K Y B E R N E T I K A — V O L U M E *42* (2 0 0 6) , N U M B E R 1 , P A G E S 9 5 – 1 1 0

ROBUST CONTROLLER DESIGN FOR LINEAR POLYTOPIC SYSTEMS

Vojtech Veselý

The paper addresses the problem of the robust output feedback controller design with a guaranteed cost and parameter dependent Lyapunov function for linear continuous time polytopic systems. Two design methods based on improved robust stability conditions are proposed. Numerical examples are given to illustrate the effectiveness of the proposed methods. The obtained results are compared with other three design procedures.

Keywords: robust control, linear polytopic systems, output feedback, LMI approach

AMS Subject Classification: 93D15

1. INTRODUCTION

The field of robust control methods based on the small-gain-like robustness condition started two decades ago when the robust control paradigm was defined as an optimization problem. Only at the end of the eighties a practical solution to this problem was found. It is worth to mention some algebraic approaches which followed the seminal works of both linear interval systems stability analysis and stability analysis of convex polytopic uncertain systems. Description of uncertain systems using the convex polytope-type uncertainty has found its natural framework in the linear matrix inequality (LMI) formalism, Boyd et al. [4]. The LMI stability analysis of such systems is based upon the notion of quadratic stability. To reduce quadratic stability conservatism in the robust controller design procedure the parameter dependent Lyapunov function has been introduced, Apkarian et al. [1], de Oliveira et al. [14], Henrion et al. [10, 11], Peaucelle et al. [15], Dettori and Scherer [6], Bachelier et al. [2], Grman et al. [8], and others. In this paper, two new robust static output feedback controller design methods to stabilize continuous linear polytopic uncertain systems are presented that guarantee the parameter dependent quadratic stability (PDQS), the cost and in the first proposed design method that the closed-loop eigenvalues are situated in D-stability region, Henrion et al. [10].

In the spirit of convexifying approach (de Oliveira et al. [14]) a linearization algorithm has been used to solve the nonconvex design problem. The proposed procedure leads to an iterative LMI based algorithm.

The paper is organized as follows. In Section 2 the problem formulation and some preliminary results are brought. The main results are given in Section 3. In Section 4 the obtained theoretical results are applied and compared with other known three robust static output feedback controller design methods.

2. PROBLEM FORMULATION AND PRELIMINARIES

We shall consider the following affine linear time invariant continuous time uncertain systems

$$\begin{aligned}\dot{x}(t) &= A(\theta)\,x(t) + B(\theta)u(t) \\ y(t) &= C(\theta)\,x(t), \quad x(0) = x_0\end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the plant state; $u(t) \in R^m$ is the control input; $y(t) \in R^l$ is the output vector of the system; $A(\theta), B(\theta), C(\theta)$ are matrices of appropriate dimensions and

$$A(\theta) = A_0 + A_1\theta_1 + \cdots + A_p\theta_p$$

$$\begin{aligned}B(\theta) &= B_0 + B_1\theta_1 + \cdots + B_p\theta_p \\ C(\theta) &= C_0 + C_1\theta_1 + \cdots + C_p\theta_p\end{aligned} \tag{2}$$

where $\theta = [\theta_1 \cdots \theta_p] \in R^p$ is a vector of uncertain and possibly time varying real parameters. There are two particular cases of robust stability analysis problem.

- The uncertain parameter vector θ is a fixed but unknown element of a given parameter set.
- The uncertain parameter θ is a time varying function $\theta : R \rightarrow R^p$ which belongs to some set defined in R^p . Equation (1) is then to be interpreted in the sense of a time variant system.

The first case typically appears in models in which the physical parameters are fixed but only approximately known up to some accuracy. For these uncertain parameters (1) defines a linear time invariant system.

Note that, in order to keep the polytope affine property, either matrix $B(\theta)$ or $C(\theta)$ must be precisely known. In the following we assume that $C(\theta)$ is known and equal to matrix C. The following performance index is associated with system (1)

$$J = \int_0^\infty (x(t)^T Q x(t) + u(t)^T R u(t))\,\mathrm{d}t \tag{3}$$

where $Q = Q^T \geq 0,\ R = R^T > 0$ are matrices of compatible dimensions.

