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A NECESSITY MEASURE OPTIMIZATION APPROACH TO LINEAR PROGRAMMING PROBLEMS WITH OBLIQUE FUZZY VECTORS

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In this paper, a necessity measure optimization model of linear programming problems with fuzzy oblique vectors is discussed. It is shown that the problems are reduced to linear fractional programming problems. Utilizing a special structure of the reduced problem, we propose a solution algorithm based on Bender's decomposition. A numerical example is given.

Keywords: fuzzy linear programming, oblique fuzzy vector, necessity measure, Bender's decomposition

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1. INTRODUCTION

Fuzzy linear programming has been developed under an implicit assumption that all uncertain coefficients are non-interactive one another. This assumption makes the reduced problem very tractable. The tractability can be seen as one of advantages of fuzzy linear programming approaches. However, it is observed that in a simple problem, such as a portfolio selection problem, solutions of models are often intuitively unacceptable because of the implicit assumption (see Inuiguchi and Tanino [5]). This implies that the non-interaction assumption is not sufficient to model all real world problems.

From this point of view, it is an open problem to solve fuzzy linear programming problems with interactive uncertain variables. However, generally, the reduced problem becomes intractable because it often loses the linearity and sometimes even the convexity. In order to overcome such intractableness, we restrict our considerations into special models of interaction so that the reduced problems preserve the tractability which is the advantage of fuzzy programming approach. Several attempts [1, 3, 4, 6, 9] have been done. Among them, the results of linear programming (LP) problems with fuzzy polytopes are arranged in Table 1. Results in oblique fuzzy vector and general fuzzy polytope cases of Table 1 are obtained when necessity measures are adopted.

Table 1. Results in linear programming with fuzzy polytopes.

	Fractile Optimization	Modality Optimization
non-interactive fuzzy numbers	– LP problem	– linear fractional programming problem under L-L fuzzy numbers – LP + bisection method in general cases
oblique fuzzy vectors	– LP problem – Bender's decomposition	*open problem*
general fuzzy polytope	– semi-infinite LP problem – relaxation procedure	– semi-infinite programming problem – relaxation procedure + bisection method

A general fuzzy polytope is a fuzzy set whose h-level sets are all polytopes. An oblique fuzzy vector is a fuzzy vector which is expressed by an obliquity matrix and non-interactive fuzzy numbers. Those models of interactive fuzzy numbers were proposed by Inuiguchi et al. [1, 3, 7]. Inuiguchi et al. [3] proposed oblique fuzzy vectors to treat the interaction among uncertain variables. They showed that the necessity fractile optimization models of LP problems with oblique fuzzy vectors can be reduced to LP problems with dual block angular structures. A solution algorithm based on Bender's decomposition method was proposed. Inuiguchi and Tanino [7] extended the results to LP problems with general fuzzy polytopes. They showed that the necessity fractile optimization models can be reduced to semi-infinite LP problems. A solution algorithm based on a relaxation procedure is proposed. Inuiguchi [1] has discussed the necessity measure optimization models of LP problems with general fuzzy polytopes. He has shown that the problem can be reduced to semi-infinite programming problems. A solution algorithm based on a bisection method and a relaxation procedure has been proposed. In the algorithm, a bisection method and a relaxation procedure converge simultaneously.

However, there is no discussion on necessity measure optimization models of LP problems with oblique fuzzy vectors. We discuss this open problem in this paper. Especially, we investigate the case when oblique fuzzy vectors are defined by L-L fuzzy numbers and clarify whether the solution procedure becomes simpler than general cases or not.

This paper is organized as follows. In the next section, we describe the problem treated in this paper. In Section 3, we discuss the reduction of the formulated problems. Since the reduced problem has a special structure, a solution algorithm based on Bender's decomposition is given in Section 4. A numerical example is given in Section 5. In the last section, the results and future topics are described.

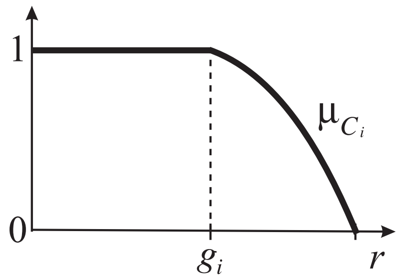

Fig. 1. A fuzzy constraint.

2. PROBLEM STATEMENT

Let us consider the following LP problem with oblique fuzzy vectors [3]:

$$\begin{aligned} \text{maximize} \quad & \boldsymbol{a}_0^{\mathrm{T}}\boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x} \lesssim g_i, \ i=1,2,\dots,m, \\ & Q\boldsymbol{x} \leq \boldsymbol{p}, \end{aligned} \tag{1}$$

where $\boldsymbol{x} = (x_1, x_2, \dots, x_n)^{\mathrm{T}}$ is a decision variable vector. Q is a $q \times n$ constant crisp matrix and $\boldsymbol{p} = (p_1, p_2, \dots, p_q)^{\mathrm{T}}$ is a constant crisp vector. $\boldsymbol{a}_i$, $i = 0, 1, \dots, m$ are uncertain parameters that take values in ranges given by oblique fuzzy vectors $\boldsymbol{A}_i$, $i = 0, 1, \dots, m$, respectively. Notation $r \lesssim g_i$ stands for 'r is substantially smaller than g_i' and is represented by a fuzzy constraint C_i. It is also assumed that each C_i has an upper semi-continuous non-increasing membership function μ_{C_i} such that $\mu_{C_i}(g_i) = 1$. A fuzzy constraint C_i is depicted in Figure 1. Note that a crisp constraint $\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x} \leq g_i$ is a special case when a fuzzy constraint C_i is an interval $(-\infty, g_i]$.

