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OPEN PROBLEMS POSED AT THE EIGHTH INTERNATIONAL CONFERENCE ON FUZZY SET THEORY AND APPLICATIONS (FSTA 2006, LIPTOVSKÝ JÁN, SLOVAKIA)

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Several open problems posed during FSTA 2006 (Liptovský Ján, Slovakia) are presented. These problems concern the classification of strict triangular norms, Lipschitz t-norms, interval semigroups, copulas, semicopulas and quasi-copulas, fuzzy implications, means, fuzzy relations, MV-algebras and effect algebras.

Keywords: triangular norm, copula, fuzzy implication, fuzzy relation, MV algebra, effect algebra

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1. INTRODUCTION

The publication of open problems sometimes had a great impact on the development of several areas of mathematics. Maybe the most famous was the formulation of D. Hilbert's problems published in [13]. In the domain of fuzzy sets, fuzzy logic and related areas several open problems were published in monographs, for example in [6, 17, 30]. A paper devoted purely to open problems was [20] which has influenced the study of several problems related to triangular norms, for example in [5, 14]. Among recently published open problems, recall the problems stated in [2] (some of them were solved in the meantime, see [22, 28]) and problems collected at the 24th Linz Seminar on Fuzzy Set Theory 2003 ("Triangular Norms and Related Operators in Many-Valued Logics" held on February 4–8, 2003 in Linz, Austria) in [18] (again some of them were already solved, see [12, 15, 27]). The aim of this paper is the presentation of open problems posed during the conference FSTA 2006 ("Eighth International Conference on Fuzzy Set Theory and Applications" held from January 30 to February 3, 2006 in Liptovský Ján, Slovakia). Before presenting those problems in detail, we will introduce some basic notions and terminology used in this paper.

Triangular norms are, on the one hand, special semigroups and, on the other hand, solutions of some functional equations [1, 17, 29, 30]. This mixture quite often requires new approaches to answer questions about the nature of triangular norms.

A triangular norm (t-norm for short) $T\colon [0,1]^2 \to [0,1]$ is an associative, commutative, non-decreasing function such that 1 acts as a neutral element [29]. Observe that each continuous Archimedean t-norm T can be represented by means of a continuous additive generator [17, 19], i. e., a strictly decreasing continuous function $t : [0,1] \to [0,\infty]$ with $t(1) = 0$ such that

$$T(x,y) = t^{(-1)}(t(x) + t(y)),$$

where the pseudo-inverse $t^{(-1)} : [0,\infty] \to [0,1]$ in this special case is given by

$$t^{(-1)}(u) = t^{-1}(\min(u, t(0))).$$

A function $I : [0,1]^2 \to [0,1]$ is called a (*fuzzy*) *implication* (see [4, Definition 1] or [10, Chapter 1]) if it is non-increasing in its first component, non-decreasing in its second component and satisfies $I(1,0) = 0$ and, for each $x \in [0,1]$, the property $I(0,x) = I(x,1) = 1$.

A (*two-dimensional*) *copula* (first mentioned in [31], for an excellent survey see [26]) is a function $C : [0,1]^2 \to [0,1]$ such that $C(0,x) = C(x,0) = 0$ and $C(1,x) = C(x,1) = x$ for all $x \in [0,1]$, and C is 2-increasing, i. e., for all $x_1, x_2, y_1, y_2 \in [0,1]$ with $x_1 \leq x_2$ and $y_1 \leq y_2$ for the volume V_C of the rectangle $[x_1,x_2] \times [y_1,y_2]$ we have

$$V_C([x_1,x_2] \times [y_1,y_2]) = C(x_1,y_1) - C(x_1,y_2) + C(x_2,y_2) - C(x_2,y_1) \geq 0. \quad (1)$$

Copulas play a key role in the analysis of bivariate distribution functions with given marginals. The basic result in this context is Sklar's Theorem [31, 32] showing that the joint distribution of a random vector and the corresponding marginal distributions are linked by some copula.