The problem studied in this paper can be formulated as follows: For a continuous time invariant system described by (1) design a static output feedback controller with the gain matrix F and control algorithm

$$u(t) = Fy(t) = FCx(t) \tag{4}$$

so that the closed loop system

$$\dot{x} = (A(\theta) + B(\theta)FC)\,x(t) = A_c(\theta)\,x(t) \tag{5}$$

is PDQS with a guaranteed cost.

Definition 1. Consider the system (1). If there exists a control law u^* and a positive scalar J^* such that the closed loop system (5) is stable and the closed loop value cost function (3) satisfies $J \leq J^*$, then J^* is said to be the guaranteed cost and u^* is said to be the guaranteed cost control law for system (1).

The system represented by (5) is a polytope of linear affine systems which can be described by a list of its vertices

$$\dot{x}(t) = A_{ci}x(t), \quad i = 1, 2, \ldots, N \tag{6}$$

where $N = 2^p$ and

$$A_{ci} = A_{vi} + B_{vi}FC$$

where A_{vi} and B_{vi} are vertices corresponding to (2). The linear uncertain system described by (6) belongs to a convex polytopic set

$$\dot{x}(t) = A_c(\alpha)\, x(t) \tag{7}$$

where

$$S = \left\{ A_c(\alpha) : A_c(\alpha) = \sum_{i=1}^{N} A_{ci}\alpha_i \sum_{i=1}^{N} \alpha_i = 1 \quad \alpha_i \geq 0 \right\}. \tag{8}$$

Using the concept of Lyapunov stability it is possible to formulate the following definition and lemmas.

Definition 2. System (7) is robustly stable in the uncertainty box (8) if and only if there exists a matrix $P(\alpha) = P(\alpha)^T > 0$ such that

$$A_c^T(\alpha)P(\alpha) + P(\alpha)A_c(\alpha) < 0 \tag{9}$$

for all α such that $A_c(\alpha) \in S$.

According to de Oliveira et al. [14] there is no general and systematic way to formally determine $P(\alpha)$ as a function of $A_c(\alpha)$. Such a matrix $P(\alpha)$ is called the parameter dependent Lyapunov matrix (PDLM) and for a particular structure of $P(\alpha)$ inequality (9) defines the parameter dependent quadratic stability (PDQS). A new formal approach to determine $P(\alpha)$ for real convex polytopic uncertainty can be found in the references, Apkarian et al. [1], Bachelier et al. [2], de Oliveira et al. [14, 13], Peaucelle et al. [15], Dettori and Scherer [6], Henrion et al. [10], Takahashi et al. [18], Rosinová and Veselý [16], Grman et al. [8]. The following LMI based robust stability conditions for a linear uncertain polytopic system are considered.

Lemma 1. (Boyd et al. [4]) Uncertain system (7) is quadratically stable in the uncertain box (8) if and only if there exists a matrix $P(\alpha) = P > 0$ such that

$$A_{ci}^T P + PA_{ci} < 0, \quad i = 1, 2, \ldots, N. \tag{10}$$

Unfortunately this approach generally provides quite conservative results. To reduce the conservatism when (1) is affine in θ and system parameters are time-invariant, a PDLM $P(\theta)$ has been introduced, Gahinet et al. [7]

$$P(\theta) = P_0 + P_1\theta_1 + \cdots + P_p\theta_p > 0. \tag{11}$$

Affine quadratic stability encompasses quadratic stability. Recently the following PDLM has been considered

$$P(\alpha) = \sum_{i=1}^{N} P_i\alpha_i \quad \sum_{i=1}^{N} \alpha_i = 1 \tag{12}$$

$$P_i = P_i^T > 0, \quad i = 1, 2, \dots, N$$

which has to be positive definite for all values of α such that $A_c(\alpha) \in S$.

Lemma 2. (Takahashi et al. [18], Veselý [20]) The continuous-time system (7) with PDLM (12) is PDQS if there exists a positive definite matrix P_i

$$A_{ci}^T P_i + P_i A_{ci} < -M, \quad i = 1, 2, \dots, N \tag{13}$$

$$A_{ck}^T P_j + P_j A_{ck} + A_{cj}^T P_k + P_k A_{cj} < \frac{2}{N-1} M$$

$$k = 1, 2, \dots, N-1, \quad j = k+1, \dots, N$$

where $M = M^T > 0$ is some positive definite matrix.

In the original version of Lemma 2, Takahashi et al. [18] $M = I$.

