KYBERNETIKA — VOLUME *42* (2006), NUMBER 3, PAGES 303 – 318

A CONTOUR VIEW ON UNINORM PROPERTIES

Koen C. Maes and Bernard De Baets

Any given increasing $[0,1]^2 \to [0,1]$ function is completely determined by its contour lines. In this paper we show how each individual uninorm property can be translated into a property of contour lines. In particular, we describe commutativity in terms of orthosymmetry and we link associativity to the portation law and the exchange principle. Contrapositivity and rotation invariance are used to characterize uninorms that have a continuous contour line.

Keywords: uninorm, contour line, orthosymmetry, portation law, exchange principle, contrapositive symmetry, rotation invariance, self quasi-inverse property

AMS Subject Classification: 26B40, 06F05, 03B52

1. INTRODUCTION

A *uninorm* U is an increasing, commutative and associative $[0,1]^2 \to [0,1]$ function with *neutral element* $e \in [0,1]$, i.e. $U(x,e) = x$, for every $x \in [0,1]$. Uninorms were introduced by Yager and Rybalov [23] as a generalization of t-norms ($e = 1$) and t-conorms ($e = 0$) [18]. They are important from a practical as well as a theoretical point of view. In multicriteria decision making, for example, they are used to aggregate the evaluation of alternatives, taking into account some level of satisfaction e [23]. Uninorms with $e \in \,]0,1[$ convert the structures $([0,1], \sup, U)$ and $([0,1], \inf, U)$ into distributive semirings in the sense of Golan [10]. For any given $[0,e] \to [0,1]$ bijection ϕ and $[e,1] \to [0,1]$ bijection ψ, we can extract from a uninorm U a t-norm T and a t-conorm S such that

$$(\forall (x,y) \in [0,e]^2)(U(x,y) = \phi^{-1}(T(\phi(x),\phi(y))))\,,$$
$$(\forall (x,y) \in [e,1]^2)(U(x,y) = \psi^{-1}(S(\psi(x),\psi(y))))\,.$$

On the other parts of the unit square it always holds that $T_{\mathbf{M}} \le U \le S_{\mathbf{M}}$, where $T_{\mathbf{M}} = \min$ and $S_{\mathbf{M}} = \max$ [9]. Furthermore, it always holds that either $U(0,1) = U(1,0) = 0$ or $U(0,1) = U(1,0) = 1$. In the first case U is called *conjunctive*, in the second case we talk about a *disjunctive* uninorm [9]. Important classes of uninorms comprise $U_{\mathbf{min}}$ and $U_{\mathbf{max}}$ [5], the representable uninorms [7, 9] and the idempotent uninorms [3].

Studying the horizontal cuts of a uninorm U, i. e. the intersections of its graph by planes parallel to the domain $[0,1]^2$, instead of the uninorm itself, we will give the description of uninorms a new impetus. The *contour lines* of U are defined as the upper and lower limits of these horizontal cuts. So far, it is still unknown how the characteristic properties of a uninorm are reflected in the behaviour of its contour lines. The knowledge of this relationship, however, is the key to fathom the structure of other increasing $[0,1]^2 \to [0,1]$ functions fulfilling one or more of these properties. With each increasing $[0,1]^2 \to [0,1]$ function A we associate four types of contour lines:

$$\begin{aligned} C_a &: [0,1] \to [0,1] : x \mapsto \sup\{t \in [0,1] \mid A(x,t) \leq a\}, \\ D_a &: [0,1] \to [0,1] : x \mapsto \inf\{t \in [0,1] \mid A(x,t) \geq a\}, \\ \tilde{C}_a &: [0,1] \to [0,1] : x \mapsto \sup\{t \in [0,1] \mid A(t,x) \leq a\}, \\ \tilde{D}_a &: [0,1] \to [0,1] : x \mapsto \inf\{t \in [0,1] \mid A(t,x) \geq a\}, \end{aligned}$$

with $a \in [0,1]$. It will be clear from the context which function A we are considering. Note that it is absolutely necessary to use an increasing A. Otherwise C_a, D_a, $\tilde{C}_a$ and $\tilde{D}_a$ would lose their geometrical meaning. Considering the ensemble of contour lines, we can associate two additional functions to each type of contour line. For example, the contour lines of the type C_a, $a \in [0,1]$, are totally determined by the $[0,1]^2 \to [0,1]$ function C that maps a couple (x,a) to $C_a(x)$. Hence, contour lines of the type C_a are partial functions of C. The partial functions obtained by fixing the first argument of C will be denoted $C_\bullet(x)$, with $x \in [0,1]$. A similar argument applies to the other types of contour lines. Dealing with a conjunctive uninorm U, the binary function C can be understood as a generalization of the Boolean implication. In this case C is usually referred to as the *residual implicator* of U and is denoted as I_U [4, 8, 21]. If U is disjunctive, then $J_U := D$ is known as its *residual coimplicator* [2, 4, 8, 21]. The contour lines C_a of a continuous t-norm T are also called *level functions* [19]. Based on the contour lines of a left-continuous t-norm T, Jenei [16] provides sufficient conditions for T to be the Łukasiewicz t-norm $T_{\mathbf{L}}$ ($T_{\mathbf{L}}(x,y) = \max(x+y-1,0)$), resp. the algebraic product $T_{\mathbf{P}}$ ($T_{\mathbf{P}}(x,y) = x\,y$).

In this paper, we provide new insights into the geometrical structure of a uninorm by examining its contour lines C_a, D_a, $\tilde{C}_a$ and $\tilde{D}_a$ in an analytical way. In Section 2, considering increasing $[0,1]^2 \to [0,1]$ functions A, we figure out how continuity, commutativity, associativity and the existence of a neutral element can be expressed in terms of properties of contour lines. Section 3 tackles the continuous contour lines of uninorms. We try to grasp the influence of the continuity of a contour line on the structure of a uninorm.

