KYBERNETIKA — VOLUME 42 (2006), NUMBER 4, PAGES 453 – 460

TOP RESPONSIVENESS AND NASH STABILITY IN COALITION FORMATION GAMES

Dinko Dimitrov and Shao Chin Sung

Top responsiveness was shown by Alcalde and Revilla [1] to guarantee the existence of core stable partitions in hedonic coalition formation games. In this paper we prove the existence of Nash stable partitions under top responsiveness when a mutuality condition is imposed.

Keywords: coalition formation, hedonic games, Nash stability, top responsiveness

AMS Subject Classification: C71, C72, C78, D71

1. INTRODUCTION

In a hedonic coalition formation game each player's preferences over coalitions depend solely on the composition of members of her coalition (cf. Drèze and Greenberg [5]). Given a hedonic game, one is usually interested in conditions that guarantee the existence of stable outcomes (partitions of the set of players). For instance, Banerjee et al. [2] focus on the existence of core stable partitions, while Bogomolnaia and Jackson [3] present sufficient conditions for the existence of Nash stable partitions as well.

Top responsiveness is introduced by Alcalde and Revilla [1] as a condition on players' preferences, which captures the idea of how each player believes that others could complement her in the formation of research teams. As shown by the authors, this condition suffices for the existence of core stable partitions in hedonic games. This result is provided constructively, i. e., an algorithm, called the *top covering algorithm*, is proposed for generating a core stable partition. A simplified version of the top covering algorithm, which always returns the same outcome as the top covering algorithm, is proposed by Dimitrov and Sung [4] (a preliminary version of this paper). We have shown in that paper that the top responsiveness condition guarantees the existence of strictly core stable partitions in hedonic games.

In this paper we elaborate on the question whether the top responsiveness condition guarantees the existence of *Nash stable* partitions in hedonic games. We provide an example to illustrate that Nash stable partitions may fail to exist, and prove that imposing a mutuality condition turns out to be sufficient for the existence.

2. SETUP

2.1. Hedonic games

Let $N = \{1, 2, \ldots, n\}$ be a finite set of players. Each nonempty subset of N is called a *coalition*. Each player i is endowed with preferences $\succeq_i$ over the set $\mathcal{A}^i = \{X \subseteq N \mid i \in X\}$ of all possible coalitions she may belong to, i. e., each $\succeq_i$ is a complete pre-ordering over $\mathcal{A}^i$. A *hedonic game* is described by a pair $\langle N, \succeq \rangle$, where $\succeq$ is a profile of players' preferences, i. e., $\succeq = (\succeq_1, \succeq_2, \ldots, \succeq_n)$. An *outcome* Π for $\langle N, \succeq \rangle$ is a partition of the player set N, i. e., Π is a collection of nonempty pairwise disjoint coalitions whose union is N. For each partition Π of N and for each player $i \in N$, we denote by $\Pi(i)$ the coalition in Π containing i.

Given a hedonic game $\langle N, \succeq \rangle$ and a partition Π of N, the notion of core stability is based on the absence of coalitional deviations in which every player is strictly better off in comparison to his corresponding coalition in the partition Π. On the other hand, Π is said to be strictly core stable if there is no coalitional deviations in which every player is weakly better off and at least one player is strictly better off in comparison to his corresponding coalition in the partition Π. Moreover, Π is said to be Nash stable if there is no player who would like to leave his current coalition in Π and stay either single or join another coalition in Π. In the formal definitions given below these three stability notions are defined in a "positive" way that will be very useful when providing our existence proof.

- A partition Π is *core stable* in the game $\langle N, \succeq \rangle$ if, for each nonempty $X \subseteq N$,
  - $\Pi(i) \succeq_i X$ for some $i \in X$;
- A partition Π is *strictly core stable* in the game $\langle N, \succeq \rangle$ if, for each nonempty $X \subseteq N$,
  - $\Pi(i) \succ_i X$ for some $i \in X$, or
  - $\Pi(i) \succeq_i X$ for every $i \in X$;
- A partition Π is *Nash stable* in the game $\langle N, \succeq \rangle$ if, for each $X \in \Pi \cup \{\emptyset\}$ and for each $i \in N$,
  - $\Pi(i) \succeq_i X \cup \{i\}$.

Observe that strict core stability implies core stability. On the other hand, there are no implications between Nash stability and the other two stability notions: neither core stability or strict core stability implies Nash stability, nor Nash stability implies core stability or strict core stability.

