n players we mean an ordered pair $(N, \mathcal{W})$ in which $N = \{1, 2, \ldots, n\}$, and $\mathcal{W}$ is a nonempty collection of subsets of N such that $S \in \mathcal{W}$ together with $S \subset S' \subset N$ implies $S' \in \mathcal{W}$.

Members of N are called *players*, nonempty subsets of N are called *coalitions*, members of $\mathcal{W}$ are called *winning coalitions*, and coalitions that are not in $\mathcal{W}$ are called *losing coalitions*. A winning coalition all of whose proper subsets are losing is called a *minimal winning coalition*.

A simple game is said to be *weighted* if there exists a real valued function w on N and a real number q such that a coalition is winning precisely when the sum of the weights (values of w) of the players in the coalition meets or exceeds q. To simplify the notation, we use w_k instead of $w(k)$ to denote the weight of player k and $w(S)$ to denote the sum of the weights of all players in coalition S. Under this setting, a particular weighted simple game can be described by $[q; w_1, w_2, \ldots, w_n]$.

Following Peleg [4], when referring to the "majority of the count rule", we have in mind the common simple majority voting rule, that is, a simple game in which a coalition is winning if and only if it contains more than half of all players. Similarly, when referring to the "majority of the account rule", we mean a simple game in which a coalition is winning if and only if the sum of the weights of its members exceeds half the sum of all weights. Finally, given a simple game $G^a = (N, \mathcal{W}^a)$ representing the majority of the account rule and a simple game $G^c = (N, \mathcal{W}^c)$

representing the majority of the count rule, the majority of the count and account rule is represented by the simple game $G^* = (N, \mathcal{W}^*)$ in which a coalition is winning if and only if it is winning in both G^a and G^c, that is, if and only if $\mathcal{W}^* = \mathcal{W}^a \cap \mathcal{W}^c$.

While both the count rule and the account rule can be represented by weighted simple games, the following example due to Peleg [4] shows that the compound game G^* representing the majority of the count and account rule is not necessarily a weighted game.

Example. Let G^* be the simple game with seven players $1, 2, \ldots, 7$ composed from the simple majority game and the weighted game

$$[q; w_1, w_2, \ldots, w_7] = [25; 18, 8, 8, 7, 4, 2, 2].$$

To show that this compound game is not weighted, suppose to the contrary that there exists a weighted simple game $[q^*; w_1^*, w_2^*, \ldots, w_7^*]$ that is equal to G^*. Now consider the following four coalitions

$$S_1 = \{2, 3, 4, 5\}, \quad S_2 = \{1, 5, 6, 7\}, \quad S_3 = \{1, 4, 5\}, \quad S_4 = \{2, 3, 5, 6, 7\}.$$

Easy calculations show that coalitions S_1 and S_2 are winning whereas coalitions S_3 and S_4 are losing. Hence,

$$w_1^* + w_4^* + w_5^* < q^* \leq w_2^* + w_3^* + w_4^* + w_5^*,$$

and

$$w_2^* + w_3^* + w_5^* + w_6^* + w_7^* < q^* \leq w_1^* + w_5^* + w_6^* + w_7^*.$$

It follows that

$$w_1^* < w_2^* + w_3^*, \quad \text{and} \quad w_2^* + w_3^* < w_1^*.$$

Since the last two inequalities contradict each other, we conclude that G^* is not weighted.

3. POWER INDICES

It is well known that the power of players in a weighted simple game need not be proportional to players' weights. For example, consider the simple game of three players in which a coalition is winning if and only if it has more than one member. Since this game can be given in the form of weighted game $[2; 1, 1, 1]$, it is reasonable to claim that all players have the same power. However, this game can also be given as a weighted game $[51; 49, 48, 3]$ with unequal weights. In this simple case, the distribution of power among players is transparent, but generally it is not clear how to measure the power of players, and in consequence, several quantitative measures for evaluating power of a player have been proposed.

We are concerned with the Shapley–Shubik, Banzhaf, Johnston, and Deegan–Packel power indices of players in simple games. To recall their definitions we need the notions of vulnerable coalitions and essential players.