A (*two-dimensional*) *quasi-copula* (introduced in [3] and conveniently characterized in [11]) is a function $Q : [0,1]^2 \to [0,1]$ such that $Q(0,x) = Q(x,0) = 0$ and $Q(1,x) = Q(x,1) = x$ for all $x \in [0,1]$, Q is non-decreasing (in each component), and Q is 1-Lipschitz. Obviously, each copula is a quasi-copula but not vice versa. The set of quasi-copulas is a lattice-theoretic completion of the set of copulas in the sense that each quasi-copula is the supremum of a suitable set of copulas.

For each $n \in \mathbb{N} \setminus \{1\}$, an n-copula is a non-decreasing mapping $C : [0,1]^n \to [0,1]$ such that

(i) $C(x_1,\ldots,x_n) = 0$ whenever $\min(x_1,\ldots,x_n) = 0$;

(ii) $C(x_1,\ldots,x_n) = x_j$ whenever $\min(x_1,\ldots,x_n) = x_j$ and, for all $i \neq j$, $x_i = 1$;

(iii) for all $(x_1,\ldots,x_n)$, $(y_1,\ldots,y_n) \in [0,1]^n$ such that $x_i \leq y_i$, $i = 1,2,\ldots,n$, the C-volume of $[x_1,y_1] \times \cdots \times [x_n,y_n]$ is non-negative, i. e.,

$$\sum\nolimits_{e \in \{-1,1\}^n} (-1)^{\mathrm{card}(\{i \mid e_i = -1\})} \cdot C(z_{1,e_1},\ldots,z_{n,e_n}) \geq 0,$$

where $z_{i,1} = y_i$ and $z_{i,-1} = x_i$. For more details about copulas see [26].

2. STRICT t-NORMS

For a strict t-norm T let $\mathcal{F}_T$ be the smallest set of all functions from $[0,1]$ to $[0,1]$ containing the identity function and being closed under T, the standard negation n given by $n(x) = 1 - x$, and under limits of comonotone sequences. It was shown in [7] that for the product $T_{\mathbf{P}}$, $\mathcal{F}_{T_{\mathbf{P}}}$ equals the class of all Borel measurable functions from $[0,1]$ to $[0,1]$. This is no more true for the *Hamacher product* $T^{\mathbf{H}}$ given by

$$T^{\mathbf{H}}(x,y) = \frac{xy}{x+y-xy}$$

whenever $(x,y) \neq (0,0)$, as well as for all so-called *nearly Hamacher t-norms*, i. e., strict t-norms being isomorphic to $T^{\mathbf{H}}$ by means of an isomorphism which generates also the standard negation n.

Problem 2.1. (D. Butnariu, E. P. Klement, R. Mesiar, and M. Navara) Is there a strict t-norm T such that $\mathcal{F}_T \neq \mathcal{F}_{T_{\mathbf{P}}}$ and $\mathcal{F}_T \neq \mathcal{F}_{T^{\mathbf{H}}}$? If $\mathcal{F}_T = \mathcal{F}_{T^{\mathbf{H}}}$ for some strict t-norm T, is T then necessarily nearly Hamacher?

3. LIPSCHITZ t-NORMS

Problem 3.1. (R. Mesiar) Characterize all t-norms which are 1-Lipschitz (or, generally, k-Lipschitz with $k \geq 1$) with respect to the L_p-norm for $p \in]1,\infty[$.

Observe that in the case $p = 1$, the 1-Lipschitz property of an Archimedean t-norm is equivalent to the convexity of its additive generator, see [23] and [17, 26].

Conjecture 3.2. An Archimedean t-norm T is 1-Lipschitz with respect to the L_p-norm if and only if each of its additive generators is a p-power of a convex function.