2.2. Choice sets and top responsiveness

Let $i \in N$ and $X \in \mathcal{A}^i$. We denote by $Ch(i, X) \subseteq 2^X \cap \mathcal{A}^i$ the set of maximals of i on X under $\succeq_i$, i. e.,

$$Ch(i, X) = \{Y \in 2^X \cap \mathcal{A}^i \mid Y \succeq_i Z \text{ for each } Z \in 2^X \cap \mathcal{A}^i\}.$$

Observe that each $Y \in Ch(i, X)$ satisfies $i \in Y \subseteq X$. Moreover, for each $Y, Z \subseteq 2^X \cap \mathcal{A}^i$, we have $Y \succ_i Z$ if $Y \in Ch(i, X)$ and $Z \notin Ch(i, X)$.

As in the work of Alcalde and Revilla [1], we assume that players' preferences satisfy *top responsiveness*, i. e., we assume that, for each $i \in N$, the following three conditions are satisfied:

Condition 1: For each $X \in \mathcal{A}^i$, $|Ch(i, X)| = 1$.

By $ch(i, X)$ we denote the unique maximal set of player i on X under $\succeq_i$, i. e., $Ch(i, X) = \{ch(i, X)\}$. Then,

Condition 2: For each pair $X, Y \in \mathcal{A}^i$, $X \succ_i Y$ if $ch(i, X) \succ_i ch(i, Y)$;

Condition 3: For each pair $X, Y \in \mathcal{A}^i$, $X \succ_i Y$ if $ch(i, X) = ch(i, Y)$ and $X \subset Y$.

As shown by Alcalde and Revilla [1] and Dimitrov and Sung [4], the top responsiveness condition suffices for the existence of strictly core stable partitions in hedonic games.

2.3. The simplified top covering algorithm

Here we present the algorithm proposed by Dimitrov and Sung [4]. The algorithm is a simplified version of the *top covering algorithm* introduced by Alcalde and Revilla [1]. It can be seen as a generalization of Gale's top trading cycle (see Shapley and Scarf [6] for more details).

Let t be a positive integer. We define a function $\mathcal{C}^t : N \times 2^N \to 2^N$ as follows. For each $i \in N$ and for each $X \in \mathcal{A}^i$,

- $\mathcal{C}^1(i, X) = ch(i, X)$, and
- $\mathcal{C}^{t+1}(i, X) = \bigcup_{j \in \mathcal{C}^t(i,X)} ch(j, X)$ for each positive integer t.

Let $i \in N$ and $X \in \mathcal{A}^i$. Observe that $j \in ch(j, X) \subseteq X$ if $j \in X$ (i. e., $X \in \mathcal{A}^j$), and thus, $i \in \mathcal{C}^1(i, X) \subseteq X$. Let t be a positive integer, and suppose $\mathcal{C}^t(i, X) \subseteq X$. Then, $j \in ch(j, X) \subseteq X$ for each $j \in \mathcal{C}^t(i, X)$, and by definition, $\mathcal{C}^t(i, X) \subseteq \mathcal{C}^{t+1}(i, X) \subseteq X$. It follows that $\mathcal{C}^{|N|+1}(i, X) = \mathcal{C}^{|N|}(i, X)$. By $\mathcal{CC}(i, X)$ we denote $\mathcal{C}^{|N|}(i, X)$.

Now we are ready to describe the simplified top covering algorithm.

Simplified top covering algorithm:

Given: A hedonic game $\langle N, \succeq \rangle$ satisfying top responsiveness.

Step 1: Set $R^1 := N$ and $\Pi^1 := \emptyset$.

Step 2: For $k := 1$ to $|N|$:

Step 2.1: Select an $i \in R^k$ satisfying $|\mathcal{CC}(i, R^k)| \leq |\mathcal{CC}(j, R^k)|$ for each $j \in R^k$.

Step 2.2: Set $S^k := \mathcal{CC}(i, R^k)$, $\Pi^k := \Pi^{k-1} \cup \{S^k\}$, and $R^{k+1} := R^k \setminus S^k$.

Step 2.3: If $R^{k+1} = \emptyset$, then goto Step 3.

Step 3: Return Π^k as outcome.

We denote by $\Pi^{TC}_{\langle N,\succeq\rangle}$ the outcome obtained by applying the simplified top covering algorithm to $\langle N,\succeq\rangle$. The following proposition is shown by Dimitrov and Sung [4].