Let S be a winning coalition in a simple game $G = (N, \mathcal{W})$. We say that coalition S is *vulnerable* if, among its members, there is at least one player whose defection would result in a losing coalition. Such a player will be called *essential* for coalition S. The collection of the vulnerable coalitions for which a player k is essential will be denoted by $\mathcal{W}(k)$, that is,

$$\mathcal{W}(k) = \{S \in \mathcal{W} : \ k \in S \ \text{ and } \ S \setminus \{k\} \notin \mathcal{W}\}.$$

Now we are ready to define the mentioned power indices for an arbitrary simple game $G = (N, \mathcal{W})$.

Shapley–Shubik power index: Let $g_\varphi : \mathcal{P}(N) \to (0, \infty)$ be defined by

$$g_\varphi(S) = (\sharp S - 1)!(n - \sharp S)!.$$

For each $k \in N$, the Shapley–Shubik power index φ_k of player k in game G is

$$\varphi_k = \frac{1}{n!} \sum\nolimits_{S \in \mathcal{W}(k)} g_\varphi(S).$$

Banzhaf power index: Let $g_\beta : \mathcal{P}(N) \to (0, \infty)$ be defined by

$$g_\beta(S) = 1 \quad \text{for each } S.$$

For each $k \in N$, the absolute Banzhaf power index β_k of player k in game G is

$$\beta_k = \frac{1}{2^{n-1}} \sum\nolimits_{S \in \mathcal{W}(k)} g_\beta(S).$$

Johnston power index: Let $d(S)$ be the number of essential players in S, and let $g_\gamma : \mathcal{P}(N) \to (0, \infty)$ be defined by

$$g_\gamma(S) = \frac{1}{d(S)}.$$

For each $k \in N$, the absolute Johnston power index γ_k of player k in game G is

$$\gamma_k = \sum\nolimits_{S \in \mathcal{W}(k)} g_\gamma(S).$$

Deegan–Packel power index: For each player k, the absolute Deegan–Packel power index δ_k of player k in game G is defined in a similar way as the absolute Johnston index of player k. The only difference is that the sum is taken over the set of minimal winning coalitions containing player k instead of over the set of vulnerable coalitions for which player k is essential.

In what follows, by a *power index of a game* we mean an ordered n-tuple of players' power indices of the same type. For example, the n-tuple $(\gamma_1, \gamma_2, \ldots, \gamma_n)$ denotes the absolute Johnston index of a game of n players.

4. MONOTONICITY

To introduce some auxiliary results useful in proving our main result on power shifts, we prove in this section that, for G^a, all power indices under consideration, with the exception of Deegan–Packel indices, are monotone with respect to weights. It is well known that this is true for all weighted simple games. We have seen in the previous section that game G^*, which is completely determined by its weights, need not be a weighted game. We also show that the same monotonicity result holds for G^*.

For the simplicity of notation, we assume that

- all weights are positive integers,
- $w_1 \leq w_2 \leq \cdots \leq w_n$,
- $\bar{w}$ denotes the number $\frac{1}{2}(w_1 + w_2 + \cdots + w_n)$.

Then a power index of a game is said to be *monotone* if and only if the power index of player i is greater than or equal to the index of player j whenever $i > j$.

By the definition of games G^a and G^*, we have

$$\mathcal{W}^a = \{S \subset N : \ w(S) > \bar{w}\} \quad \text{and} \quad \mathcal{W}^* = \{S \subset N : \ w(S) > \bar{w} \ \& \ \sharp S > \frac{n}{2}\}.$$

Hence, for each player k,

$$\begin{aligned} S \in \mathcal{W}^a(k) \quad &\text{iff} \quad w(S) > \bar{w} \ \& \ k \in S \ \& \ w(S\backslash\{k\}) \leq \bar{w}, \\ S \in \mathcal{W}^*(k) \quad &\text{iff} \quad w(S) > \bar{w} \ \& \ k \in S \ \& \ \sharp S > \tfrac{n}{2} \\ &\qquad \& \ [\, w(S\backslash\{k\}) \leq \bar{w} \ \text{ or } \ \sharp(S\backslash\{k\}) \leq \tfrac{n}{2}\,]. \end{aligned}$$

To analyse relations between power indices, it is helpful to introduce the sets $\mathcal{W}^+(k)$ and $\mathcal{W}^-(k)$ of coalitions defined as follows:

$$\begin{aligned} S \in \mathcal{W}^-(k) \quad &\text{iff} \quad w(S) > \bar{w} \ \& \ k \in S \ \& \ \sharp S > \tfrac{n}{2} \ \& \ w(S\backslash\{k\}) \leq \bar{w}\ , \\ S \in \mathcal{W}^+(k) \quad &\text{iff} \quad k \in S \ \& \ \tfrac{n}{2} < \sharp S \leq \tfrac{n}{2} + 1 \ \& \ w(S\backslash\{k\}) > \bar{w}. \end{aligned}$$