4. EXTENSIONS OF DISCRETE t-NORMS

Given a discrete t-norm T on the set $I_n = \{0, \frac{1}{n}, \frac{2}{n}, \dots, 1\}$ (see [9, 17]), one can extend it to a t-norm on $[0,1]$ by a piecewise constant extension

$$T^\star(x,y) = \begin{cases} \min(x,y) & \text{if } \max\{x,y\} = 1, \\ T\left(\frac{\lfloor xn \rfloor}{n}, \frac{\lfloor yn \rfloor}{n}\right) & \text{elsewhere}. \end{cases}.$$

Denote $\tau_n^\star$ the set of all such extensions of all discrete t-norms on I_n and put

$$\tau^\star = \bigcup\nolimits_{n\in\mathbb{N}} \tau_n^\star.$$

Problem 4.1. (B. De Baets, P. Sarkoci) What is the closure of the set $\tau^\star$ with respect to the norm of uniform convergence? In particular, are the sets of all strict t-norms, of all continuous t-norms and of all t-norms inside?

5. INTERVAL SEMIGROUPS

A mapping $T : [0, 1]^2 \to [0, 1]$ is called a triangular subnorm (t-subnorm, for short) if it fulfills the following properties: (i) associativity; (ii) commutativity; (iii) non-decreasing monotonicity; (iv) $T \leq \min$. Recently, an ever increasing number of papers regarding various aspects of t-subnorms has appeared in literature, as evidence of their importance in many other related topics (see, for instance, the problem of construction of left-continuous t-norms).

A continuous t-subnorm T is *Archimedean* if $1^n \to 0$, where x^n denotes the usual n-power of $x \in [0, 1]$. This property allows a first sharp classification of the class of continuous Archimedean t-subnorms into two subclasses, the non idempotent, for which $1^n \neq 0$ for every $n \in \mathbb{N}$, and the idempotent otherwise.

In a more general algebraic context, adopting the notation $x * y = T(x, y)$, we deal with linearly ordered semigroups of the kind $S = (I, *, \leq)$, with 0 as annihilator but no neutral elements, where $I = [0, 1]$. Such an algebraic structure will be called an *interval semigroup*: in the sequel, it is intended that S is a continuous Archimedean interval semigroup. We say that a non-decreasing mapping $f : I \to \mathbb{R}$ is a multiplicative generator of S if $x * y = f^{(-1)}(f(x) \cdot f(y))$ for all $x, y \in I$, where $f^{(-1)}$ denotes the pseudo-inverse of f. Recently, the best characterization for non nilpotent S has been formulated in terms of the *diagonal* $d(x) = x^2$.

Theorem 5.1. Let S be non nilpotent. Then it admits a continuous multiplicative generator if and only if d is strictly increasing on $[0, 1]^2$.

Problem 5.2. (R. Ghiselli Ricci) Characterize the nilpotent interval semigroups which admit a continuous multiplicative generator.

Problem 5.3. (R. Ghiselli Ricci) Characterize the nilpotent interval semigroups which exclusively admit non continuous multiplicative generators.

Problem 5.4. (R. Ghiselli Ricci) Characterize the non nilpotent interval semigroups which do not admit any multiplicative generator.

We emphasize that explicit examples of interval semigroups belonging to the subclasses mentioned in Problems 5.3 and 5.4 have been provided, for example, in [21].

6. COPULAS

Problem 6.1. (C. Sempi) Each 3-copula C induces three marginal 2-copulas C_{12}, C_{23}, and C_{13} given by $C_{12}(x, y) = C(x, y, 1)$, $C_{23}(y, z) = C(1, y, z)$, and $C_{13}(x, z) = C(x, 1, z)$. The triplet (C_{12}, C_{23}, C_{13}) is then called compatible. For a given triplet of 2-copulas (C, D, E), find a characterization ensuring its compatibility. Similarly, for a fixed 2-copula E, find all pairs of 2-copulas (C, D) such that the triplet (C, D, E)

is compatible. Moreover, for fixed 2-copulas C, D, identify all 2-copulas E such that the triplet (C, D, E) is compatible.