Lemma 3. (Henrion et al. [10], de Oliveira et al. [13]) The continuous time system (7) with PDLM (12) is PDQS if there exists a matrix E and matrices $P_i > 0$ satisfying LMI

$$\begin{bmatrix} E^T A_{ci} + A_{ci}^T E^T - r_{11} P_i & * \\ -A_{ci} - E - r_{12}^* P_i & 2I - r_{22} P_i, \quad i = 1, 2, \dots, N \end{bmatrix} > 0 \tag{14}$$

within a stability region in the complex plane defined as

$$D = \left\{ s \in C : \begin{bmatrix} 1 \\ s \end{bmatrix}^* \begin{bmatrix} r_{11} & r_{12} \\ r_{12}^* & r_{22} \end{bmatrix} \begin{bmatrix} 1 \\ s \end{bmatrix} < 0 \right\} \tag{15}$$

where the asterisk denotes the transpose conjugate.

Stability within region D with PDLM is denoted as D–PDQS. Standard choices for D are the left half-plane ($r_{11} = 0, r_{12} = 1, r_{22} = 0$) or the unit circle ($r_{11} = -1, r_{12} = 0, r_{22} = 1$) for discrete-time systems.

Lemma 4. (Peaucelle et al. [15], Dettori and Scherer [6]) The continuous-time system (7) with PDLM (12) is D–PDQS if there exist matrices H, G and N positive definite matrices P_i such that

$$\begin{bmatrix} r_{11}P_i + HA_{ci} + A_{ci}^T H^T & * \\ (r_{12}P_i + A_{ci}^T G - H)^T & r_{22}P_i - G - G^T \end{bmatrix} < 0, \quad i = 1, 2, \dots, N. \tag{16}$$

Lemma 5. (Grman et al. [8]) The continuous-time system (7) with PDLM (12) is PDQS if there exist positive definite matrices P_i such that the following LMI are satisfied

$$A_{ci}^T P_i + P_i A_{ci} < -v_{ii} I, \quad i = 1, 2, \dots, N \tag{17}$$

$$A_{cj}^T P_k + P_k A_{cj} + A_{ck}^T P_j + P_j A_{ck} < 2v_{jk} I$$

$$j = 1, 2, \dots, N-1, \quad k = j+1, \dots, N$$

where $v_{ii} > 0,\ v_{ij} = v_{ji} \geq 0$ for all $i \neq j$ and $V = \{v_{ij}\}$ is positive definite.

Finally we introduce the well known results from LQ theory [12].

Lemma 6. Consider the continuous-time system (7) with control algorithm (4).The control algorithm (4) is the guaranteed cost control law for system (7) if and only if the following condition holds.

$$A_c(\alpha)^T P(\alpha) + P(\alpha) A_c(\alpha) + Q + C^T F^T RFC < 0. \tag{18}$$

Note that if in Lemma 3 or Lemma 4 one substitutes $P_i = P_j = P = P^T > 0$, the new quadratic stability conditions are obtained with less conservative results than those given by Lemma 1.

3. ROBUST CONTROLLER DESIGN

In this paragraph we present two new procedures to design a static output feedback controller for polytopic continuous-time linear system (7) with control law (4) which ensure the guaranteed cost using PDQS based on Lemma 4 and Lemma 5. Because of term $A_{ci}^T G$ (16), immediate application of Lemma 4 to design a robust controller is strenuous. One main result that enables to avoid this problem is summarized in the following theorem.

Theorem 1. Consider the continuous-time system (7) with guaranteed cost control law (4) and PDLM (12), then the following statements are equivalent:

- The control algorithm (4) is the guaranteed cost control law for the system (7) and PDLM (12).

- There exists PDLM $P(\alpha) > 0$ (12) and matrices G, H and F such that the following condition holds:

$$\begin{bmatrix} S_{11}(\alpha) & S_{12}(\alpha) \\ (S_{12}(\alpha))^T & r_{22}P(\alpha) - G - G^T \end{bmatrix} < 0$$

$$S_{11}(\alpha) = r_{11}P(\alpha) + HA_c(\alpha) + A_c(\alpha)^T H^T + Q + C^T F^T RFC \qquad (19)$$
$$S_{12}(\alpha) = r_{12}P(\alpha) + A_c(\alpha)^T G - H.$$