It can easily be seen that:

$$\begin{aligned} &\mathcal{W}^*(k) = \mathcal{W}^+(k) \cup \mathcal{W}^-(k), \ \ \mathcal{W}^+(k) \cap \mathcal{W}^-(k) = \emptyset, \\ &\mathcal{W}^-(k) \subset \mathcal{W}^a(k), \quad \text{and} \quad \mathcal{W}^+(k) \cap \mathcal{W}^a(k) = \emptyset. \end{aligned}$$

It will also be helpful to introduce the mappings $f_{ij} : \mathcal{P}(N) \to \mathcal{P}(N)$ defined, for each i and j from N, by:

$$f_{ij}(S) = \begin{cases} S & \text{if} \quad i \in S, \\ (S\backslash\{j\}) \cup \{i\} & \text{if} \quad i \notin S. \end{cases}$$

Often we will use the following two properties of mappings f_{ij} :

(a) For each $S \subset N$, if $j \in S$, then $\sharp f_{ij}(S) = \sharp S$,

(b) For each $S, S' \subset N$, if $j \in S \cap S'$ and $i \neq j$, then $S \neq S'$ implies $f_{ij}(S) \neq f_{ij}(S')$.

Lemma 1. If $i > j$, then there is a one-to-one mapping

$$f : \mathcal{W}^a(j) \to \mathcal{W}^a(i)$$

such that, for each $S \in \mathcal{W}^a(j)$, $\sharp f(S) = \sharp S$ and $d(S) \geq d(f(S))$.

Proof. We verify that the restriction f of f_{ij} to the set $\mathcal{W}^a(j)$ satisfies all requirements. In view of the mentioned properties (a) and (b), we know that $\sharp f(S) = \sharp S$, and that f is a one-to-one mapping of $\mathcal{W}^a(j)$ into $\mathcal{P}(N)$. Thus it remains to prove that, for each $S \in \mathcal{W}^a(j)$, we have $f(S) \in \mathcal{W}^a(i)$ and $d(S) \geq d(f(S))$.

Let S be a member of $\mathcal{W}^a(j)$. Then $w(S) > \bar{w}$ and $w(S\backslash\{j\}) \leq \bar{w}$.

(i) If $i \in S$, then $w(S\backslash\{i\}) \leq w(S\backslash\{j\})$ because $i > j$ implies $w_i \geq w_j$. Since $f(S) = S$ in this case, we obtain $i \in f(S)$, $w(f(S)) = w(S) > \bar{w}$, and $w(f(S)\backslash\{i\}) = w(S\backslash\{i\}) \leq w(S\backslash\{j\}) \leq \bar{w}$. Therefore $f(S) \in \mathcal{W}^a(i)$.

(ii) If $i \notin S$, then $f(S) = (S\backslash\{j\}) \cup \{i\}$, and we obtain $i \in f(S)$. Moreover, since $i > j$ implies $w_i \geq w_j$, we have $w((S\backslash\{j\}) \cup \{i\}) \geq w(S)$. It follows $w(f(S)) \geq w(S) > \bar{w}$. Since $f(S)\backslash\{i\} = S\backslash\{j\}$, we obtain $w(f(S)\backslash\{i\}) = w(S\backslash\{j\}) \leq \bar{w}$. Consequently, again $f(S) \in \mathcal{W}^a(i)$.

The validity of the inequality $d(S) \geq d(f(S))$ is obvious when i is in S. Thus suppose $i \notin S$. For this case, we have already proved that $w(S) \leq w(f(S))$. Let player k be essential for $f(S)$.

(i) If $k \neq i$, then $k \in S$. In this case, we can show that k is essential also for S. Since we know that $w(S) \leq w(f(S))$, we also know that $w(S \setminus \{k\}) \leq w(f(S)) \setminus \{k\})$, because k belongs both to S and $f(S)$. Since k is essential for $f(S)$, we have $w(f(S)) \setminus \{k\}) \leq \bar{w}$. It follows that $w(S \setminus \{k\}) \leq \bar{w}$, and therefore k is essential also for S.