2. UNINORM PROPERTIES

Let A be an increasing $[0,1]^2 \to [0,1]$ function. Then its contour lines C_a, D_a, $\tilde{C}_a$ and $\tilde{D}_a$ are clearly decreasing. For any $a_1 \leq a_2$ it holds that $C_{a_1} \leq C_{a_2}$, $D_{a_1} \leq D_{a_2}$, $\tilde{C}_{a_1} \leq \tilde{C}_{a_2}$ and $\tilde{D}_{a_1} \leq \tilde{D}_{a_2}$. Before studying the characteristic properties of uninorms, we first discuss some continuity conditions that are crucial for our further results. The partial functions $A(x,\bullet)$ and $A(\bullet,x)$, $x \in [0,1]$, are obtained by fixing the first, resp. the second argument of A. Note that A will be called left-continuous if all of its partial functions are left-continuous.

2.1. Continuity

Let I_1 and I_2 be two closed subintervals of $[-\infty,+\infty]$ and consider two real functions $F : I_1 \to I_2$ and $G : I_2 \to I_1$. Then (F,G) is called a *Galois connection* [1] if $F(x) \leq y \;\Leftrightarrow\; x \leq G(y)$ holds for every $(x,y) \in I_1 \times I_2$. A t-norm T is left-continuous if and only if $(T(x,\bullet), C_\bullet(x))$ is a Galois connection for every $x \in [0,1]$, i. e. $T(x,y) \leq a \;\Leftrightarrow\; y \leq C_a(x)$ is satisfied for every $(x,y,a) \in [0,1]^3$ (see e. g. [8]). Dealing with an arbitrary increasing $[0,1]^2 \to [0,1]$ function A, the following characterization holds:

Theorem 1. For every $x \in [0,1]$ the following assertions hold:

(i) $A(x,\bullet)$ is left-continuous if and only if

$$A(x,y) \leq a \quad\Leftrightarrow\quad y \leq C_a(x) \tag{1}$$

holds for every $(y,a) \in [0,1]^2$, with $0 < y$.

(ii) $A(x,\bullet)$ is right-continuous if and only if

$$D_a(x) \leq y \quad\Leftrightarrow\quad a \leq A(x,y) \tag{2}$$

holds for every $(y,a) \in [0,1]^2$, with $y < 1$.

(iii) $A(\bullet,x)$ is left-continuous if and only if

$$A(y,x) \leq a \quad\Leftrightarrow\quad y \leq \tilde{C}_a(x) \tag{3}$$

holds for every $(y,a) \in [0,1]^2$, with $0 < y$.

(iv) $A(\bullet,x)$ is right-continuous if and only if

$$\tilde{D}_a(x) \leq y \quad\Leftrightarrow\quad a \leq A(y,x) \tag{4}$$

holds for every $(y,a) \in [0,1]^2$, with $y < 1$.

P r o o f. We will prove the first case of the theorem only, the other cases being similar. Note that by definition $A(x,y) \leq a$ always implies $y \leq C_a(x)$. Suppose that $A(x,\bullet)$ is left-continuous and consider arbitrary $(y,a) \in [0,1]^2$, $0 < y$. If $y \leq C_a(x)$, then for every $\epsilon \in \,]0,y]$ we know that $A(x,y-\epsilon) \leq a$. The left-continuity of $A(x,\bullet)$ then ensures that $A(x,y) \leq a$. Conversely, suppose that equation (1) holds and that $A(x,\bullet)$ is not left-continuous. Then there exists $(y,a) \in [0,1]^2$, $0 < y$, such that $A(x,y-\epsilon) \leq a$ and $a < A(x,y)$, for every $\epsilon \in \,]0,y[$. However, from equation (1) we obtain that $y-\epsilon \leq C_a(x)$, for every $\epsilon \in \,]0,y[$, and therefore $y \leq C_a(x)$. Appyling equation (1) once again, leads to the contradiction $A(x,y) \leq a$. □

The continuity of A also affects the continuity of its contour lines.

Corollary 1. Take arbitrary $(x,a) \in [0,1]^2$. If A is left-continuous, then $C_a(x)$ and $\tilde{C}_a(x)$ are left-continuous in x and right-continuous in a. If A is right-continuous, then $D_a(x)$ and $\tilde{D}_a(x)$ are right-continuous in x and left-continuous in a.

Proof. Let A be left-continuous. We only prove the continuity properties of $C_a(x)$. Suppose that there exists a triplet $(x,y,a) \in [0,1]^3$ such that $0 < x$, $0 < y$ and $C_a(x) < y \leq C_a(x-\epsilon)$, for every $\epsilon \in]0,x]$. Applying equation (1), we then know that $A(x-\epsilon,y) \leq a < A(x,y)$, for every $\epsilon \in]0,x]$. This contradicts the left-continuity of A and, hence, C_a must be left-continuous. Suppose now that there exists a triplet $(x,y,a) \in [0,1]^3$ such that $0 < y$, $a < 1$ and $C_a(x) < y \leq C_{a+\epsilon}(x)$, for every $\epsilon \in]0,1-a]$. From equation (1) it then follows that $a < A(x,y) \leq a+\epsilon$, for every $\epsilon \in]0,1-a]$. Taking the limit $\epsilon \overset{>}{\to} 0$ leads to the contradiction $a < a$. We conclude that $C_a(x)$ is right-continuous in a. $\square$

Remark 1. In order to prove the right-continuity of $C_\bullet(x)$, it is sufficient to invoke the left-continuity of the partial functions $A(x,\bullet)$ only. However, when proving the left-continuity of C_a, also the left-continuity of the partial functions $A(\bullet,x)$ is needed. For example, if $A(1,y) = 1$, for every $y \in [0,1]$, and $A(x,y) = 0$, elsewhere, then $C_0(x) = 1$ for every $x \in [0,1[$ and $C_0(1) = 0$. The contour line C_0 is, in contrast to the partial functions $A(x,\bullet)$, not left-continuous. Note that the converse implications of Corollary 1 do not hold. If $A(1,1) = 1$ and $A(x,y) = 0$ elsewhere, then $C_a(x) = 1$, for every $(x,a) \in [0,1]^2$. A is not left-continuous, although $C_a(x)$ is continuous in both x and a.

Taking a closer look at equations (1)–(4), it strikes that only the restrictions on y prevent them from being fully interpreted as Galois connections. In the following theorem we figure out under which conditions these restrictions on y become superfluous.