Proposition 1. Let $\langle N,\succeq\rangle$ be a hedonic game satisfying top responsiveness. When applied to $\langle N,\succeq\rangle$, the simplified top covering algorithm ends in finite steps and its outcome $\Pi^{TC}_{\langle N,\succeq\rangle}$ is a partition of N. Moreover, $\Pi^{TC}_{\langle N,\succeq\rangle}$ is strictly core stable in $\langle N,\succeq\rangle$.

By K we denote the number of the repetitions of Step 2. Then, according to the algorithm, we have

- $\emptyset = R^{K+1} \subset R^K \subset \cdots \subset R^1 = N$,
- $S^k \subseteq R^k$ for each $1 \leq k \leq K$, and
- $\Pi^{TC}_{\langle N,\succeq\rangle} = \{S^1, S^2, \ldots, S^K\}$.

In order to analyze the partition $\Pi^{TC}_{\langle N,\succeq\rangle}$, for each $i \in N$, we denote by $k(i)$ the number such that $i \in S^{k(i)}$. In other words, the coalition $\Pi^{TC}_{\langle N,\succeq\rangle}(i) = S^{k(i)}$ is included into $\Pi^{TC}_{\langle N,\succeq\rangle}$ at the $k(i)$th iteration of Step 2. Since $\Pi^{TC}_{\langle N,\succeq\rangle}$ is a partition of N, the number $k(i)$ is well-defined for each $i \in N$. The following proposition is also shown by Dimitrov and Sung [4].

Proposition 2. Let $\langle N,\succeq\rangle$ be a hedonic game satisfying top responsiveness. Then, $\Pi^{TC}_{\langle N,\succeq\rangle}(i) = \mathcal{CC}(i, R^{k(i)})$ for each $i \in N$.

Now let us summarize the properties of $\Pi^{TC}_{\langle N,\succeq\rangle}(i)$ according to Proposition 2 and the definition of $\mathcal{CC}(\cdot,\cdot)$. Since $ch(i, R^{k(i)}) = \mathcal{C}^1(i, R^{k(i)}) \subseteq \mathcal{CC}(i, R^{k(i)})$, we have

$$ch(i, R^{k(i)}) = ch(i, \Pi^{TC}_{\langle N,\succeq\rangle}(i)) \subseteq \Pi^{TC}_{\langle N,\succeq\rangle}(i). \tag{1}$$

As a special case, when $ch(i, R^{k(i)}) = ch(i, \Pi^{TC}_{\langle N,\succeq\rangle}(i)) = \{i\}$, we have

$$\{i\} = ch(i, R^{k(i)}) = \mathcal{C}^1(i, R^{k(i)}) = \cdots = \mathcal{C}^{|N|}(i, R^{k(i)}) = \mathcal{CC}(i, R^{k(i)}).$$

Hence,

$$\Pi^{TC}_{\langle N,\succeq\rangle}(i) = \{i\} \quad \text{when} \quad ch(i, \Pi^{TC}_{\langle N,\succeq\rangle}(i)) = \{i\}. \tag{2}$$

3. INDIVIDUAL RATIONALITY

In order to show our main result, here we show that the outcome $\Pi^{TC}_{\langle N,\succeq\rangle}$ of the simplified top covering algorithm is individually rational, where a partition Π is *individually rational* in the game $\langle N,\succeq\rangle$ if $\Pi(i) \succeq_i \{i\}$ for each $i \in N$.

Observe that (strict) core stability implies individual rationality, and a proof of the core stability of $\Pi^{TC}_{\langle N,\succeq\rangle}$ can be found in the work of Alcalde and Revilla [1] (or in the work of Dimitrov and Sung [4]). Here, a short proof of individual rationality is provided.

Theorem 1. Let $\langle N,\succeq\rangle$ be a hedonic game satisfying top responsiveness. Then, $\Pi^{TC}_{\langle N,\succeq\rangle}$ is individually rational for $\langle N,\succeq\rangle$.

Proof. Let $i \in N$. Then, we have $ch(i, \Pi^{TC}_{\langle N,\succeq\rangle}(i)) \succeq_i ch(i,\{i\}) = \{i\}$ from $i \in \Pi^{TC}_{\langle N,\succeq\rangle}(i)$.

- When $ch(i, \Pi^{TC}_{\langle N,\succeq\rangle}(i)) \neq \{i\}$, we have $ch(i, \Pi^{TC}_{\langle N,\succeq\rangle}(i)) \succ_i ch(i,\{i\})$ from Condition 1. Then, from Condition 2, we have $\Pi^{TC}_{\langle N,\succeq\rangle}(i) \succ_i \{i\}$.
- When $ch(i, \Pi^{TC}_{\langle N,\succeq\rangle}(i)) = \{i\}$, we have $\Pi^{TC}_{\langle N,\succeq\rangle}(i) = \{i\}$ from (2), and hence, we have $\Pi^{TC}_{\langle N,\succeq\rangle}(i) \sim_i \{i\}$.