We assume that oblique fuzzy vectors $\boldsymbol{A}_i$, $i = 0, 1, \dots, m$ have obliquity matrices D_i, $i = 1, 2, \dots, m$ and non-interactive L-L fuzzy numbers $B_{ij} = (b_{ij}^{\mathrm{L}}, b_{ij}^{\mathrm{R}}, \beta_{ij}^{\mathrm{L}}, \beta_{ij}^{\mathrm{R}})_{L_i L_i}$, $i = 0, 1, \dots, m$, $j = 1, 2, \dots, n$. Then an oblique fuzzy vector $\boldsymbol{A}_i$ is characterized by a membership function defined by (see [3])

$$\mu_{\boldsymbol{A}_i}(\boldsymbol{r}) = \min_{j=1,2,\dots,n} \mu_{B_{ij}}(\boldsymbol{d}_{ij}^{\mathrm{T}}\boldsymbol{r}), \tag{2}$$

where $\boldsymbol{d}_{ij}^{\mathrm{T}}$ are jth row of D_i and an obliquity matrix is nonsingular. An L-L fuzzy number $B_{ij} = (b_{ij}^{\mathrm{L}}, b_{ij}^{\mathrm{R}}, \beta_{ij}^{\mathrm{L}}, \beta_{ij}^{\mathrm{R}})_{L_i L_i}$ has the following membership function;

$$\mu_{B_{ij}}(r) = \begin{cases} L_i\left(\dfrac{b_{ij}^{\mathrm{L}} - r}{\beta_{ij}^{\mathrm{L}}}\right) & \text{if } r < b_{ij}^{\mathrm{L}}, \\ 1 & \text{if } b_{ij}^{\mathrm{L}} \leq r \leq b_{ij}^{\mathrm{R}}, \\ L_i\left(\dfrac{r - b_{ij}^{\mathrm{R}}}{\beta_{ij}^{\mathrm{R}}}\right) & \text{if } r > b_{ij}^{\mathrm{R}}, \end{cases} \tag{3}$$

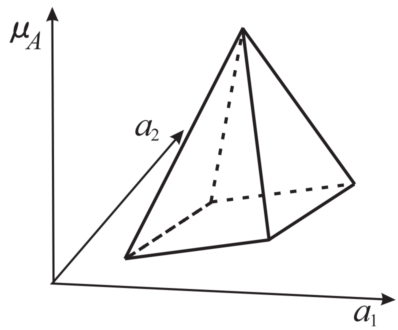

Fig. 2. An oblique fuzzy vector ($n = 2$).

where we assume $b_{ij}^{\mathrm{L}} \le b_{ij}^{\mathrm{R}}$, $\beta_{ij}^{\mathrm{L}} > 0$ and $\beta_{ij}^{\mathrm{R}} > 0$. $L_i : [0, +\infty) \to [0, 1]$ is a reference function such that

(L1) $L_i(0) = 1$,

(L2) L_i is upper semi-continuous,

(L3) L_i is non-increasing,

(L4) $\lim_{r \to +\infty} L_i(r) = 0$.

An oblique fuzzy vector $\boldsymbol{A}_i$ with $n = 2$ is depicted in Figure 2. In what follows, 'oblique fuzzy vector' is abbreviated to OFV.

The range $\boldsymbol{a}_i^{\mathrm{T}} \boldsymbol{x}$ may take is given by a fuzzy set $Y_i(\boldsymbol{x})$ with a membership function

$$\mu_{Y_i(\boldsymbol{x})}(y) = \sup_{\boldsymbol{r} : \boldsymbol{r}^{\mathrm{T}} \boldsymbol{x} = y} \mu_{\boldsymbol{A}_i}(\boldsymbol{r}). \tag{4}$$

Since $\boldsymbol{A}_i$ is an OFV with non-interactive L-L fuzzy numbers B_{ij}, $j = 1, 2, \ldots, n$, by the discussion in [3], $Y_i(\boldsymbol{x})$ is also an L-L fuzzy number $(y_i^{\mathrm{L}}(\boldsymbol{x}), y_i^{\mathrm{R}}(\boldsymbol{x}), \gamma_i^{\mathrm{L}}(\boldsymbol{x}), \gamma_i^{\mathrm{R}}(\boldsymbol{x}))_{L_i L_i}$ with

$$y_i^{\mathrm{L}}(\boldsymbol{x}) = \sum_{j : k_{ij}(\boldsymbol{x}) \ge 0} b_{ij}^{\mathrm{L}} k_{ij}(\boldsymbol{x}) + \sum_{j : k_{ij}(\boldsymbol{x}) < 0} b_{ij}^{\mathrm{R}} k_{ij}(\boldsymbol{x}), \tag{5}$$