Theorem 2. Consider $x \in [0,1]$. Each of the following four sets consists of four equivalent assertions.

1. (i) $A(x,\bullet)$ is left-continuous and fulfills $A(x,0) = 0$.
   (ii) $(A(x,\bullet), C_\bullet(x))$ is a Galois connection.
   (iii) For every $a \in [0,1]$ it holds that $A(x, C_a(x)) \leq a$.
   (iv) For every $a \in [0,1]$ it holds that $C_a(x) = \max\{t \in [0,1] \mid A(x,t) \leq a\}$.
2. (i) $A(x,\bullet)$ is right-continuous and fulfills $A(x,1) = 1$.
   (ii) $(D_\bullet(x), A(x,\bullet))$ is a Galois connection.
   (iii) For every $a \in [0,1]$ it holds that $a \leq A(x, D_a(x))$.
   (iv) For every $a \in [0,1]$ it holds that $D_a(x) = \min\{t \in [0,1] \mid A(x,t) \geq a\}$.
3. (i) $A(\bullet,x)$ is left-continuous and fulfills $A(0,x) = 0$.
   (ii) $(A(\bullet,x), \tilde{C}_\bullet(x))$ is a Galois connection.
   (iii) For every $a \in [0,1]$ it holds that $A(\tilde{C}_a(x), x) \leq a$.
   (iv) For every $a \in [0,1]$ it holds that $\tilde{C}_a(x) = \max\{t \in [0,1] \mid A(t,x) \leq a\}$.

4. (i) $A(\bullet, x)$ is right-continuous and fulfills $A(1, x) = 1$.
   (ii) $(\tilde{D}_\bullet(x), A(\bullet, x))$ is a Galois connection.
   (iii) For every $a \in [0,1]$ it holds that $a \leq A(\tilde{D}_a(x), x)$.
   (iv) For every $a \in [0,1]$ it holds that $\tilde{D}_a(x) = \min\{t \in [0,1] \mid A(t, x) \geq a\}$.

Proof. We will only prove the equivalence in the first set, the other cases being similar. Taking into account Theorem 1, assertion (1)(i) will be equivalent with assertion (1)(ii) if we can show that the boundary condition $A(x, 0) = 0$ is equivalent with $A(x, 0) \leq a \;\Leftrightarrow\; 0 \leq C_a(x)$, for every $a \in [0,1]$. As $0 \leq C_a(x)$ is always true, this amounts to the trivial equivalence $A(x, 0) = 0 \;\Leftrightarrow\; A(x, 0) \leq a$, for every $a \in [0,1]$. By definition $A(x, y) \leq a$ always implies $y \leq C_a(x)$ and $y < C_a(x)$ always implies $A(x, y) \leq a$. Therefore, assertion (1)(ii) is satisfied if and only if $y = C_a(x)$ implies $A(x, y) \leq a$. The latter is expressed by assertion (1)(iii). It is clear that assertion (1)(iv) implies assertion (1)(ii). Also the converse is true because otherwise there would exist $(y, a) \in [0,1]^2$ such that $C_a(x) = y$ and $A(x, y) > a$, which contradicts assertion (1)(ii). □

For a conjunctive uninorm U, the inequality

$$U(x, I_U(x, a)) = U(x, C_a(x)) = U(\tilde{C}_a(x), x) \leq a$$

is also known as the *generalized modus ponens* [4]. Dually, if U is disjunctive, we obtain the inequality $a \leq U(x, D_a(x)) = U(\tilde{D}_a(x), x) = U(x, J_U(x, a))$.

2.2. Commutativity

As can be seen from their definition and from Theorem 2, the contour lines, resp. C_a, D_a, $\tilde{C}_a$, and $\tilde{D}_a$ are particularly suited to describe increasing $[0,1]^2 \to [0,1]$ functions A that have, respectively, left-continuous partial functions $A(x, \bullet)$, right-continuous partial functions $A(\bullet, x)$, left-continuous partial functions $A(x, \bullet)$ and right-continuous partial functions $A(\bullet, x)$. If A is *commutative*, i. e. $A(x, y) = A(y, x)$ for every $(x, y) \in [0,1]^2$, both partial functions $A(x, \bullet)$ and $A(\bullet, x)$ must always have the same type of continuity. For left- or right-continuous increasing binary operators A, commutativity can be easily expressed in terms of contour lines.

Theorem 3. The following assertions hold:

(i) If A is left-continuous, then A is commutative if and only if $C_a = \tilde{C}_a$, for every $a \in [0,1]$.

(ii) If A is right-continuous, then A is commutative if and only if $D_a = \tilde{D}_a$, for every $a \in [0,1]$.

Proof. For a left-continuous A it suffices to prove the sufficient conditions, the necessary conditions being trivially fulfilled. Suppose that $A(x, y) < A(y, x)$, for some $(x,y) \in [0,1]^2$ with $0 < x$ or $0 < y$. Then it follows from equation (1) that $C_{A(x,y)}(y) < x$, whenever $0 < x$, and from equation (3) that $\tilde{C}_{A(x,y)}(x) < y$, whenever $0 < y$. Since $C_{A(x,y)} = \tilde{C}_{A(x,y)}$, this leads to $\tilde{C}_{A(x,y)}(y) < x$, whenever $0 < x$, and $C_{A(x,y)}(x) < y$, whenever $0 < y$. Applying equation (1) or (3) once again, we obtain in both cases the contradiction $A(x,y) < A(x,y)$. □

The commutativity of an increasing $[0,1]^2 \to [0,1]$ function A does not always ensure the symmetry of its contour lines. If a contour line is not bijective, its inverse is not a $[0,1] \to [0,1]$ function, hence, the contour line can never coincide with its inverse. Introducing some new kind of symmetry, we can illustrate that the commutativity of A shows up through the symmetry of its contour lines.