- There exists PDLM $P(\alpha) = \sum_{i=1}^{N} P_i\alpha_i > 0$ (12) and matrices F, H such that for the ith vertex, $i = 1, 2, \dots, N$ the following two conditions hold:

$$r_{12}(A_{ci}^T P_i + P_i A_{ci}) + r_{11}P_i + r_{22}A_{ci}^T P_i A_{ci} + Q + C^T F^T RFC < 0$$
$$r_{11}P_i + A_{ci}^T H^T + HA_{ci} + Q + C^T F^T RFC < 0 \qquad (20)$$

$i = 1, 2, \dots, N.$

Proof. Consider inequality (16) in the form of (19). Since the matrix $[I \quad A_c(\alpha)]$ has a full rank, (19) implies that

$$\begin{aligned} &[I \quad A_c(\alpha)^T]\{\text{LHS (eq. (19))}\}([I \quad A_c(\alpha)^T])^T \\ &= r_{11}P(\alpha) + (P(\alpha)A_c(\alpha)r_{12})^T + P(\alpha)A_c(\alpha)r_{12} \\ &+ r_{22}A_c(\alpha)^T P(\alpha)A_c(\alpha) + Q + C^T F^T RFC < 0 \end{aligned} \qquad (21)$$

which for the continuous-time system ($r_{11} = r_{22} = 0$, $r_{12} = 1$) is exactly the expression (18). Proof of necessity is done in a similar way to (Peaucelle and et al. [15]). Equation (19) implies the necessary and sufficient condition for the guaranteed cost control law (18). The next step is to prove that (19) implies inequalities (20). Because of linearity for the ith vertex of continuous-time system (7) and PDLM (12) condition (19) reads as

$$\begin{bmatrix} r_{11}P_i + HA_{ci} + A_{ci}^T H^T + Q + C^T F^T RFC & * \\ r_{12}P_i - H^T & r_{22}P_i \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & A_{ci}^T G \\ G^T A_{ci} & -(G + G^T) \end{bmatrix}, \quad i = 1, 2, \dots, N. \qquad (22)$$

Applying projection lemma (Skelton et al. [17]) to eliminate G from (22) we obtain inequalities (20), which completes the proof. □

It follows from the preceding theorem that for PDLM given by (12), although the first two above statements are equivalent, the stability and guaranteed cost control of the closed loop system (7), (4) are provided only with sufficient conditions. In the above theorem the first two statements are equivalent for any PDLM. Note that the first inequality in (20) represents the necessary and sufficient condition for the guaranteed cost control law for individual vertices while the second one "binds" the whole polytopic system through common matrix H. Obviously, condition (20) includes quadratic stability as a special case with $P_i = P = H^T = H$, $i = 1, 2, \ldots, N$. In general there are no requirements on matrix H, therefore (20) is less conservative than quadratic stability.

The second procedure of the robust controller design is based on Lemma 5. The results are summarized in the following theorem.

Theorem 2. Consider the continuous-time system (7) with control law (4). The control algorithm (4) is a guaranteed control law if the following two conditions hold:

$$A_{ci}^T P_i + P_i A_{ci} + Q + C^T F^T R_1 F C \leq -v_{ii} I$$

$$A_{cj}^T P_k + P_k A_{cj} + A_{ck}^T P_j + P_j A_{ck} + C^T F^T R_2 F C \leq 2 v_{jk} I \tag{23}$$

matrix $V = \{v_{jk}\}_{N \times N}$ is symmetric positive definite and

$$R_1 = R\frac{N+1}{2N} \quad R_2 = \frac{R}{2}.$$

P r o o f. Inequality (18) can be rewritten using (7), (12) as follows

$$\left(\sum_{i=1}^{N} A_{ci}\alpha_i\right)^T \sum_{i=1}^{N} P_i\alpha_i + \sum_{i=1}^{N} P_i\alpha_i$$

$$\sum_{i=1}^{N} A_{ci}\alpha_i + Q + C^T F^T R F C < 0$$

or equivalently

$$\sum_{i=1}^{N} N_{ii}\alpha_i^2 + \sum_{j=1}^{N-1} \sum_{k=j+1}^{N} 2\alpha_j\alpha_k N_{jk} + Q + C^T F^T R F C < 0 \tag{24}$$