$$y_i^{\mathrm{R}}(\boldsymbol{x}) = \sum_{j : k_{ij}(\boldsymbol{x}) \ge 0} b_{ij}^{\mathrm{R}} k_{ij}(\boldsymbol{x}) + \sum_{j : k_{ij}(\boldsymbol{x}) < 0} b_{ij}^{\mathrm{L}} k_{ij}(\boldsymbol{x}), \tag{6}$$

$$\gamma_i^{\mathrm{L}}(\boldsymbol{x}) = \sum_{j : k_{ij}(\boldsymbol{x}) \ge 0} \beta_{ij}^{\mathrm{L}} k_{ij}(\boldsymbol{x}) - \sum_{j : k_{ij}(\boldsymbol{x}) < 0} \beta_{ij}^{\mathrm{R}} k_{ij}(\boldsymbol{x}), \tag{7}$$

$$\gamma_i^{\mathrm{R}}(\boldsymbol{x}) = \sum_{j : k_{ij}(\boldsymbol{x}) \ge 0} \beta_{ij}^{\mathrm{R}} k_{ij}(\boldsymbol{x}) - \sum_{j : k_{ij}(\boldsymbol{x}) < 0} \beta_{ij}^{\mathrm{L}} k_{ij}(\boldsymbol{x}), \tag{8}$$

and $k_{ij}(\boldsymbol{x})$ is defined by

$$k_{ij}(\boldsymbol{x}) = \sum_{l=1}^{n} d_{ilj}^{*} x_l. \tag{9}$$

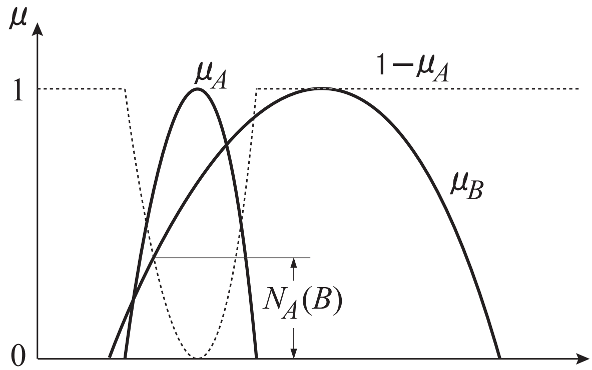

Fig. 3. Necessity measure.

d^*_{ilj} is the (l, j) component of D_i^{-1}. Consider a strong h-level set $(Y_i(\boldsymbol{x}))_h$ of $Y_i(\boldsymbol{x})$, i. e., $(Y_i(\boldsymbol{x}))_h = \{y \in \mathbb{R} \mid \mu_{Y_i(\boldsymbol{x})}(y) > h\}$. Then, its closure $\mathrm{cl}(Y_i(\boldsymbol{x}))_h$ is obtained as

$$\mathrm{cl}(Y_i(\boldsymbol{x}))_h = \left[y_i^{\mathrm{L}}(\boldsymbol{x}) - L_i^{\#}(h)\gamma_i^{\mathrm{L}}(\boldsymbol{x}), y_i^{\mathrm{R}}(\boldsymbol{x}) + L_i^{\#}(h)\gamma_i^{\mathrm{R}}(\boldsymbol{x})\right], \tag{10}$$

where we define $L_i^{\#}(h) = \sup\{r \in \mathbb{R} \mid L_i(r) > h\}$.

Based on a necessity measure optimization model [2], Problem (1) can be formulated as

$$\begin{array}{ll} \text{maximize} & N_{Y_0(\boldsymbol{x})}([z^0, +\infty)), \\ \text{subject to} & N_{Y_i(\boldsymbol{x})}(C_i) \geq h^i, \ i = 1, 2, \ldots, m, \\ & Q\boldsymbol{x} \leq \boldsymbol{p}, \end{array} \tag{11}$$

where $N_A(B) = \inf_r \max(1 - \mu_A(r), \mu_B(r))$ is a necessity measure of a fuzzy set B under a fuzzy set A (see Figure 3). A constant real number z^0 is a target objective function value specified by the decision maker. $h^i \in (0, 1]$, $i = 1, 2, \ldots, m$ are constants determined by the decision maker. The higher h^i is, the more certain $\boldsymbol{x}$ satisfies the constraint $\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x} \lesssim g_i$.

Let $[A]_h$ and $(A)_h$ be an h-level set of A and a strong h-level set of A, i. e., $[A]_h = \{r \mid \mu_A(r) \geq h\}$ and $(A)_h = \{r \mid \mu_A(r) > h\}$. Since we have

$$N_A(B) \geq h \ \text{ if and only if } \ (A)_{1-h} \subseteq [B]_h, \tag{12}$$

Problem (11) is reduced to

$$\begin{array}{ll} \text{maximize} & h, \\ \text{subject to} & \inf(Y_0(\boldsymbol{x}))_{1-h} \geq z^0, \\ & \sup(Y_i(\boldsymbol{x}))_{1-h^i} \leq \sup[C_i]_{h^i}, \ i = 1, 2, \ldots, m, \\ & Q\boldsymbol{x} \leq \boldsymbol{p}. \end{array} \tag{13}$$

From (10), this problem is rewritten as

$$\begin{array}{rl} \text{maximize} & h, \\ \text{subject to} & y_i^{\mathrm{L}}(\boldsymbol{x}) - L_i^{\#}(1-h)\gamma_i^{\mathrm{L}}(\boldsymbol{x}) \ge z^0, \\ & y_i^{\mathrm{R}}(\boldsymbol{x}) + L_i^{\#}(1-h^i)\gamma_i^{\mathrm{R}}(\boldsymbol{x}) \le c_i(h^i),\ i=1,2,\dots,m, \\ & Q\boldsymbol{x} \le \boldsymbol{p}, \end{array} \tag{14}$$

where, for convenience, we define $c_i(h^i) = \sup[C_i]_{h^i}$.