(ii) If $k = i$, then $k \notin S$, $k \neq j$ and k is inessential for S.

Since j is essential for S and inessential for $f(S)$, it is clear from (i) and (ii) that the number of players essential for S cannot be less than the number of players essential for $f(S)$. □

Lemma 2. If $i > j$, then there is a one-to-one mapping

$$f : \mathcal{W}^*(j) \to \mathcal{W}^*(i)$$

such that, for each $S \in \mathcal{W}^*(j)$, $\sharp f(S) = \sharp S$ and $d(S) \geq d(f(S))$.

Proof. Using analogous arguments to those used in the proof of Lemma 1, we can easily prove that the restriction f of f_{ij} to the set $\mathcal{W}^*(j)$ has all required properties. As an illustration, let us verify that

$$w(f(S)\backslash\{i\}) \leq \bar{w} \ \text{ or } \ \sharp(f(S)\backslash\{i\}) \leq \frac{n}{2}$$

whenever $S \in \mathcal{W}^*(j)$. Since $S \in \mathcal{W}^*(j)$, we know that

$$w(S\backslash\{j\}) \le \bar{w} \text{ or } \sharp(S\backslash\{j\}) \le \frac{n}{2}.$$

Suppose that $\sharp(S\backslash\{j\}) \le \frac{n}{2}$. If $i \in S$, then $\sharp(f(S)\backslash\{i\}) = \sharp(S\backslash\{i\}) = \sharp(S\backslash\{j\}) \le \frac{n}{2}$. If $i \notin S$, then $\sharp(f(S) \setminus \{i\}) = \sharp(((S \setminus \{j\}) \cup \{i\}) \setminus \{i\}) = \sharp(S \setminus \{j\}) \le \frac{n}{2}$. If our assumption is not satisfied, that is, if $\sharp(S\backslash\{j\}) > \frac{n}{2}$, then $w(S\backslash\{j\}) \le \bar{w}$. If $i \in S$, then $w(f(S)\backslash\{i\}) = w(S\setminus\{i\}) = w(S) - w_i \le w(S) - w_j = w(S\setminus\{j\}) \le \bar{w}$. If $i \notin S$, then $w(f(S)\backslash\{i\}) = w(((S \setminus \{j\}) \cup \{i\}) \setminus \{i\}) = w(S \setminus \{j\}) \le \bar{w}$. □

Lemma 3. If $i > j$, then

$$\sum\nolimits_{S\in\mathcal{W}^a(j)} g(S) \le \sum\nolimits_{S\in\mathcal{W}^a(i)} g(S) \text{ and } \sum\nolimits_{S\in\mathcal{W}^*(j)} g(S) \le \sum\nolimits_{S\in\mathcal{W}^*(i)} g(S)$$

for each nonnegative real-valued function g on $\mathcal{P}(N)$ such that $g(S) = g(S')$ whenever $\sharp S = \sharp S'$.

Proof. According to Lemma 1, there exists a one-to-one mapping f of $\mathcal{W}^a(j)$ into $\mathcal{W}^a(i)$ such that $\sharp f(S) = \sharp S$ for each $S \in \mathcal{W}^a(j)$. Since $g(S) = g(S')$ whenever $S = S'$, we have $\sum_{S\in\mathcal{W}^a(j)} g(S) = \sum_{S\in\mathcal{W}^a(j)} g(f(S))$. However, $\{f(S) : S \in \mathcal{W}^a(j)\}$ is a subset of $\mathcal{W}^a(i)$ and f is a one-to-one mapping. Therefore, $\sum_{S\in\mathcal{W}^a(j)} g(f(S)) \le \sum_{S\in\mathcal{W}^a(i)} g(S)$. Consequently, we have $\sum_{S\in\mathcal{W}^a(j)} g(S) \le \sum_{S\in\mathcal{W}^a(i)} g(S)$. Similarly, using Lemma 2 instead of Lemma 1, we can obtain the inequality $\sum_{S\in\mathcal{W}^*(j)} g(S) \le \sum_{S\in\mathcal{W}^*(i)} g(S)$. □

Theorem 1. All game power indices under consideration, with the exception of Deegan–Packel index, are monotone both in G^a and G^*.

Proof. First we note that, for each k, each power index of player k can be written in the form

$$\sum\nolimits_{S\in\mathcal{W}^a(k)} g(S) \text{ and } \sum\nolimits_{S\in\mathcal{W}^*(k)} g(S)$$

where g is a nonnegative real valued function on $\mathcal{P}(N)$. The monotonicity of the Shapley–Shubik index and the monotonicity of the Banzhaf index then follow directly from Lemma 3.