Observe that for all 2-copulas C and D, the triplet $(C, D, C * D)$ is compatible, where $C * D$ is the product of copulas introduced in [8], see also [17, 26].

Problem 6.2. (C. Sempi) Copulas generated by means of additive generators are called Archimedean copulas. Each associative 2-copula $C : [0,1]^2 \to [0,1]$ is Archimedean whenever $C(x,x) < x$ for all $x \in]0,1[$. Is there an algebraic characterization for Archimedean 3-copulas (or, generally, for n-copulas with $n > 2$)?

Problem 6.3. (C. Sempi) Replacing property (iii) of copulas by the 1-Lipschitz property, i. e.,

(iii') for all $(x_1, \dots, x_n)$, $(y_1, \dots, y_n) \in [0,1]^n$

$$|(Q(x_1, \dots, x_n) - Q(y_1, \dots, y_n)| \leq \sum\nolimits_{i=1}^{n} |x_i - y_i|,$$

(and keeping properties (i) and (ii)), the class of n-quasi-copulas is defined. Generated quasi-copulas are called Archimedean quasi-copulas and, for $n = 2$, Archimedean quasi-copulas are necessarily copulas (in general, n-copulas form a proper subclass of n-quasi-copulas for all $n = 2, 3, \dots$). Is there an algebraic characterization for Archimedean 3-quasi-copulas (or, generally, for n-quasi-copulas with $n > 2$)?

Problem 6.4. (C. Sempi) Each n-copula C can be represented by means of n measure preserving transformations $\varphi_i : [0,1] \to [0,1]$, $i = 1, \dots, n$, via

$$C(x_1, \dots, x_n) = \lambda({\varphi_1}^{-1}([0, x_1]) \cap \dots \cap {\varphi_n}^{-1}([0, x_n])),$$

where λ is the standard Lebesgue measure on the Borel subsets of $[0,1]$. Are there any particular properties of an n-copula C which are represented by means of n ergodic (or n weakly mixing) transformations?

Problem 6.5. (C. Sempi) Is there a norm or topology on $[0,1]^2$ such that the product $*$ of copulas introduced in [8] is jointly continuous?

7. SEMICOPULAS AND QUASI–COPULAS

A mapping $S : [0,1]^2 \to [0,1]$ is called a *semicopula* (*conjunctor*) if it is non-decreasing in both coordinates and if 1 is the neutral element of S, i. e., $S(x,1) = S(1,x) = x$ for all $x \in [0,1]$. Observe that each semicopula which is 1-Lipschitz (with respect to the L_1-norm) is a quasi-copula, and, moreover, it is 2-Lipschitz with respect to the Chebyschev norm L_∞.

Problem 7.1. (A. Mesiarová)

(i) Characterize all associative semicopulas which are 2-Lipschitz with respect to L_∞.

(ii) Characterize all quasi-copulas which are exactly 2-Lipschitz with respect to L_∞, i. e., which are not Lipschitz with respect to L_∞ for any constant $c < 2$.

(iii) Similarly, for each constant $c \in [1, 2]$, characterize all quasi-copulas which are exactly c-Lipschitz with respect to L_∞.

8. FUZZY IMPLICATIONS

Problem 8.1. (M. Baczyński, J. Balasubramaniam) Let I be a fuzzy implication and T be a t-norm.

(i) For a given (continuous) t-norm T, characterize all fuzzy implications which satisfy the law of importation with respect to T, i. e., for all $x, y, z \in [0, 1]$

$$I(x, I(y, z)) = I(T(x, y), z).$$

(ii) Since T is commutative, the law of importation implies the exchange principle, i. e., for all $x, y, z \in [0, 1]$

$$I(x, I(y, z)) = I(y, I(x, z)).$$

Is the converse also true, i. e., does the exchange principle imply that there exists a t-norm such that the law of importation holds? If yes, is the t-norm uniquely determined? If not, give an example and characterize all fuzzy implications for which the converse is true.