The first constraint of Problem (14) is equivalent to

$$\frac{y_0^{\mathrm{L}}(\boldsymbol{x}) - z^0}{\gamma_0^{\mathrm{L}}(\boldsymbol{x})} \ge L_0^{\#}(1-h). \tag{15}$$

Because L is non-increasing, maximizing h is equivalent to maximizing $L^{\#}(1-h)$, Problem (14) is reduced to

$$\begin{array}{rl} \text{minimize} & \dfrac{y_0^{\mathrm{L}}(\boldsymbol{x}) - z^0}{\gamma_0^{\mathrm{L}}(\boldsymbol{x})}, \\ \text{subject to} & y_i^{\mathrm{R}}(\boldsymbol{x}) + L_i^{\#}(1-h^i)\gamma_i^{\mathrm{R}}(\boldsymbol{x}) \le c_i(h^i),\ i=1,2,\dots,m, \\ & Q\boldsymbol{x} \le \boldsymbol{p}. \end{array} \tag{16}$$

By (5)–(8), this problem can be rewritten as

$$\begin{array}{rl} \text{maximize} & \dfrac{\displaystyle\sum_{j:k_{0j}(\boldsymbol{x})\ge 0} b_{0j}^{\mathrm{L}} k_{0j}(\boldsymbol{x}) + \sum_{j:k_{0j}(\boldsymbol{x})<0} b_{0j}^{\mathrm{R}} k_{0j}(\boldsymbol{x}) - z^0}{\displaystyle\sum_{j:k_{0j}(\boldsymbol{x})\ge 0} \beta_{0j}^{\mathrm{L}} k_{0j}(\boldsymbol{x}) - \sum_{j:k_{0j}(\boldsymbol{x})<0} \beta_{0j}^{\mathrm{R}} k_{0j}(\boldsymbol{x})}, \\ \text{subject to} & \displaystyle\sum_{j:k_{ij}(\boldsymbol{x})\ge 0} \bar{b}_{ij}^{\mathrm{R}}(1-h^i) k_{ij}(\boldsymbol{x}) + \sum_{j:k_{ij}(\boldsymbol{x})<0} \bar{b}_{ij}^{\mathrm{L}}(1-h^i) k_{ij}(\boldsymbol{x}) \le c_i(h^i), \\ & \qquad\qquad\qquad\qquad\qquad\qquad i=1,2,\dots,m, \\ & Q\boldsymbol{x} \le \boldsymbol{p}, \end{array} \tag{17}$$

where we define $\bar{b}_{ij}^{\mathrm{L}}(h) = b_{ij}^{\mathrm{L}} - \beta_{ij}^{\mathrm{L}} L^{\#}(h)$, $\bar{b}_{ij}^{\mathrm{R}}(h) = b_{ij}^{\mathrm{R}} - \beta_{ij}^{\mathrm{R}} L^{\#}(h)$. Let $\boldsymbol{k}_i(\boldsymbol{x}) = (k_{i1}(\boldsymbol{x}), k_{i2}(\boldsymbol{x}), \dots, k_{in}(\boldsymbol{x}))^{\mathrm{T}}$, $i = 0,1,\dots,m$. Then we have $\boldsymbol{k}_i(\boldsymbol{x}) = {D_i^{-1}}^{\mathrm{T}}\boldsymbol{x}$, $i = 0,1,\dots,m$. Erasing $\boldsymbol{x}$ and $\boldsymbol{k}_i(\boldsymbol{x})$, $i = 0,1,\dots,m$ by introduction of deviational variable vectors $\boldsymbol{y}_i^+$, $\boldsymbol{y}_i^- \ge \boldsymbol{0}$ such that $\boldsymbol{k}_i(\boldsymbol{x}) = \boldsymbol{y}^+ - \boldsymbol{y}_i^-$ and ${\boldsymbol{y}^+}^{\mathrm{T}}\boldsymbol{y}^- = 0$, we can rewrite Problem (17) as the following problem with complementary conditions

${\boldsymbol{y}_i^+}^{\mathrm{T}} \boldsymbol{y}_i^- = 0$, $i = 0, 1, \ldots, m$:

$$
\begin{array}{ll}
\text{maximize} & \dfrac{\displaystyle\sum_{j=1}^{n} b_{0j}^{\mathrm{L}} y_{0j}^{+} - \sum_{j=1}^{n} b_{0j}^{\mathrm{R}} y_{0j}^{-} - z^0}{\displaystyle\sum_{j=1}^{n} \beta_{0j}^{\mathrm{L}} y_{0j}^{+} + \sum_{j=1}^{n} \beta_{0j}^{\mathrm{R}} y_{0j}^{-}}, \\
\text{subject to} & \displaystyle\sum_{j=1}^{n} \bar{b}_{ij}^{\mathrm{R}} (1-h^i) y_{ij}^{+} - \sum_{j=1}^{n} \bar{b}_{ij}^{\mathrm{L}} (1-h^i) y_{ij}^{-} \leq c_i(h^i),\ i = 1, 2, \ldots, m, \\
& D_i^{\mathrm{T}} (\boldsymbol{y}_i^+ - \boldsymbol{y}_i^-) = D_0^{\mathrm{T}} (\boldsymbol{y}_0^+ - \boldsymbol{y}_0^-),\ i = 1, 2, \ldots, m, \\
& Q D_0^{\mathrm{T}} (\boldsymbol{y}_0^+ - \boldsymbol{y}_0^-) \leq \boldsymbol{p}, \\
& \boldsymbol{y}_i^+ \geq \boldsymbol{0},\ \boldsymbol{y}_i^- \geq \boldsymbol{0},\ i = 0, 1, \ldots, m.
\end{array}
\tag{18}
$$

We can obtain an optimal solution of Problem (18) satisfying complementary conditions ${\boldsymbol{y}_i^+}^{\mathrm{T}} \boldsymbol{y}_i^- = 0$, $i = 0, 1, \ldots, m$ by a certain modification of an arbitrary optimal solution of Problem (18) as shown in the following theorem.

Theorem. Assume that the optimal value of Problem (17) is non-negative. An optimal solution of Problem (18) satisfying complementary conditions ${\boldsymbol{y}_i^+}^{\mathrm{T}} \boldsymbol{y}_i^- = 0$, $i = 0, 1, \ldots, m$ can be obtained from any optimal solution of Problem (18).

P r o o f. Let $\hat{\boldsymbol{y}}_i^+$ and $\hat{\boldsymbol{y}}_i^-$, $i = 0, 1, \ldots, m$ be an arbitrary optimal solution to Problem (18). Define $\bar{\boldsymbol{y}}_i^+ = (\bar{y}_{1i}^+, \bar{y}_{2i}^+, \ldots, \bar{y}_{in}^+)^{\mathrm{T}}$ and $\bar{\boldsymbol{y}}_i^- = (\bar{y}_{1i}^-, \bar{y}_{2i}^-, \ldots, \bar{y}_{in}^-)^{\mathrm{T}}$ by $\bar{y}_{ij}^+ = \max(0, \hat{y}_{ij}^+ - \hat{y}_{ij}^-)$ and $\bar{y}_{ij}^- = \max(0, \hat{y}_{ij}^- - \hat{y}_{ij}^+)$, $i = 0, 1, \ldots, m$, $j = 1, 2, \ldots, n$. Then, from $\bar{y}_{ij}^+ \leq \hat{y}_{ij}^+$, $\bar{y}_{ij}^- \leq \hat{y}_{ij}^-$, $\bar{b}_{ij}^{\mathrm{L}}(1-h^i) \leq \bar{b}_{ij}^{\mathrm{R}}(1-h^i)$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ and $\hat{\boldsymbol{y}}_i^+ - \hat{\boldsymbol{y}}_i^- = \bar{\boldsymbol{y}}_i^+ - \bar{\boldsymbol{y}}_i^-$, $i = 0, 1, \ldots, n$, we know $\bar{\boldsymbol{y}}_i^+$ and $\bar{\boldsymbol{y}}_i^-$, $i = 0, 1, \ldots, m$ can be a feasible solution. Moreover, from $b_0^{\mathrm{L}} \leq b_0^{\mathrm{R}}$, $\beta_{0j}^{\mathrm{L}} > 0$ and $\beta_{0j}^{\mathrm{R}} > 0$, we have

$$
\frac{\displaystyle\sum_{j=1}^{n} b_{0j}^{\mathrm{L}} \hat{y}_{0j}^{+} - \sum_{j=1}^{n} b_{0j}^{\mathrm{R}} \hat{y}_{0j}^{-} - z^0}{\displaystyle\sum_{j=1}^{n} \beta_{0j}^{\mathrm{L}} \hat{y}_{0j}^{+} + \sum_{j=1}^{n} \beta_{0j}^{\mathrm{R}} \hat{y}_{0j}^{-}} \leq \frac{\displaystyle\sum_{j=1}^{n} b_{0j}^{\mathrm{L}} \bar{y}_{0j}^{+} - \sum_{j=1}^{n} b_{0j}^{\mathrm{R}} \bar{y}_{0j}^{-} - z^0}{\displaystyle\sum_{j=1}^{n} \beta_{0j}^{\mathrm{L}} \bar{y}_{0j}^{+} + \sum_{j=1}^{n} \beta_{0j}^{\mathrm{R}} \bar{y}_{0j}^{-}}.
$$

Hence, we obtain another optimal solution composed of $\bar{\boldsymbol{y}}_i^+$ and $\bar{\boldsymbol{y}}_i^-$, $i = 0, 1, \ldots, m$ which satisfies the complementary conditions. □