Therefore, $\Pi^{TC}_{\langle N,\succeq\rangle}(i) \succeq_i \{i\}$ for each $i \in N$, i.e., $\Pi^{TC}_{\langle N,\succeq\rangle}$ is individually rational for $\langle N,\succeq\rangle$. □

4. NASH STABILITY

Nash stability is the strongest stability notion that is based on (the absence of) individual deviations since in its definition neither the reaction of the welcoming coalition nor the reaction of the coalition a player leaves is taken into account. Unfortunately, top responsiveness does not guarantee the existence of a Nash stable partition as exemplified next.

Example 1. Let $N = \{1,2,3\}$ and players' preferences be as follows:

$$\begin{array}{ccccccc} \{1\} & \succ_1 & \{1,2\} & \sim_1 & \{1,3\} & \succ_1 & \{1,2,3\}, \\ \{2\} & \succ_2 & \{1,2\} & \sim_2 & \{2,3\} & \succ_2 & \{1,2,3\}, \\ \{1,2,3\} & \succ_3 & \{1,3\} & \sim_3 & \{2,3\} & \succ_3 & \{3\}. \end{array}$$

The reader can easily check that this game satisfies top responsiveness. Notice that any partition in which player 1 and player 2 are not single will be blocked by the corresponding player. Hence, we have to check only the partition $\Pi = \{\{1\},\{2\},\{3\}\}$. However, $\{1,3\} \succ_3 \{3\}$ (and $\{2,3\} \succ_3 \{3\}$). Therefore, a Nash stable partition does not exist for this game.

In order to guarantee the existence of a Nash stable partition for a hedonic game $\langle N, \succeq \rangle$ we will require, in addition to top responsiveness, $\langle N, \succeq \rangle$ to satisfy *mutuality*, i. e., the following condition:

- For each $i, j \in N$ and for each $X \in \mathcal{A}^i \cap \mathcal{A}^j$, $i \in ch(j, X)$ if and only if $j \in ch(i, X)$.

In other words, mutuality requires that, for any group X of players, the members of every player's maximal on X mutually complement each other. In the formulation of this condition we were inspired by the existence result in the seminal paper of Bogomolnaia and Jackson [3]. These authors show that the combination of additive separability and symmetry (a stronger version of mutuality) guarantees the existence of Nash stable partitions. Here, we show that, by imposing both top responsiveness and mutuality, the simplified top covering algorithm generates a Nash stable partition. Note that one can easily construct a hedonic game in which players' preferences satisfy top responsiveness but are not additive separable.

We would like to mention finally a common feature between the setup in which additive separability and symmetry (mutuality) are imposed together and the setup in which top responsiveness and mutuality are imposed together: both additive separability and top responsiveness are conditions on individual's preference itself, while mutuality (and hence, symmetry) is a condition on a preference profile.

Theorem 2. Let $\langle N, \succeq \rangle$ be a hedonic game satisfying top responsiveness and mutuality. Then, $\Pi^{TC}_{\langle N, \succeq \rangle}$ is Nash stable for $\langle N, \succeq \rangle$.

P r o o f. According to Theorem 1, $\Pi^{TC}_{\langle N, \succeq \rangle}$ is individually rational, i. e.,

$$\Pi^{TC}_{\langle N, \succeq \rangle}(i) \succeq_i \{i\} \quad \text{for each } i \in N.$$

Hence, in order to show that $\Pi^{TC}_{\langle N, \succeq \rangle}$ is Nash stable, it suffices to show that

$$\Pi^{TC}_{\langle N, \succeq \rangle}(i) \succeq_i X \cup \{i\} \quad \text{for each } i \in N \text{ and for each } X \in \Pi^{TC}_{\langle N, \succeq \rangle}.$$

Let $i \in N$ and $X \in \Pi^{TC}_{\langle N, \succeq \rangle}$. Observe that X is nonempty. Moreover, let k be the number such that $X = S^k$, i. e., X is included in $\Pi^{TC}_{\langle N, \succeq \rangle}$ at the kth iteration of Step 2 in the simplified top covering algorithm.