Let $f : [0,1] \to [0,1]$ be a monotone function. Adding vertical segments, we complete the graph of f to a continuous curve from the point $(0,0)$ to the point $(1,1)$ whenever f is increasing and from the point $(0,1)$ to the point $(1,0)$ whenever f is decreasing. We construct the classical inverse of this 'completed' curve in which we delete all but one point from any vertical segment. The set of $[0,1] \to [0,1]$ functions obtained in this way is denoted by $Q(f)$. Whenever f is bijective, it holds that $Q(f) = \{f^{-1}\}$. For a constant function $\boldsymbol{\alpha} : [0,1] \to [0,1] : x \mapsto \alpha$, with $\alpha \in [0,1]$, the set $Q(\boldsymbol{\alpha})$ contains functions constructed from the increasing completion of $\boldsymbol{\alpha}$ as well as functions constructed from the decreasing completion of $\boldsymbol{\alpha}$. For increasing functions f the construction of the set $Q(f)$ is ascribed to Schweizer and Sklar [22]. Some additional results for monotone functions are due to Klement *et al.* [17, 18].

We call the monotone function f *orthosymmetrical* if $f \in Q(f)$. A motivation for this terminology is given in [20]. For a decreasing function f, the $[0,1] \to [0,1]$ functions $\overline{f}$ and $\underline{f}$, defined by

$$\begin{aligned} \overline{f}(x) &= \sup\{t \in [0,1] \mid f(t) > x\}\,, \\ \underline{f}(x) &= \inf\{t \in [0,1] \mid f(t) < x\}\,, \end{aligned}$$

totally determine its orthosymmetry. Note that, for a non-constant f, the function $\overline{f}$, is also known as the *pseudo-inverse* $f^{(-1)}$ of f [18].

Theorem 4. ([20]) A decreasing $[0,1] \to [0,1]$ function f is orthosymmetrical if and only if $\overline{f} \le f \le \underline{f}$.

A constant $[0,1] \to [0,1]$ function $\boldsymbol{\alpha}$ with $\alpha \in [0,1]$ is orthosymmetrical if and only if $\alpha \in \{0,1\}$ [20]. In the following theorem we try to lay bare the connection between the commutativity of A and the orthosymmetry of its contour lines.

Theorem 5. If A is commutative, then all contour lines C_a, D_a, $\tilde{C}_a$ and $\tilde{D}_a$, $a \in [0,1]$, are orthosymmetrical.

P r o o f. For a commutative A it always holds that $C_a = \tilde{C}_a$ and $D_a = \tilde{D}_a$. We will prove that each contour line C_a, $a \in [0,1]$, is orthosymmetrical. By definition it holds that

$$\begin{aligned}\overline{C_a}(x) &= \sup\{t \in [0,1] \mid C_a(t) > x\}\,,\\ C_a(x) &= \sup\{t \in [0,1] \mid A(x,t) \leq a\}\,,\\ \underline{C_a}(x) &= \sup\{t \in [0,1] \mid C_a(t) \geq x\}\,.\end{aligned}$$

The commutativity of A guarantees that

$$C_a(t) > x \quad \Rightarrow \quad A(t,x) = A(x,t) \leq a \quad \Rightarrow \quad C_a(t) \geq x\,,$$

which leads to $\overline{C_a} \leq C_a \leq \underline{C_a}$. As C_a is decreasing, it follows from Theorem 4 that it is orthosymmetrical. □

Remark 2. The function A defined by $A(x,0) = 0$, for all $x \in [0,1]$, and $A(x,y) = 1$, elsewhere, is left-continuous and not commutative. It is easily verified that in this example all contour lines C_a, D_a, $\tilde{C}_a$ and $\tilde{D}_a$ are orthosymmetrical. Hence, orthosymmetry (of all contour lines) is not sufficient to obtain commutativity.

To better understand the relationship between orthosymmetry and commutativity, we need to recall the following result.

Theorem 6. ([20]) For each decreasing $[0,1] \to [0,1]$ function f the following assertions hold:

(i) If f is left-continuous and fulfills $f(0) = 1$, then f is orthosymmetrical if and only if $f = \underline{f}$.

(ii) If f is right-continuous and fulfills $f(1) = 0$, then f is orthosymmetrical if and only if $f = \overline{f}$.

Based on this characterization of orthosymmetry, we are able to present an alternative description of commutativity for left- or right-continuous increasing functions A that satisfy some additional boundary condition.

Theorem 7.

1. If A is left-continuous and $A(0,1) = A(1,0) = 0$, then the following assertions are equivalent:

   (i) A is commutative.

   (ii) C_a is orthosymmetrical for every $a \in [0,1]$.

   (iii) $\tilde{C}_a$ is orthosymmetrical for every $a \in [0,1]$.

2. If A is right-continuous and $A(0,1) = A(1,0) = 1$, then the following assertions are equivalent:

   (i) A is commutative.

(ii) D_a is orthosymmetrical for every $a \in [0,1]$.

(iii) $\tilde{D}_a$ is orthosymmetrical for every $a \in [0,1]$.

Proof. We only prove the first part of the theorem. Note that the increasingness of A implies that $A(x,0) = A(0,x) = 0$, for every $x \in [0,1]$. Because assertion (1)(i) implies assertions (1)(ii) and (1)(iii) (Theorem 5), we only have to prove the converse. Assume that C_a is orthosymmetrical for every $a \in [0,1]$. The left-continuity of A ensures that every C_a is left-continuous (Corollary 1). $C_a(0) = 1$ due to the boundary condition $A(0,1) = 0$. Hence, applying Theorem 6, the orthosymmetry of C_a is equivalent with $C_a = \underline{C_a}$. Suppose now that A is not commutative. Then $A(y,x) < A(x,y)$, for some $(x,y) \in [0,1]^2$. As $(A(x,\bullet), C_\bullet(x))$ forms a Galois connection (Theorem 2), this inequality is equivalent with $C_{A(y,x)}(x) < y$ and therefore

$$\underline{C_{A(y,x)}}(x) = \inf\{t \in [0,1] \mid C_{A(y,x)}(t) < x\} < y\,.$$

Since contour lines are decreasing, the latter implies that $C_{A(y,x)}(y) < x$ which leads to the contradiction $A(y,x) < A(y,x)$. We conclude that A must be commutative. In a similar way it can be shown that assertion (1)(iii) ensures the commutativity of A. □

2.3. Associativity

We call A *associative* if $A(A(x,y),z) = A(x,A(y,z))$ holds for every $(x,y,z) \in [0,1]^3$. Assuming some continuity and boundary conditions, we can use contour lines to express the associativity of A.