This theorem implies that we can obtain an optimal solution of necessity measure optimization problem (11) by solving Problem (18). If the optimal value of Problem (18) is negative, the optimal value of Problem (11) is zero. In this case, the value z^0 is improper and should be replaced with a smaller value. Problem (18) is a linear fractional programming problem. Applying Charnes and Cooper's reduction, Problem (18) is reduced to the following LP problem:

$$\begin{aligned}
\text{maximize} \quad & \sum_{j=1}^{n} b_{0j}^{\mathrm{L}} v_{0j}^{+} - \sum_{j=1}^{n} b_{0j}^{\mathrm{R}} v_{0j}^{-} - z^0 t, \\
\text{subject to} \quad & \sum_{j=1}^{n} \beta_{0j}^{\mathrm{L}} v_{0j}^{+} + \sum_{j=1}^{n} \beta_{0j}^{\mathrm{R}} v_{0j}^{-} = 1, \\
& \sum_{j=1}^{n} \bar{b}_{ij}^{\mathrm{R}} (1-h^i) v_{ij}^{+} - \sum_{j=1}^{n} \bar{b}_{ij}^{\mathrm{L}} (1-h^i) v_{ij}^{-} \ \leq c_i(h^i) t, \ i = 1, 2, \dots, m, \\
& D_i^{\mathrm{T}} (\boldsymbol{v}_i^{+} - \boldsymbol{v}_i^{-}) = D_0^{\mathrm{T}} (\boldsymbol{v}_0^{+} - \boldsymbol{v}_0^{-}), \ i = 1, 2, \dots, m, \\
& Q D_0^{\mathrm{T}} (\boldsymbol{v}_0^{+} - \boldsymbol{v}_0^{-}) \leq \boldsymbol{p} t, \\
& t \geq 0, \ \boldsymbol{v}_i^{+} \geq \boldsymbol{0}, \ \boldsymbol{v}_i^{-} \geq \boldsymbol{0}, \ i = 0, 1, \dots, m.
\end{aligned} \tag{19}$$

Let $\hat{\boldsymbol{v}}_i^{+}$, $\hat{\boldsymbol{v}}_i^{-}$, $i = 0, 1, \dots, m$ and $\hat{t}$ be an optimal solution to Problem (19). Then an optimal solution of Problem (11) is obtained as

$$\hat{\boldsymbol{x}} = \frac{D_0^{\mathrm{T}} (\hat{\boldsymbol{v}}_i^{+} - \hat{\boldsymbol{v}}_i^{-})}{\hat{t}}. \tag{20}$$

3. A SOLUTION ALGORITHM

Problem (19) has a special structure called a dual block angular structure [8]. Thus Problem (19) can be solved by the following algorithm based on Bender's decomposition.

Algorithm.

Step 1. Set $s = 0$. Select $\boldsymbol{x}^0$ such that $Q\boldsymbol{x}^0 \leq \boldsymbol{p}$, arbitrarily. The initial solution to Problem (19) is obtained by

$$\boldsymbol{y}_0 = {D_0^{-1}}^{\mathrm{T}} \boldsymbol{x}^s, \ t^0 = \frac{1}{\displaystyle\sum_{j:y_{0j} \geq 0} \beta_{0j}^{\mathrm{L}} y_{0j} - \sum_{j:y_{0j} < 0} \beta_{0j}^{\mathrm{R}} y_{0j}},$$

$${v_{0j}^{+}}^{0} = t^0 \max(0, y_{0j}), \ {v_{0j}^{-}}^{0} = t^0 \max(0, -y_{0j}), \ j = 1, 2, \dots, n,$$

where y_{0j}, ${v_{0j}^{+}}^{0}$ and ${v_{0j}^{-}}^{0}$ are the jth components of $\boldsymbol{y}_0$, ${\boldsymbol{v}_0^{+}}^{0}$ and ${\boldsymbol{v}_0^{-}}^{0}$, respectively.

Step 2. Calculate

$$\begin{aligned}
\boldsymbol{v}_i &= {D_i^{-1}}^{\mathrm{T}} D_0^{\mathrm{T}} ({\boldsymbol{v}_0^{+}}^{s} - {\boldsymbol{v}_0^{-}}^{s}), \ i = 0, 1, \dots, m, \\
b_i &= \sum_{j:v_{ij} \geq 0} \bar{b}_{ij}^{\mathrm{R}} (1-h^i) v_{ij} + \sum_{j:v_{ij} < 0} \bar{b}_{ij}^{\mathrm{L}} (1-h^i) v_{ij}, \ i = 1, 2, \dots, m,
\end{aligned}$$

where v_{ij} is the jth element of $\boldsymbol{v}_i$.

Step 3. If the following formula is satisfied, terminate the algorithm:

$$s > 0 \text{ and } b_i \leq c_i(h^i)t^s, \; i = 1, 2, \ldots, m.$$

In this case, ${\boldsymbol{v}_0^+}^s$, ${\boldsymbol{v}_0^-}^s$ and t^s compose an optimal solution to Problem (11).