9. MEANS

Let $a = (a_1, \dots, a_n)$ be an n-dimensional vector of positive real numbers, $n \in \mathbb{N}$. Consider the quasi-arithmetic mean defined by

$$M_f(a) = f^{-1}\Big(\frac{1}{n}\sum\nolimits_{k=1}^{n} f(a_k)\Big),$$

where $f :]0, \infty[\to \mathbb{R}$ is a continuous and strictly monotone function.

Problem 9.1. (O. Hutník) Find all pairs (f, g), (u, v) such that

$$M_f(a) - M_g(a) \le M_u(a) - M_v(a).$$

In connection with the generalized weighted quasi-arithmetic means in integral form we introduce the generalized weighted mean operator defined in (3) and state the following *general mean-type inequality problem*:

Problem 9.2. (O. Hutník)

(iii) Let $p > 0$, $0 < q < \infty$ and define $W :]0, \infty[\to \mathbb{R}$ by

$$W(x) = \int_0^x w(s)\,\mathrm{d}s,$$

where $w :]0, \infty[\to \mathbb{R}$ is a positive Lebesgue integrable function.

Find necessary and sufficient conditions on the positive measurable functions u, v (weights) and establish a class of functions g (a real continuous and strictly monotone function with its inverse g^{-1}) such that the following general mean-type inequality

$$\Big(\int_0^\infty u(x)\big([\mathbf{M}_w^g f](x)\big)^q\,\mathrm{d}x\Big)^{\frac{1}{q}} \le C\Big(\int_0^\infty v(x) f(x)^p\,\mathrm{d}x\Big)^{\frac{1}{p}}, \quad f \ge 0, \qquad (2)$$

holds for a positive finite constant C, where

$$[\mathbf{M}_w^g f](x) = g^{-1}\Big(\frac{1}{W(x)}\int_0^x w(t) g(f(t))\,\mathrm{d}t\Big) \qquad (3)$$

denotes the *generalized weighted mean operator* of a positive Lebesgue integrable function $f :]0, \infty[\to \mathbb{R}$.

(iv) Estimate the best possible constant C in (2).

Let $L_1^+([a, b])$ be the vector space of all real positive Lebesgue integrable functions on $[a, b]$ with $a, b \in \mathbb{R}$ and $a < b$. In what follows, $\|p\|_{[a,b]}$ denotes the finite L_1-norm of a function $p \in L_1^+([a, b])$.

Let $(p, f) \in L_1^+([a, b]) \times L_1^+([a, b])$ such that $f : [a, b] \to [\alpha, \beta]$ and $\varphi : [\alpha, \beta] \to \mathbb{R}$ is a real continuous and strictly monotone function. The *generalized weighted quasi-arithmetic mean $M_{[a,b],\varphi}(p, f)$ of a function f with respect to a weight function p* is given by

$$M_{[a,b],\varphi}(p, f) = \varphi^{-1}\Big(\frac{1}{\|p\|_{[a,b]}}\int_a^b p(x)\varphi(f(x))\,\mathrm{d}x\Big),$$

where φ^{-1} is the inverse function to the function φ.

Problem 9.3. (O. Hutník) Determine all those strictly monotone and continuous functions $\varphi, \psi : [\alpha, \beta] \to \mathbb{R}$ for which the equation

$$M_{[a,b],\varphi}(p, f) + M_{[a,b],\psi}(p, g) = p(a)f(a) + p(b)g(b)$$

holds for every continuous functions f, g on $[a, b]$.

10. FUZZY RELATIONS

Let $\mathcal{A}$ be the set of all symmetric matrices defined on the real vector space $\mathbb{R}^n$, such that for each $A \in \mathcal{A}$ we have $AA = A$. Let $\mathbf{I}$ be the identity matrix and let $\mathbf{0}$ be the zero matrix.