Theorem 8. The following assertions hold:

(i) If $A(x,\bullet)$ is left-continuous for every $x \in [0,1]$ and $A(1,0) = 0$, then A is associative if and only if

$$C_a(A(x,y)) = C_{C_a(x)}(y) \tag{5}$$

holds for every $(x,y,a) \in [0,1]^3$.

(ii) If $A(x,\bullet)$ is right-continuous for every $x \in [0,1]$ and $A(0,1) = 1$, then A is associative if and only if

$$D_a(A(x,y)) = D_{D_a(x)}(y) \tag{6}$$

holds for every $(x,y,a) \in [0,1]^3$.

(iii) If $A(\bullet,x)$ is left-continuous for every $x \in [0,1]$ and $A(0,1) = 0$, then A is associative if and only if

$$\tilde{C}_a(A(x,y)) = \tilde{C}_{\tilde{C}_a(y)}(x) \tag{7}$$

holds for every $(x,y,a) \in [0,1]^3$.

(iv) If $A(\bullet, x)$ is right-continuous for every $x \in [0,1]$ and $A(1,0) = 1$, then A is associative if and only if

$$\tilde{D}_a(A(x,y)) = \tilde{D}_{\tilde{D}_a(y)}(x) \tag{8}$$

holds for every $(x,y,a) \in [0,1]^3$.

Proof. It suffices to prove the first assertion. Note that the boundary condition $A(0,1) = 0$ is equivalent with $A(0,x) = 0$, for every $x \in [0,1]$. This proof makes extensive use of the first set of equivalent assertions in Theorem 2. If A is associative, then we know that $C_a(A(x,y)) =$

$$\max\{t \in [0,1] \mid A(A(x,y),t) \leq a\} = \max\{t \in [0,1] \mid A(x, A(y,t))) \leq a\}\,,$$

for every $(x,y,a) \in [0,1]^3$. Because $A(x, A(y,t))) \leq a$ is equivalent with $A(y,t) \leq C_a(x)$, we can rewrite this equality as follows:

$$C_a(A(x,y)) = \max\{t \in [0,1] \mid A(y,t) \leq C_a(x)\} = C_{C_a(x)}(y)\,.$$

Conversely, if equation (5) holds, we need to prove that $A(A(x,y),z) = A(x, A(y,z))$, for every $(x,y,z) \in [0,1]^3$. Since

$$C_{C_{A(A(x,y),z)}(x)}(y) = C_{A(A(x,y),z)}(A(x,y)) \geq z\,,$$

we obtain that $A(y,z) \leq C_{A(A(x,y),z)}(x)$ and, hence, $A(x, A(y,z)) \leq A(A(x,y),z)$. If $A(x, A(y,z)) < A(A(x,y),z)$, then it follows that $C_{A(x,A(y,z))}(A(x,y)) < z$. Applying equation (5) yields $C_{C_{A(x,A(y,z))}(x)}(y) < z$ and thus $C_{A(x,A(y,z))}(x) < A(y,z)$. Finally, we obtain the contradiction $A(x, A(y,z)) < A(x, A(y,z))$. □

Remark 3. The continuity and boundary conditions are indispensable in the proof of the above theorem. For example, consider the increasing function A defined by $A(x,1) = 1/2$, for every $x \in [0,1]$, and $A(x,y) = 0$, elsewhere. The partial functions $A(x,\bullet)$ are not left-continuous, and for every $(x,a) \in [0,1]^2$ it holds that $C_a(x) = 1$. Equation (5) is then trivially fulfilled although A is not associative (e.g. $A(A(1,1),1) = A(1/2,1) = 1/2 > 0 = A(1,1/2) = A(1, A(1,1))$). To illustrate the importance of the boundary conditions, consider the increasing function A defined by $A(1,y) = 1$, for every $y \in [0,1]$, and $A(x,y) = 0$, elsewhere. All partial functions $A(x,\bullet)$ are continuous but $A(1,0) = 1$. It is easily verified that A is associative. However, $C_{1/2}(A(1,0)) = C_{1/2}(1) = 0 < 1 = C_0(0) = C_{C_{1/2}(1)}(0)$, which contradicts equation (5).

For a left-continuous t-norm T, taking into account the correspondence between its residual implicator I_T and its contour lines C_a, equation (5) coincides with the *portation law* [11]: $I_T(T(x,y),z) = I_T(y, I_T(x,z))$, for every $(x,y,z) \in [0,1]^3$. Theorem 8 implies that for a left-continuous t-norm T, this portation law is always fulfilled and is equivalent with its associativity. Due to the commutativity of a t-norm, the portation law also implies the *exchange principle* [2, 5, 6]: $I_T(x, I_T(y,z)) = I_T(y, I_T(x,z))$, for every $(x,y,z) \in [0,1]^3$. Dealing with a commutative A, this property can also be used to express associativity.

Theorem 9. If A is commutative, then the following assertions hold:

(i) If A left-continuous and $A(0,1)=0$, then A is associative if and only if

$$C_{C_a(x)}(y) = C_{C_a(y)}(x) \tag{9}$$

holds for every $(x,y,a) \in [0,1]^3$.

(ii) If A is right-continuous and $A(1,0)=1$, then A is associative if and only if

$$D_{D_a(x)}(y) = D_{D_a(y)}(x) \tag{10}$$

holds for every $(x,y,a) \in [0,1]^3$.