Step 4. Update $s = s+1$. Generate the following linear functions of $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)^{\mathrm{T}}$:

$$f_{is}(\boldsymbol{v}) = \sum_{l=1}^{n} \left(\sum_{j: v_{ij} \geq 0} \bar{b}_{ij}^{\mathrm{R}} (1 - h^i) d_{ilj}^* + \sum_{j: v_{ij} < 0} \bar{b}_{ij}^{\mathrm{L}} (1 - h^i) d_{ilj}^* \right) v_l, \quad i = 1, 2, \ldots, m. \tag{21}$$

Step 5. Solve the following LP problem:

$$\begin{aligned} \text{maximize} \quad & \sum_{j=1}^{n} b_{0j}^{\mathrm{L}} v_{0j}^{+} - \sum_{j=1}^{n} b_{0j}^{\mathrm{R}} v_{0j}^{-} - z^0 t, \\ \text{subject to} \quad & \sum_{j=1}^{n} \beta_{0j}^{\mathrm{L}} v_{0j}^{+} + \sum_{j=1}^{n} \beta_{0j}^{\mathrm{R}} v_{0j}^{-} = 1, \\ & f_{ij}(D_0^{\mathrm{T}}(\boldsymbol{v}_0^{+} - \boldsymbol{v}_0^{-})) \leq c_i(h^i)t, \; i = 1, \ldots, m, \; j = 1, \ldots, s, \\ & QD_0^{\mathrm{T}}(\boldsymbol{v}_0^{+} - \boldsymbol{v}_0^{-}) \leq \boldsymbol{p}t, \\ & t \geq 0, \; \boldsymbol{v}_0^{+} \geq \boldsymbol{0}, \; \boldsymbol{v}_0^{-} \geq \boldsymbol{0}. \end{aligned} \tag{22}$$

Let ${\boldsymbol{v}_0^+}^s$, ${\boldsymbol{v}_0^-}^s$ and t^s be the obtained optimal solution. If the optimal value of Problem (22) is negative, terminate the algorithm and z^0 should be changed with a smaller value. Otherwise, return to Step 2.

4. A NUMERICAL EXAMPLE

As an example, let us consider Problem (1) with the following parameters:

$$\begin{aligned} & n = 2, \; m = 3, \; q = 2, \; z^0 = 25, \; g_1 = 20, \; g_2 = 14, \; g_3 = 24, \; Q = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \\ & D_0 = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}, \; D_1 = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \; D_2 = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}, \; D_3 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \\ & B_{01} = (1, 1, 1, 1)_{LL}, \quad B_{02} = (12, 12, 3, 3)_{LL}, \quad B_{11} = (4, 4, 0.2, 0.2)_{LL}, \\ & B_{12} = (7, 7, 0.1, 0.1)_{LL}, \; B_{21} = (3, 3, 0.5, 0.5)_{LL}, \; B_{22} = (2, 2, 1, 1)_{LL}, \\ & B_{31} = (4, 4, 0.2, 0.2)_{LL}, \; B_{32} = (3, 3, 0.1, 0.1)_{LL}, \; L(r) = \max(1 - r, 0). \end{aligned} \tag{23}$$

The membership functions of fuzzy constraints are defined by

$$\mu_{C_i}(r) = \begin{cases} 1, & \text{if } r \leq g_i, \\ 1 - \dfrac{r - g_i}{4}, & \text{if } g_i < r \leq g_i + 4, \\ 0, & \text{if } r > g_i + 4, \end{cases} \qquad i = 1, 2, 3. \tag{24}$$

Solving the necessity measure optimization model with $h^i = 0.4$, $i = 1, 2, 3$ and the initial solution $(1, 4)^{\mathrm{T}}$, the algorithm terminates at the third iteration. Then we obtain an optimal solution as

$$\hat{\boldsymbol{x}} = \frac{D_0^{\mathrm{T}}(\boldsymbol{v}_0^{+2} - \boldsymbol{v}_0^{-2})}{t^2} = (7.770, 3.330)^{\mathrm{T}}. \tag{25}$$

The solution procedure is shown in Table 2.

Table 2. The solution procedure in the numerical example.

Step 1. $s = 0$, $\boldsymbol{x}^0 = (1,2)^{\mathrm{T}}$: $\boldsymbol{y}^0 = (-1,1)^{\mathrm{T}}$, $t^0 = 1$, $\boldsymbol{v}_0^{+0} = (0,1)^{\mathrm{T}}$, $\boldsymbol{v}_0^{-0} = (-1,0)^{\mathrm{T}}$.

Step 2. $\boldsymbol{v}_1 = (5,-2)^{\mathrm{T}}$, $\boldsymbol{v}_2 = (1.4, 0.4)^{\mathrm{T}}$, $\boldsymbol{v}_3 = (-3,7)^{\mathrm{T}}$, $b_1 = 6.72$, $b_2 = 5.66$, $b_3 = 9.78$.

Step 3. $s = 0$. Continue.

Step 4. $s = 1$. $f_{11}(\boldsymbol{v}) = 1.52v_1 + 1.3v_2$, $f_{21}(\boldsymbol{v}) = 0.94v_1 + 1.18v_2$ and $f_{31}(\boldsymbol{x}) = 0.82v_1 + 2.24v_2$.