Regarding the Johnston index, by the same logic as in the proof of Lemma 3, we first observe that by Lemma 1, $\sum_{S\in\mathcal{W}^a(j)} g(f(S)) \le \sum_{S\in\mathcal{W}^a(i)} g(S)$. From Lemma 1 we also know $1/d(S) \le 1/d(f(S))$ and therefore, for the corresponding g, we have $g(S) \le g(f(S))$ for each S from $\mathcal{W}^a(j)$. Consequently $\sum_{S\in\mathcal{W}^a(j)} g(S) \le \sum_{S\in\mathcal{W}^a(j)} g(f(S))$. Therefore $\sum_{S\in\mathcal{W}^a(j)} g(S) \le \sum_{S\in\mathcal{W}^a(i)} g(S)$ as required. Thus the Johnston index of game $G^a$is monotone. Using Lemma 2 analogously, we obtain the monotonicity of the Johnston index of G^*. □

The following example shows that the absolute Deegan–Packel index of game G^a need not be monotonic.

Example. Consider G^a with five players whose weights are $w_1 = w_2 = w_3 = 1$ and $w_4 = 2, w_5 = 3$. Let $\mathcal{M}^a(k)$ be the set of minimal coalitions in G^a containing player k. Then we have: $\mathcal{M}^a(1) = \{125, 135, 1234\}$, $\mathcal{M}^a(2) = \{125, 235, 1234\}$, $\mathcal{M}^a(3) = \{135, 235, 1234\}$, $\mathcal{M}^a(4) = \{45, 1234\}$, $\mathcal{M}^a(5) = \{45, 125, 135, 235\}$, where inner brackets are omitted when specifying coalitions. The resulting absolute Deegan–Packel power index is then as follows:

$$(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5) = \left(\frac{11}{12}, \frac{11}{12}, \frac{11}{12}, \frac{9}{12}, \frac{18}{12}\right).$$

5. POWER SHIFT THEOREM

In this section we prove that a certain shift of power occurs when the voting by account is replaced by voting by count and account. We will need the following two lemmata, the proofs of which we omit, because they are analogous to the proofs of previous Lemma 1 and Lemma 2.

Lemma 4. If $i > j$, then there is a one-to-one mapping

$$f : \mathcal{W}^a(j)\backslash\mathcal{W}^-(j) \to \mathcal{W}^a(i)\backslash\mathcal{W}^-(i)$$

such that, for each $S \in \mathcal{W}^a(j)\backslash\mathcal{W}^-(j)$, $\sharp f(S) = \sharp S$ and $d(S) \geq d(f(S))$.

Lemma 5. If $i > j$, then there is a one-to-one mapping

$$f : \mathcal{W}^+(i) \to \mathcal{W}^+(j)$$

such that, for each $S \in \mathcal{W}^+(i)$, $\sharp f(S) = \sharp S$ and $d(S) \geq d(f(S))$.

For each k, let $\varphi_k^a, \beta_k^a, \gamma_k^a$ and $\varphi_k^*, \beta_k^*, \gamma_k^*$ denote the corresponding power indices of player k in games G^a and G^*.

Theorem 2. If $i > j$, then

$$\varphi_i^* - \varphi_j^* \leq \varphi_i^a - \varphi_j^a, \quad \beta_i^* - \beta_j^* \leq \beta_i^a - \beta_j^a, \quad \gamma_i^* - \gamma_j^* \leq \gamma_i^a - \gamma_j^a.$$

P r o o f. Since, for each player k,

$$\mathcal{W}^+(k) \cap \mathcal{W}^-(k) = \emptyset, \quad \mathcal{W}^*(k) = \mathcal{W}^+(k) \cup \mathcal{W}^-(k), \quad \mathcal{W}^-(k) \subset \mathcal{W}^a(k),$$

we obtain

$$\begin{aligned} & \sum\nolimits_{S\in\mathcal{W}^*(k)} g(S) - \sum\nolimits_{S\in\mathcal{W}^a(k)} g(S) \\ = \quad & \sum\nolimits_{S\in\mathcal{W}^+(k)} g(S) + \sum\nolimits_{S\in\mathcal{W}^-(k)} g(S) - \sum\nolimits_{S\in\mathcal{W}^a(k)} g(S), \end{aligned}$$