Proof. The commutativity of A allows us to consider contour lines of the types C_a and D_a only. Assume that A is left-continuous and $A(0,1)=0$. If A is associative, then equation (9) follows immediately from Theorem 8. Conversely, suppose that equation (9) is satisfied. If A is not associative, there exists a triplet $(x,y,z) \in [0,1]^3$ such that

$$A(y, A(x,z)) = A(A(x,z),y) < A(x, A(z,y)) = A(x, A(y,z))\,.$$

Consider $a \in \,]A(y,A(x,z)), A(x,A(y,z))[$. From Theorem 2 it then follows that $A(x,z) \le C_a(y)$ and $C_a(x) < A(y,z)$. Applying Theorem 2 a second time leads to $z \le C_{C_a(y)}(x)$ and $C_{C_a(x)}(y) < z$. We obtain the contradiction $C_{C_a(x)}(y) < C_{C_a(y)}(x)$. □

Remark 4. Note that the commutativity of A plays a key role in the above theorem. For example, define a non-commutative A by $A(x,0)=0$, for every $x \in [0,1]$, and $A(x,y)=x$, elsewhere. Although A is associative, left-continuous and satisfies $A(0,1)=0$ it holds that $C_{C_{1/2}(1)}(1/2) = 0 < 1 = C_{C_{1/2}(1/2)}(1)$.

2.4. Neutral element

Recall that $e \in [0,1]$ is called a *neutral element* of A if $A(x,e)=A(e,x)=x$, for every $x \in [0,1]$. In the following theorem we investigate, for a fixed $x \in [0,1]$, the conditions $A(x,e)=x$ and $A(e,x)=x$.

Theorem 10. For every $x \in [0,1]$ the following assertions hold:

(i) If $A(x,\bullet)$ is left-continuous, then $A(x,e)=x$ holds for some $e \in \,]0,1]$ if and only if

$$e \le C_a(x) \quad \Leftrightarrow \quad x \le a \tag{11}$$

holds for every $a \in [0,1]$.

(ii) If $A(x,\bullet)$ is right-continuous, then $A(x,e)=x$ holds for some $e \in [0,1[$ if and only if

$$D_a(x) \le e \quad \Leftrightarrow \quad a \le x \tag{12}$$

holds for every $a \in [0,1]$.

(iii) If $A(\bullet, x)$ is left-continuous, then $A(e, x) = x$ holds for some $e \in \,]0, 1]$ if and only if

$$e \leq \tilde{C}_a(x) \quad \Leftrightarrow \quad x \leq a \tag{13}$$

holds for every $a \in [0, 1]$.

(iv) If $A(\bullet, x)$ is right-continuous, then $A(e, x) = x$ holds for some $e \in [0, 1[$ if and only if

$$\tilde{D}_a(x) \leq e \quad \Leftrightarrow \quad a \leq x \tag{14}$$

holds for every $a \in [0, 1]$.

Proof. We prove (i). The necessary conditions immediately follow from equation (1) (take $y = e$). Conversely, if equation (11) holds, then we obtain that $e \leq C_x(x)$. Applying equation (1) leads to $A(x, e) \leq x$. In case $A(x, e) < x$, there exists $\epsilon \in \,]0, x[$ such that $A(x, e) \leq x - \epsilon$. Hence, $e \leq C_{x-\epsilon}(x)$, which is equivalent with the contradiction $x \leq x - \epsilon$. □

Remark 5. In the above theorem there are some restrictions on e. The first assertion, for example, deals with $e \in \,]0, 1]$ only. For $e = 0$ the equivalence between $A(x, 0) = x$ and equation (11) reduces to $A(x, 0) = x \;\Leftrightarrow\; x = 0$. The latter is incorrect as it does not hold for $A = S_{\mathbf{M}}$. A left-continuous (resp. right- continuous) increasing function A will have a neutral element $e \in \,]0, 1]$ (resp. $e \in [0, 1[$) if and only if equations (11) and (13) (resp. equations (12) and (14)) are fulfilled for every $x \in [0, 1]$.

3. CONTINUOUS CONTOUR LINES

Depending on the continuity of the partial functions $U(x, \bullet)$ and $U(\bullet, x)$ of a uninorm U, its contour lines fulfill several of the properties stated in the previous section. Uninorms can have discontinuous as well as continuous contour lines. For example, all but one contour line of $T_{\mathbf{M}}$ contain a unique discontinuity point. On the other hand, the algebraic product $T_{\mathbf{P}}$ has only one discontinuous contour line. So far it has not been uncovered how the continuity of the contour lines affects the structure of the uninorm. To tackle this problem, we first need to link the continuity of the contour lines to their involutivity. A decreasing $[0, 1] \to [0, 1]$ function f is called *involutive* on an interval $[a, b] \subseteq [0, 1]$ if $f(f(x)) = x$ is satisfied for every $x \in [a, b]$.

Theorem 11. A decreasing $[0, 1] \to [0, 1]$ function f is orthosymmetrical and continuous if and only if it is involutive on $[f(1), f(0)]$, with $f(0) = 1$ or $f(1) = 0$.

Proof. The functions $\mathbf{0}$ and $\mathbf{1}$ are the only orthosymmetrical, constant decreasing functions [20]. They are trivially continuous. Consider an orthosymmetrical, continuous, non-constant, decreasing $[0, 1] \to [0, 1]$ function f. Take $x \in [f(1), f(0)]$ and denote $l := \min\{t \in [0, 1] \mid f(t) = x\}$ and $u := \max\{t \in [0, 1] \mid f(t) = x\}$. Then, due to Theorem 4 and the continuity of f, we obtain that $l = \overline{f}(x) \leq f(x) \leq \underline{f}(x) = u$

and, hence, $f(f(x)) = x$, for every $x \in [f(1), f(0)]$. Suppose now that $f(0) < 1$, then $0 = \overline{f}(1) \leq f(1) \leq \underline{f}(1) = 0$ leads to $f(1) = 0$. In a similar way $0 < f(1)$ implies that $f(0) = 1$.

Let f be a decreasing $[0,1] \to [0,1]$ function that is involutive on $[f(1), f(0)]$, with $f(0) = 1$ or $f(1) = 0$. Clearly, $f([f(1), f(0)]) = [f(1), f(0)]$ and $f(x) = f(y)$, with $(x,y) \in [f(1), f(0)]^2$, can only occur if $x = y$. We conclude that the restriction of f to $[f(1), f(0)]$ is a decreasing bijection and thus f must be continuous on $[0,1]$. From Theorem 6 it follows that f will be orthosymmetrical if $f = \underline{f}$ when $f(0) = 1$, resp. $f = \overline{f}$ when $f(1) = 0$. If $f(0) = 1$, the bijectivity of f on $[f(1), f(0)]$ ensures that $\underline{f}(x) = f(x)$, whenever $x \in [f(1), 1]$. For every $x \in [0, f(1)[$ it holds that $\underline{f}(x) = f(x) = 1$. In a similar way it can be shown that $f(1) = 0$ implies $f = \overline{f}$. This concludes the proof. □

The continuous contour lines of a left- or right-continuous uninorm are now characterized in the following way:

Theorem 12. Consider a uninorm U with neutral element $e \in [0,1]$. The following statements hold:

1. If U is left-continuous and conjunctive, then, for every $a \in [0,1]$, the following assertions are equivalent:

   (i) C_a is continuous.