where

$$N_{ii} = A_{ci}^T P_i + P_i A_{ci}, \quad i = 1, 2, \ldots, N$$

$$N_{jk} = \frac{1}{2}(A_{cj}^T P_k + P_k A_{cj} + A_{ck}^T P_j + P_j A_{ck})$$

$$j = 1, 2, \ldots, N-1, \quad k = j+1, \ldots, N$$

because

$$\sum_{i=1}^{N} \alpha_i^2 \in \langle 1/N, 1\rangle, \quad \sum_{j=1}^{N-1} \sum_{k=j+1}^{N} 2\alpha_j\alpha_k \leq \frac{N-1}{N}$$

and R can be splitted as

$$R = R_1 + \frac{N-1}{N} R_2.$$

Let $R_1 = R\frac{N+1}{2N}$, then for R_2 one obtains $R_2 = \frac{R}{2}$. Inequality (24) can be rewritten as follows

$$\sum_{i=1}^{N} \alpha_i^2 (N_{ii} + Q + C^T F^T R_1 FC) + \sum_{j=1}^{N-1} \sum_{k=j+1}^{N} 2\alpha_j \alpha_k (N_{jk} + C^T F^T R_2 FC) < 0 \quad (25)$$

which can be splitted into inequalities (23). □

Consider the first inequality of (20). If one substitutes $A_{ci} = A_{vi} + B_{vi}FC$ to (20), after some small manipulations one obtains

$$A_{vi}^T P_i + P_i A_{vi} + Q + G_i^T R G_i - P_i B_{vi} R^{-1} B_{vi}^T P_i < 0 \quad (26)$$

where

$$G_i = FC + R^{-1} B_{vi}^T P_i.$$

Matrix inequality (26) is nonconvex. There are the following possible approaches to solve this nonconvex problem:

- linearization of a nonconvex term, Yong–Yan Cao and Yon–Xian Sun [21], de Oliveira et al. [14], Han and Skelton [9], Rosinová and Veselý [16], Veselý [20],
- using the two-step algorithm proposed in Veselý [19], and
- convexifying algorithm (de Oliveira et al. [14].

The linearization algorithm is less conservative than the two-step algorithm. Note that with the linearization algorithm only local optimality is guaranteed. It should be mentioned that the linearization is a convexifying algorithm, in the spirit of de Oliveira et al. [14]. A convexifying algorithm must find a convexifying potential function. There might exist many canditates for convexifying potential functions for a given nonconvex matrix inequality. Finding a nice convexifying function is generally difficult. Linearization approach may provide such a nice convexifying potential function.

Lemma 7. (Han and Skelton [9]) Let a matrix $W \in R^{n\times n} > 0$ be given. Then the following statements are true.

i) The linearization of $X^{-1} \in R^{n\times n}$ around value $X_k > 0$ is

$$\text{lin}(X^{-1}, X_k) = X_k^{-1} - X_k^{-1}(X - X_k)X_k^{-1}. \quad (27)$$

ii) The linearization of $XWX \in R^{n\times n}$ around value of $X_k > 0$ is

$$\text{lin}(XWX, X_k) = -X_k W X_k + XWX_k + X_k WX \quad (28)$$

where $\text{lin}(\cdot, X_k)$ is the linearization operator at given point X_k.

It is easy to show that

$$-XWX \leq -\text{lin}(XWX, X_k)$$

$$X^{-1} - X_k^{-1} + X_k^{-1}(X - X_k)X_k^{-1} \geq 0. \tag{29}$$

On the basis of Lemma 7 the results of Theorem 1 and Theorem 2 can be reformulated for continuous-time system in the following LMI conditions, respectively

$$\begin{bmatrix} A_{vi}^T P_i + P_i A_{vi} + Q + \text{lin}_i 1 & G_i^T \\ G_i & -R^{-1} \end{bmatrix} < 0$$

$$\text{lin}_i 1 = -\text{lin}(P_i B_{vi} R^{-1} B_{vi}^T P_i) \tag{30}$$

$$\begin{bmatrix} A_{vi}^T H^T + H A_{vi} + Q + \text{lin}_i 2 & M_i^T \\ M_i & -R^{-1} \end{bmatrix} < 0$$

$$M_i = FC + R^{-1} B_{vi}^T H^T, \quad i = 1, 2, \ldots, N \tag{31}$$

$$\text{lin}_i 2 = -\text{lin}(H B_{vi} R^{-1} B_{vi}^T H^T)$$

$$P_i < r_0 I, \quad i = 1, 2, \ldots, N$$

and

$$\begin{bmatrix} A_{vi}^T P_i + P_i A_{vi} + Q + \text{lin}_i 3 + v_{ii} I & G_{vi}^T \\ G_{vi} & -R_1^{-1} \end{bmatrix} < 0$$