and $$\sum_{S\in\mathcal{W}^-(k)} g(S) - \sum_{S\in\mathcal{W}^a(k)} g(S) = -\sum_{S\in\mathcal{W}^a(k)\backslash\mathcal{W}^-(k)} g(S)$$

for each k and each function g. Consequently

$$\sum_{S\in\mathcal{W}^*(i)} g(S) - \sum_{S\in\mathcal{W}^a(i)} g(S) = \sum_{S\in\mathcal{W}^+(i)} g(S) - \sum_{S\in\mathcal{W}^a(i)\backslash\mathcal{W}^-(i)} g(S).$$

It follows from Lemma 4 and Lemma 5 that

$$\sum_{S\in\mathcal{W}^+(i)} g(S) \leq \sum_{S\in\mathcal{W}^+(j)} g(S)$$

and $$\sum_{S\in\mathcal{W}^a(i)\backslash\mathcal{W}^-(i)} g(S) \geq \sum_{S\in\mathcal{W}^a(j)\backslash\mathcal{W}^-(j)} g(S).$$

Therefore we have

$$\begin{aligned}&\sum_{S\in\mathcal{W}^*(i)} g(S) - \sum_{S\in\mathcal{W}^a(i)} g(S) \leq \sum_{S\in\mathcal{W}^+(j)} g(S) - \sum_{S\in\mathcal{W}^a(j)\backslash\mathcal{W}^-(j)} g(S)\\ &= \sum_{S\in\mathcal{W}^*(j)} g(S) - \sum_{S\in\mathcal{W}^a(j)} g(S),\end{aligned}$$

which is equivalent to the required inequality for each of the corresponding functions g. □

Theorem 2 establishes a power shift from "bigger" players to "smaller" players in the sense that the bigger (smaller) players have less or equal (more or equal) power in the compound game than in the corresponding weighted game. Hence, in voting by count and account, the smaller player is never in a worse position than he or she is in the voting by account. The following example shows that this is not necessarily true in the case of players' Deegan–Packel indices.

Example. Consider G^a with seven players whose weights are $w_k = 1$ for $k = 1,\ldots,5$ and $w_6 = w_7 = 3$. Let $\mathcal{M}^a(k)$ (resp., $\mathcal{M}^*(k)$) be the set of all minimal coalitions in G^a (resp., G^*) containing player k. Then we obtain:

$$\begin{aligned}\mathcal{M}^a(1) &= \{1236, 1246, 1256, 1346, 1356, 1456, 1237, 1247, 1257, 1347, 1357, 1457\},\\ \mathcal{M}^a(6) &= \{1236, 1246, 1256, 1346, 1356, 1456, 2346, 2356, 2456, 3456, 67\},\\ \mathcal{M}^*(1) &= \mathcal{M}^a(1) \cup \{1267, 1367, 1467, 1567\},\\ \mathcal{M}^*(6) &= (\mathcal{M}^a(6)\backslash\{67\}) \cup \{1267, 1367, 1467, 1567, 2367, 2467, 2567, 3467, 3567, 4567\}.\end{aligned}$$

The resulting absolute Deegan–Packel indices of players' power are then as follows:

$$\begin{aligned}\delta_k^a &= \frac{1}{4}\cdot 12 = 3 \quad \text{for} \quad k = 1,\ldots,5, \quad \delta_6^a = \delta_7^a = \frac{1}{4}\cdot 10 + \frac{1}{2} = 3 \quad \text{and}\\ \delta_k^* &= \frac{1}{4}\cdot 16 = 4 \quad \text{for} \quad k = 1,\ldots,5, \quad \delta_6^* = \delta_7^* = \frac{1}{4}\cdot 20 = 5.\end{aligned}$$

Hence, in this case, the absolute Deegan–Packel index of G^* is less egalitarian than that of G^a.

Peleg [4] shows a similar power shift theorem for the Shapley–Shubik power index. In particular, he shows that $\sum_{k=1}^{i} \varphi_k^* \geq \sum_{k=1}^{i} \varphi_k$ for $i = 1, 2, \cdots, n$. In view of the Proposition, presented in the Appendix, this result can be obtained as a consequence of Theorem 2, which provides an alternative proof of this particular result of Peleg. Moreover, we have proved that analogous power shifts also occur when players' power is measured by the absolute Banzhaf or the absolute Johnston power indices.