Each element $A \in \mathcal{A}$ has the information about a measurable system for a "yes"/"no" random experiment. It means it has Boolean information. If we take two such elements $A, B \in \mathcal{A}$ we have information about two Boolean systems (i. e. two measurable systems).

A fuzzy relation $s : \mathcal{A} \times \mathcal{A} \to [0, 1]$ is called an *s-map* [24, 25, 16] if

(i) $s(\mathbf{I}, \mathbf{I}) = 1$;

- if $A, B \in \mathcal{A}$ such that $AB = BA = \mathbf{0}$, then $s(A, B) = 0$;

(ii) if $A, B \in \mathcal{A}$ such that $AB = BA = \mathbf{0}$, then for each $C \in \mathcal{A}$

$$\begin{aligned} s(A + B, C) &= s(A, C) + s(B, C), \\ s(C, A + B) &= s(C, A) + s(C, B). \end{aligned}$$

If $s(A, B) = s(B, A)$ then we can find one probability space for two different measurable systems. It means A, B are "virtually compatible". For example, we cannot study the problem of causality between these measurable systems, in this case [16].

Problem 10.1. (O. Nánásiová) Is the fuzzy relation s necessarily symmetric, i. e., $s(A, B) = s(B, A)$ for all $A, B \in \mathcal{A}$?

11. MV-ALGEBRAS

Problem 11.1. (G. Jenča) Characterize all bounded chains which are carriers of MV-algebras. Is it enough to require the existence of an involution of the given chain? Is the property of a bounded chain to be a carrier of an MV-algebra a first order property?

12. EFFECT ALGEBRAS

A partial algebra $(E; \oplus, 0, 1)$ is called an *effect algebra* if $0, 1$ are two distinct elements and $\oplus$ is a partially defined binary operation on E which satisfies the following conditions for any $a, b, c \in E$:

(Ei) $b \oplus a = a \oplus b$ if $a \oplus b$ is defined,

(Eii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ if one side is defined,

(Eiii) for every $a \in E$ there exists a unique $b \in E$ such that $a \oplus b = 1$ (we put $a' = b$),

(Eiv) if $1 \oplus a$ is defined then $a = 0$.

Let for $a, b \in E$: $a \leq b$ if and only if there is $c \in E$ with $a \oplus c = b$ (then we write $c = b \ominus a$). If $(E, \leq)$ is a lattice (complete lattice) then E is called a *lattice effect algebra* (*complete effect algebra*). We write $a \leftrightarrow b$ if and only if $a \vee b = a \oplus (b \ominus (a \wedge b))$. A maximal subset $B \subseteq E$ such that $a \leftrightarrow b$ for all $a, b \in B$ is called a *block* of E .

Set

$$\begin{aligned} \mathcal{S}(E) &= \{x \in E \mid x \wedge x' = 0\}, \\ \mathcal{B}(E) &= \bigcap \{B \subset E \mid B \text{ a block of } E\}, \\ \mathcal{C}(E) &= \mathcal{S}(E) \cap \mathcal{B}(E). \end{aligned}$$

Then $\mathcal{S}(E)$, $\mathcal{B}(E)$, $\mathcal{C}(E)$ are sub-lattices and sub-effect algebras of E.

Problem 12.1. (Z. Riečanová) Describe the family of all lattice (complete) effect algebras with one of the following properties:

(i) $\mathcal{C}(E) = \mathcal{S}(E)$,

(ii) $\mathcal{C}(E) = \mathcal{B}(E)$,

(iii) $\mathcal{C}(E) = \mathcal{S}(E) = \mathcal{B}(E)$.

For instance, we have shown that (i) holds for an Archimedean atomic lattice effect algebra E if and only if E is a subdirect product of finite chains, from which the existence of (o)-continuous states (probabilities) on E easily follows [35].

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