   (ii) $U(x,y) = C_a(C_{C_a(x)}(y))$, for every $(x,y) \in [0,1]^2$ s.t. $y > C_a(U(x,1))$.

   (iii) $C_b(x) = C_{C_a(x)}(C_a(b))$, for every $(x,b) \in [0,1] \times [C_a(1), 1]$.

   (iv) $U(x,y) \leq z \;\Leftrightarrow\; U(y, C_a(z)) \leq C_a(x)$, for every $(x,y,z) \in [C_a(1), 1]^3$.

2. If U is right-continuous and disjunctive, then, for every $a \in [0,1]$, the following assertions are equivalent:

   (i) D_a is continuous.

   (ii) $U(x,y) = D_a(D_{D_a(x)}(y))$, for every $(x,y) \in [0,1]^2$ s.t. $y < D_a(U(0,x))$.

   (iii) $D_b(x) = D_{D_a(x)}(D_a(b))$, for every $(x,b) \in [0,1] \times [0, D_a(0)]$.

   (iv) $z \leq U(x,y) \;\Leftrightarrow\; D_a(x) \leq U(y, D_a(z))$, for every $(x,y,z) \in [0, D_a(0)]^3$.

Proof. Consider a conjunctive left-continuous uninorm U. The neutral element e of U must belong to $]0,1]$. Since $C_1(x) = 1$, for every $x \in [0,1]$, assertions (1)(i), (1)(ii),(1)(iii) and (1)(iv) are trivially fulfilled if $a = 1$. Assume now that $a < 1$. The commutativity of U implies the orthosymmetry of C_a (Theorem 5) and the boundary condition $U(0,1) = 0$ is equivalent with $C_a(0) = 1$, for every $a \in [0,1]$. Taking into account Theorem 11, we know that a contour line C_a is continuous if and only if it is involutive on $[C_a(1), C_a(0)] = [C_a(1), 1]$. Since $C_a(C_a(1)) = 1$ also ensures that $C_a(C_a(C_a(1))) = C_a(1)$, it suffices that C_a is involutive on $]C_a(1), 1]$. Note also that $C_a(e) = a$.

(i) $\Leftrightarrow$ (ii) If $y > C_a(U(x,1)) = C_{C_a(1)}(x)$, then it holds that $U(x,y) > C_a(1)$. Under the assumption that C_a is involutive on $]C_a(1),1]$ it follows from equation (5) that $U(x,y) = C_a(C_a(U(x,y))) = C_a(C_{C_a(x)}(y))$. Conversely, suppose that assertion (1)(ii) holds. Let $x = e$, then $y = U(e,y) = C_a(C_{C_a(e)}(y)) = C_a(C_a(y))$. We conclude that C_a is involutive on $]C_a(1),1]$.

(i) $\Leftrightarrow$ (iii) Consider $(x,b) \in [0,1] \times [C_a(1),1]$. If C_a is involutive on $]C_a(1),1]$, then, taking into account the commutativity of U and equation (5), we obtain that

$$C_b(x) = C_{C_a(C_a(b))}(x) = C_a(U(C_a(b),x)) = C_a(U(x,C_a(b))) = C_{C_a(x)}(C_a(b))\,.$$

Conversely, if assertion (1)(iii) holds, then $b = C_b(e) = C_{C_a(e)}(C_a(b)) = C_a(C_a(b))$, for every $b \in [C_a(1),1]$.

(i) $\Leftrightarrow$ (iv) Take arbitrary $(x,y,z) \in [C_a(1),1]^3$ and assume that C_a is involutive on $]C_a(1),1]$. From Theorem 2 it follows that $U(y,C_a(z)) \leq C_a(x)$ is equivalent with $C_a(z) \leq C_{C_a(x)}(y)$. Using equation (5), the latter can be rewritten as $C_a(z) \leq C_a(U(x,y))$. Whenever $C_a(1) \leq U(x,y)$, this inequality is equivalent with $U(x,y) \leq z$. However, if $U(x,y) < C_a(1)$, then $C_a(U(x,y)) = 1$. The inequalities $C_a(z) \leq C_a(U(x,y))$ and $U(x,y) \leq z$ are in that case trivially fulfilled. Assertion (1)(iv) is indeed true. Conversely, suppose that (1)(iv) holds. As $U(C_a(1),1) = U(1,C_a(1)) \leq a$, it holds that $C_a(C_a(1)) = 1$ and therefore $C_a([C_a(1),1]) = [C_a(1),1]$. The decreasingness of U implies that $C_a(1) \leq C_a(e) = a$. If $e \leq C_a(1)$, we obtain the contradiction $a = C_a(e) \geq 1$. Thus, for every $x \in [C_a(1),1]$ it holds that both $(C_a(x),x,a)$ and $(e,C_a(C_a(x)),x)$ belong to $[C_a(1),1]^3$. Applying (1)(iv) on

$$U(C_a(x),x) = U(x,C_a(x)) \leq a$$

and

$$U(C_a(C_a(x)),C_a(x)) = U(C_a(x),C_a(C_a(x))) \leq a = C_a(e)$$

results in two inequalities:

$$U(x,C_a(a)) \leq C_a(C_a(x)) \quad \text{and} \quad C_a(C_a(x)) = U(e,C_a(C_a(x))) \leq x\,.$$

From equation (11) we know that $e \leq C_a(a)$. Weakening the first inequality to $x \leq C_a(C_a(x))$, we conclude that $x = C_a(C_a(x))$, for every $x \in [C_a(1),1]$.□

Example 1. A typical example of a uninorm that has continuous contour lines is the left-continuous, conjunctive $3\,\Pi$–operator E [9]. It is defined by

$$E(x,y) = \frac{x\,y}{(1-x)\,(1-y) + x\,y}\,,$$

for every $(x,y) \notin \{(1,0),(0,1)\}$, and $E(0,1) = E(1,0) = 0$. For every $(x,a) \in [0,1] \times]0,1[$ it holds that

$$C_a(x) = \frac{a\,(1-x)}{x\,(1-a) + a\,(1-x)}\,.$$

Furthermore, $C_0(0) = 1$ and $C_0(x) = 0$, whenever $x \in]0, 1]$. As $C_1 = \mathbf{1}$, we conclude that every contour line C_a, with $a \in]0, 1]$, is continuous.