APPENDIX

Let $\phi_1, \phi_2, \cdots, \phi_n$, $\phi_1^*, \phi_2^*, \cdots, \phi_n^*$ be non-negative real numbers such that

$$\sum_{k=1}^{n} \phi_k = 1 \quad \text{and} \quad \sum_{k=1}^{n} \phi_k^* = 1.$$

Proposition If $\phi_i^* - \phi_j^* \leq \phi_i - \phi_j$ for all i, j with $i \geq j$, then $\sum_{k=1}^{i} \phi_k^* \geq \sum_{k=1}^{i} \phi_k$ for $i = 1, \cdots, n$. The converse is not true.

Proof. (i) If $\phi_i^* - \phi_j^* \leq \phi_i - \phi_j$ for i, j $(i \geq j)$, then we have $\sum_{k=1}^{n-1} (\phi_n^* - \phi_k^*) \leq \sum_{k=1}^{n-1} (\phi_n - \phi_k)$. Hence

$$(n-1)\phi_n^* - \sum_{k=1}^{n-1} \phi_k^* \leq (n-1)\phi_n - \sum_{k=1}^{i} \phi_k.$$

Since $\phi_n^* = 1 - \sum_{k=1}^{n-1} \phi_k^*$ and $\phi_n = 1 - \sum_{k=1}^{n-1} \phi_k$,

$$(n-1)[1 - \sum_{k=1}^{n-1} \phi_k^*] - \sum_{k=1}^{n-1} \phi_k^* \leq (n-1)[1 - \sum_{k=1}^{n-1} \phi_k] - \sum_{k=1}^{n-1} \phi_k$$
$$(n-1) - n\sum_{k=1}^{n-1} \phi_k^* \leq (n-1) - n\sum_{k=1}^{n-1} \phi_k.$$

Therefore we have

$$\sum_{k=1}^{n-1} \phi_k^* \ \geq \ \sum_{k=1}^{n-1} \phi_k.$$

(ii) Again, if $\phi_i^* - \phi_j^* \leq \phi_i - \phi_j$ for i, j $(i \geq j)$, then $\sum_{k=1}^{n-2} (\phi_n^* - \phi_k^*) \leq \sum_{k=1}^{n-2} (\phi_n - \phi_k)$. Hence

$$(n-2)\phi_{n-1}^* - \sum_{k=1}^{n-2} \phi_k^* \leq (n-2)\phi_{n-1} - \sum_{k=1}^{n-2} \phi_k.$$

Since $\phi_{n-1}^* = (1 - \phi_n^*) - \sum_{k=1}^{n-1} \phi_k^*$ and $\phi_{n-1} = (1 - \phi_n) - \sum_{k=1}^{n-1} \phi_k$,

$$(n-2)(1 - \phi_n^*) - (n-1)\sum_{k=1}^{n-2} \phi_k^* \leq (n-2)(1 - \phi_n) - (n-1)\sum_{k=1}^{n-2} \phi_k.$$

From the last inequality of (i), it follows $1 - \phi_n^* \geq 1 - \phi_n$. Therefore, we have

$$\sum_{k=1}^{n-2} \phi_k^* \geq \sum_{k=1}^{n-2} \phi_k.$$

Similarly, we obtain the required inequalities

$$\sum_{k=1}^{i} \phi_k^* \geq \sum_{k=1}^{i} \phi_k$$

for the remaining values of i. $\square$

The following numbers show that the converse is not true when $n \geq 3$:

$$\phi_1 = 0.2, \quad \phi_2 = 0.3, \quad \phi_3 = 0.5, \quad \phi_1^* = 0.2, \quad \phi_2^* = 0.4, \quad \phi_3^* = 0.4.$$

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Midori Hirokawa, Faculty of Economics, Hosei University, 4342 Aihara, Machida, Tokyo 194-0298. Japan.
e-mail: mhiro@mt.tama.hosei.ac.jp

Milan Vlach, Kyoto College of Graduate Studies for Informatics, 7 Monzen-cho, Tanaka, Sakyo-ku, Kyoto 606-8225, Japan. On leave from Faculty of Mathematics and Physics, Charles University, Malostranské náměstí 25, 118 00 Praha 1, Czech Republic.
e-mails: m_vlach@kcg.ac.jp, mvlach@ksi.ms.mff.cuni.cz