Step 5.

$$\begin{aligned} \text{maximize} \quad & v_{01}^+ + 12v_{02}^+ - v_{01}^- - 12v_{02}^- - 25t, \\ \text{subject to} \quad & v_{01}^+ + 3v_{02}^+ + v_{01}^- + 3v_{02}^- = 1, \\ & 0.22v_{01}^+ + 6.94v_{02}^+ - 0.22v_{01}^- - 6.94v_{02}^- \leq 21.6t, \\ & -0.24v_{01}^+ + 5.42v_{02}^+ + 0.24v_{01}^- - 5.42v_{02}^- \leq 15.6t, \\ & -1.42v_{01}^+ + 8.36v_{02}^+ + 1.42v_{01}^- - 8.36v_{02}^- \leq 25.6t, \\ & -v_{01}^+ + v_{02}^+ + v_{01}^- - v_{02}^- \leq 0, \\ & -2v_{01}^+ + 3v_{02}^+ + 2v_{01}^- - 3v_{02}^- \leq 0, \\ & v_{01}^+, v_{02}^+, v_{01}^-, v_{02}^-, t \geq 0. \end{aligned}$$

$t^1 = 0.0748$, $\boldsymbol{v}_0^{+1} = (0.333, 0.222)^{\mathrm{T}}$, $\boldsymbol{v}_0^{-1} = (0,0)^{\mathrm{T}}$.

Step 2. $\boldsymbol{v}_1 = (-1.667, 1.222)^{\mathrm{T}}$, $\boldsymbol{v}_2 = (0.533, -0.244)^{\mathrm{T}}$, $\boldsymbol{v}_3 = (0.444, 0.111)^{\mathrm{T}}$, $b_1 = 2.162$, $b_2 = 1.418$, $b_3 = 1.504$.

Step 3. $c_1(h^1)t^1 = 1.616 < b_1$, $c_2(h^2)t^2 = 1.167 < b_2$, $c_3(h^3)t^1 = 1.616 > b_3$. Continue.

Step 4. $s = 2$. $f_{12}(\boldsymbol{v}) = 2.58v_1 + 0.7v_2$, $f_{22}(\boldsymbol{v}) = 1.42v_1 + 0.94v_2$ and $f_{32}(\boldsymbol{x}) = 1.18v_1 + 1.76v_2$.

Step 5.
$$\begin{aligned}
\text{maximize} \quad & v_{01}^+ + 12v_{02}^+ - v_{01}^- - 12v_{02}^- - 25t, \\
\text{subject to} \quad & v_{01}^+ + 3v_{02}^+ + v_{01}^- + 3v_{02}^- = 1, \\
& 0.22v_{01}^+ + 6.94v_{02}^+ - 0.22v_{01}^- - 6.94v_{02}^- \le 21.6t, \\
& -0.24v_{01}^+ + 5.42v_{02}^+ + 0.24v_{01}^- - 5.42v_{02}^- \le 15.6t, \\
& -1.42v_{01}^+ + 8.36v_{02}^+ + 1.42v_{01}^- - 8.36v_{02}^- \le 25.6t, \\
& 1.78v_{01}^+ + 7.06v_{02}^+ - 1.78v_{01}^- - 7.06v_{02}^- \le 21.6t, \\
& 0.48v_{01}^+ + 5.66v_{02}^+ - 0.48v_{01}^- - 5.66v_{02}^- \le 15.6t, \\
& -0.58v_{01}^+ + 7.64v_{02}^+ + 0.58v_{01}^- - 7.64v_{02}^- \le 25.6t, \\
& -v_{01}^+ + v_{02}^+ + v_{01}^- - v_{02}^- \le 0, \\
& -2v_{01}^+ + 3v_{02}^+ + 2v_{01}^- - 3v_{02}^- \le 0, \\
& v_{01}^+, v_{02}^+, v_{01}^-, v_{02}^-, t \ge 0.
\end{aligned}$$

$t^2 = 0.100$, ${\boldsymbol{v}_0^+}^2 = (0.333, 0.222)^{\mathrm{T}}$, ${\boldsymbol{v}_0^-}^2 = (0, 0)^{\mathrm{T}}$.

Step 2. $\boldsymbol{v}_1 = (-1.667, 1.222)^{\mathrm{T}}$, $\boldsymbol{v}_2 = (0.533, -0.244)^{\mathrm{T}}$, $\boldsymbol{v}_3 = (0.444, 0.111)^{\mathrm{T}}$, $b_1 = 2.162$, $b_2 = 1.418$, $b_3 = 1.504$.

Step 3. $c_1(h^1)t^2 = 2.162 > b_1$, $c_2(h^2)t^2 = 1.562 > b_2$, $c_3(h^3)t^2 = 2.563 > b_3$. Terminate.

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5. CONCLUDING REMARKS

In this paper, we show that a necessity measure optimization model of an LP problem with OFVs is reduced to an LP problem with a dual block angular structure if the OFV of the objective function is defined by L-L fuzzy numbers. A solution algorithm based on Bender's decomposition is given. Due to the specialty of the problem, the proposed algorithm requires much less computational effort than the algorithm proposed in Inuiguchi [1] for a more general problem. Therefore, the *open problem* in Table 1 is solved when the OFV of objective function is defined by L-L fuzzy numbers, and the solution method is 'linear fractional programming' and 'Bender's decomposition'.

However, if the OFV of the objective function is not defined by L-L fuzzy numbers, the necessity measure optimization models of LP problems with OFVs are not always solved by the proposed algorithm because of the nonlinearity of the reduced problems. We should introduce the bisection method in this case. The solution algorithm in this case will be similar to the one proposed in Inuiguchi [1].

We have not yet discussed symmetric models [2] of linear programming problems with OFVs as well as with a fuzzy polytope. The discussion about symmetric models [10] which treat an objective function as a constraint by introducing the target value will be one of our future topics.

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