For left-continuous t-norms two assertions of Theorem 12 are generalizations of well-known properties. Let $a = 0$. Assertion (1)(iii) is referred to as the *contrapositive symmetry* of I_T: $I_T(x, y) = I_T(I_T(y, 0), I_T(x, 0))$, for every $(x, y) \in [0, 1]^2$ [5, 11]. Assertion (1)(iv) expresses the *rotation invariance* of the t-norm: $T(x, y) \leq z \;\Leftrightarrow\; T(y, I_T(z, 0)) \leq I_T(x, 0)$, for every $(x, y, z) \in [0, 1]^3$ [11]. Left-continuous rotation-invariant t-norms have been studied extensively by Jenei [12, 13, 14, 15]. If $I_T(x, 0) = 1-x$, for every $x \in [0, 1]$, the graph of these t-norms remains invariant under an order 3 transformation: the rotation of $[0, 1]^3$ with angle $\frac{2\pi}{3}$ around the axis through the points $(0, 0, 1)$ and $(1, 1, 0)$. Furthermore, assertion (1)(ii) is equivalent with the *self quasi-inverse property* of T: $I_T(x, y) = z \;\Leftrightarrow\; T(x, I_T(y, 0)) = I_T(z, 0)$, for every $(x, y, z) \in [0, 1]^3$ [11].

Corollary 2. Consider a uninorm U with neutral element $e \in [0, 1]$. The following statements hold:

1. If U is left-continuous and conjunctive, then, for every $a \in [0, 1]$ fulfilling $C_a(a) = e$, C_a is continuous if and only if
$$C_y(x) = z \;\Leftrightarrow\; U(x, C_a(y)) = C_a(z) \tag{15}$$
holds for every $(x, y, z) \in [C_a(1), 1]^3$ s.t. $x > C_a(U(C_a(y), 1))$.
2. If U is right-continuous and disjunctive, then, for every $a \in [0, 1]$ fulfilling $D_a(a) = e$, D_a is continuous if and only if
$$D_y(x) = z \;\Leftrightarrow\; U(x, D_a(y)) = D_a(z) \tag{16}$$
holds for every $(x, y, z) \in [0, D_a(0)]^3$ s.t. $x < D_a(U(0, D_a(y)))$.

Proof. Let U be a left-continuous, conjunctive uninorm such that $C_a(a) = e$, for some $a \in [0, 1]$. If C_a is continuous, then assertion (1)(ii) of Theorem 12 implies that $U(x, C_a(y)) = U(C_a(y), x) = C_a(C_{C_a(C_a(y))}(x))$, for every $(x, y) \in [0, 1]^2$ such that $x > C_a(U(C_a(y), 1))$. Taking into account the involutivity of C_a on $[C_a(1), 1]$ (Theorem 11), we immediately obtain equation (15). Conversely, suppose that equation (15) is satisfied. Then also $U(x, C_a(y)) = C_a(C_y(x))$, for every $(x, y) \in [C_a(1), 1]^2$ such that $x > C_a(U(C_a(y), 1))$. In the proof of Theorem 12 we showed that $C_a(1) \leq a$. Putting $y = a$ leads to $U(x, C_a(a)) = C_a(C_a(x))$, for every $x > C_a(U(C_a(a), 1)) = C_{C_a(C_a(a))}(1)$. We obtain that C_a is involutive on $[C_a(1), 1]$ by expressing that $C_a(a) = e$. Due to Theorem 11, this concludes the proof. $\square$

Remark 6. The condition $C_a(a) = e$ is also necessary to obtain the equivalence between the continuity of a contour line C_a and equation (15). Indeed, $U(C_a(a), y) = C_a(C_a(y))$, for every $y > C_{C_a(C_a(a))}(1)$, will be equivalent with the continuity of C_a if and only if $U(C_a(a), y) = y$ holds for every $y \in]C_a(1), 1]$. Because $e > C_a(1) \geq C_a(U(C_a(a), 1))$ (see the proof of Theorem 12), we conclude that $C_a(a) = U(C_a(a), e) = e$.

4. CONCLUSION

With each increasing $[0,1]^2 \to [0,1]$ function A we have associated four types of contour lines. Depending on the continuity of A and some additional boundary conditions, different types of contour lines are of interest. We have been able to translate uninorm properties of A into properties on its contour lines. In particular, the commutativity of A is equivalent with the orthosymmetry of its contour lines. Furthermore, we have identified two conditions on contour lines that each characterize the associativity of A. These conditions are generalizations of the portation law and the exchange principle. We have laid bare how the existence of a continuous contour line affects the structure of a uninorm U. Dealing with a left-continuous and conjunctive (resp. right-continuous and disjunctive) uninorm U, a continuous contour line amounts to a kind of contrapositive symmetry of the residual implicator I_U (resp. residual coimplicator J_U). Also two alternative characterizations of continuous contour lines have been presented. One of them is closely related to the rotation invariance property of t-norms. The other one is related to the portation law. Under some additional conditions, this latter property has been linked to the self quasi-inverse property.

(Received January 26, 2006.)

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Koen C. Maes and Bernard De Baets, Department of Applied Mathematics, Biometrics and Process Control, Ghent University, Coupure links 653, B-9000 Gent. Belgium.
e-mails: Koen.Maes@UGent.be, Bernard.DeBaets@UGent.be