$$\text{lin}_i 3 = -\text{lin}(P_i B_{vi} R_1^{-1} B_{vi}^T P_i), \quad i = 1, 2, \ldots, N \tag{32}$$

$$\begin{bmatrix} LJ_{ik} + \text{lin}_i 4 + 2v_{ik} I & * & * \\ G_{ik} & -R_2^{-1} & 0 \\ G_{ki} & 0 & -R_2^{-1} \end{bmatrix} < 0$$

$$\text{lin}_i 4 = -\text{lin}(P_k B_{vi} R_2^{-1} B_{vi}^T P_k) - \text{lin}(P_i B_{vk} R_2^{-1} B_{vk}^T P_i) \tag{33}$$

$$i = 1, 2, \ldots, N-1, \quad k = i+1, \ldots, N$$

$$P_i < r_0 I, \quad i = 1, 2, \ldots, N$$

where

$$G_{vi} = FC + R_1^{-1} B_{vi}^T P_i$$

$$G_{ik} = FC + R_2^{-1} B_{vi}^T P_k \quad P_i \leq r_o I$$

$$G_{ki} = FC + R_2^{-1} B_{vk}^T P_i \quad r_o > 0$$

$$LJ_{ik} = A_{vi}^T P_k + P_k A_{vi} + A_{vk}^T P_i + P_i A_{vk}.$$

If the LMI solution of (30), (31) or (32), (33) are feasible with respect to matrices F, H, and P_i, $i = 1, 2, \ldots, N$ or F, P_i and matrix $V = \{v_{ij}\}_{N \times N}$ respectively, then the uncertain polytopic system (1) is parameter dependent quadratically stable with a guaranteed cost control algorithm

$$u = FCx = Fy \tag{34}$$

and

$$J^* = \max_i x_o^T P_i x_o$$

is the guaranteed cost for uncertain closed loop system.

4. NUMERICAL EXAMPLES

In this paragraph we present the results of numerical calculations of four examples to design a static output feedback controller with a guaranteed cost. Two designed methods proposed in this paper (PDLM1, PDLM2) have been compared with the following three methods

- parameter dependent quadratic stability (PDLM3) based on Lemma 3,
- improved quadratic stability ($P_i = P_j = P$) (IQS) based on Lemma 3,
- quadratic stability, Lemma 1 (QS).

Example 1. Example 1 has been borrowed from Benton and Smith [3] to demonstrate the use of the algorithm given by (30)–(33) on the problem of robustly stabilizing with a guaranteed cost a vertical take off and landing of a helicopter. It is known that the presented system is static output feedback stabilizable. Let matrices $(A(\theta), B(\theta), C)$ in (1) be defined as

$$A(\theta) = \begin{bmatrix} -0.036 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.010 & 0.0024 & -4.0208 \\ 0.1002 & q_1(t) & -0.707 & q_2(t) \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B(\theta) = \begin{bmatrix} 0.4422 & 0.1761 \\ q_3(t) & -7.59222 \\ -5.520 & 4.490 \\ 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$$

with parameters bounds $-0.6319 \leq q_1(t) \leq 1.3681, 1.22 \leq q_2(t) \leq 1.420$, and $2.7446 \leq q_3(t) \leq 4.3446$ for all time. The above model has been recalculated to the form (2) with $-1 \leq \theta_i \leq 1,\ i = 1,2,3$, and the following model has been obtained:

- $A_0 = A(\theta)$ with $A_0(3,2) = .3681$ and $A_0(3,4) = 1.32$; $B_0 = B(\theta)$ with $B_0(2,1) = 3.5446$,
- $A_1 = 0$ with $A_1(3,2) = 1,\ B_1 = 0$,
- $A_2 = 0$ with $A_2(3,4) = 0.1,\ B_2 = 0$,
- $A_3 = 0,\ B_3 = 0$ with $B_3(2,1) = 0.8$.

The respective eight vertices are calculated. The results of calculation are summarized in Table 1.

$r_0 = 75,\ R = rI,\ r = 1,\ Q = qI,\ q = .0001$

Table 1. The results of calculation for Example 1.