- Suppose $k = k(i)$. Then, we have $X = S^k = S^{k(i)} = \Pi^{TC}_{\langle N, \succeq \rangle}(i)$, and thus, $i \in X$. It follows that $X \cup \{i\} = X = \Pi^{TC}_{\langle N, \succeq \rangle}(i)$, and hence, $\Pi^{TC}_{\langle N, \succeq \rangle}(i) \sim_i X \cup \{i\}$.
- Suppose $k > k(i)$. Then, we have $X \subseteq R^k \subset R^{k(i)}$, and from $i \in R^{k(i)}$, we have $X \cup \{i\} \subseteq R^{k(i)}$. Hence

$$ch(i, \Pi^{TC}_{\langle N, \succeq \rangle}(i)) = ch(i, R^{k(i)}) \succeq_i ch(i, X \cup \{i\}).$$

Observe from Condition 1 that

– $ch(i, \Pi^{TC}_{\langle N,\succeq\rangle}(i)) \neq ch(i, X \cup \{i\})$ implies $ch(i, \Pi^{TC}_{\langle N,\succeq\rangle}(i)) \succ_i ch(i, X \cup \{i\})$,

and thus, $\Pi^{TC}_{\langle N,\succeq\rangle}(i) \succ_i X \cup \{i\}$ from Condition 2.

When $ch(i, \Pi^{TC}_{\langle N,\succeq\rangle}(i)) = ch(i, X \cup \{i\})$, we have

$$ch(i, \Pi^{TC}_{\langle N,\succeq\rangle}(i)) = ch(i, X \cup \{i\}) = \{i\},$$

because $X \cap \Pi^{TC}_{\langle N,\succeq\rangle}(i) = \emptyset$, and thus,

$$i \in ch(i, \Pi^{TC}_{\langle N,\succeq\rangle}(i)) = ch(i, X \cup \{i\}) \subseteq \Pi^{TC}_{\langle N,\succeq\rangle}(i) \cap (X \cap \{i\}) = \{i\}.$$

Then, from (2), we have $\Pi^{TC}_{\langle N,\succeq\rangle}(i) = \{i\}$, and moreover, from $X \neq \emptyset$ and Condition 3, we have $\Pi^{TC}_{\langle N,\succeq\rangle}(i) = \{i\} \succ_i X \cup \{i\}$.

- Suppose $k < k(i)$. Then, we have $i \in R^k \setminus X$. From (1), we have $ch(j, X) = ch(j, R^k)$ for each $j \in X$, and thus,

$$i \notin ch(j, X) = ch(j, X \cup \{i\}) \quad \text{for each } j \in X.$$

By mutuality, we have $j \notin ch(i, X \cup \{i\})$ for each $j \in X$, which implies $\{i\} = ch(i, X \cup \{i\}) \succeq_i X \cup \{i\}$. Since $\Pi^{TC}_{\langle N,\succeq\rangle}$ is individually rational, we have $\Pi^{TC}_{\langle N,\succeq\rangle}(i) \succeq_i \{i\} \succeq_i X \cup \{i\}$. □

ACKNOWLEDGEMENT

The authors gratefully acknowledge financial support from CentER, Tilburg University, where the work on this paper has started. The work of D. Dimitrov was also supported by a Humboldt Research Fellowship conducted at Bielefeld University.

(Received November 30, 2005.)

REFERENCES

[1] J. Alcalde and P. Revilla: Researching with whom? Stability and manipulation. J. Math. Econom. *40* (2004), 869–887.

[2] S. Banerjee, H. Konishi, and T. Sönmez: Core in a simple coalition formation game. Soc. Choice Welf. *18* (2001), 135–153.

[3] A. Bogomolnaia and M. O. Jackson: The stability of hedonic coalition structures. Games and Economic Behavior *38* (2002), 201–230.

[4] D. Dimitrov and S.-C. Sung: Top Responsiveness and Stable Partitions in Coalition Formation Games. IMW Working paper 365, Bielefeld University 2005. (Available at: http://www.wiwi.uni-bielefeld.de/ imw/Papers/files/imw-wp-365.pdf).

[5] J. Dréze and J. Greenberg: Hedonic coalitions: optimality and stability. Econometrica *48* (1980), 987–1003.

[6] L. Shapley and H. Scarf: On cores and indivisibility. J. Math. Econom. *1* (1974), 23–28.

Dinko Dimitrov, Institute of Mathematical Economics, Bielefeld University. Germany.

Shao Chin Sung, Department of Industrial and Systems Engineering, Aoyama Gakuin University. Japan.
e-mail: son@ise.aoyama.ac.jp