Methods	θ_m	maxEig
PDLM1	5.8	−.0623
PDLM2	6.4	−.0679
PDLM3	no	no
IQS	no	no
QS	no	no

In Table 1 and the next tables θ_m is the absolute value of the maximum uncertainty level while maintaining closed-loop stability and maxEig is the maximum closed-loop eigenvalue when $|\theta_1| = |\theta_2| = \theta_3 = 1$. The gain matrices F for the first and second methods and $|\theta| = 1$ are

PDLM1: $F^T = [-.793 \quad 1.2671]$ and PDLM2: $F^T = [-.9368 \quad 2.2113]$.

In the first example the methods PDLM3, IQS and QS do not give feasible solutions.

Example 2. The second example has been borrowed from Benton and Smith [3]. It concerns the design of a robust controller with a guaranteed cost for stabilizing the lateral axis dynamics for an aircraft L–1011. Let matrices $(A(\theta), B(\theta), C)$ be defined as

$$A(\theta) = \begin{bmatrix} -2.98 & q_1(t) & 0 & -0.0340 \\ -.9900 & -.2100 & 0.0350 & -0.0011 \\ 0 & 0 & 0 & 1 \\ .3900 & -5.555 & 0 & -1.890 \end{bmatrix}$$

$$B(\theta) = \begin{bmatrix} -.0320 & 0 & 0 & -1.600 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with parameter bound $-0.5700 \le q_1(t) \le 2.4300$ for all time. The above model has been recalculated for the case of $-1 \le \theta \le 1$ and $A_0 = A(\theta)$ with of $A_0(1,2) = .930$ and $A_1 = 0$ with $A_1(1,2) = 1.50$. Input matrices are $B_0 = B(\theta)$ and $B_1 = 0$. Two vertices are calculated. The results of calculations are summarized in Table 2.

$r_0 = 100,\ r = 1,\ q = 0.1$

Table 2. The results of calculation for Example 2.

Methods	θ_m	maxEig
PDLM1	1.3	−.2203
PDLM2	1.3	−.1685
PDLM3	1.1	–0.0966
IQS	1.1	–0.1046
QS	no	no

The gain matrices F for the first and second approaches and $|\theta| = 1$ are:

PDLM1: $F = [1.4075 \quad 2.6673]$ PDLM2: $F = [1.3879 \quad 3.8679]$.

For the second example the quadratic stability design method gives no feasible solution.

Example 3. In the third example the design techniques developed in this paper are applied to a realistic missile example, Chilali et al. [5]. The purpose is to determine the maximum admissible uncertainty level for which stability of the closed-loop system with guaranteed cost is preserved. The dynamics of the controlled missile roll axis is described by the following matrices.

$$A_0 = \begin{bmatrix} -180.0 & 0 & 0 & 0 & 0 \\ 0 & -180.0 & 0 & 0 & 0 \\ -21.23 & 0 & -.6888 & -14.7 & 0 \\ 256.7 & 0 & 122.6 & -1.793 & 0 \\ -52.33 & 304.7 & 0 & 36.7 & -9.661 \end{bmatrix}$$

$$B_0^T = \begin{bmatrix} 180 & 0 & 0 & 256.7 & 0 \\ 0 & 180 & 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 27 & 0 & 0 & 0 & 0 \\ 0 & 27.0 & 0 & 0 & 0 \\ 21.2 & 0 & .6888 & 14.96 & 0 \\ -38.6 & 0 & 122.6 & 0 & 0 \\ 52.4 & 304.8 & 0 & 36.8 & 9.66 \end{bmatrix}$$

$$B_2^T = \begin{bmatrix} 40.5 & 0 & 0 & 57.9 & 0 \\ 0 & 40.5 & 0 & 0 & 0 \end{bmatrix}$$

$A_2 = 0 \quad B_1 = 0$. The four vertices are calculated. The results of calculation are summarized in Table 3.

$r_0 = 100, \; r = 1, \; q = 0.005$

Table 3. The results of calculation for Example 3.

Methods	θ_m	maxEig*
PDLM1	.95	−20.9299
PDLM2	.95	−14.4537
PDLM3	.15	−11.2437
IQS	.05	−10.4183
QS	no	no

where maxEig* is the maximum eigenvalue of the closed-loop system for $|\theta| = .05$. The quadratic stability method does not give feasible solutions. The static output feedback gain matrices for $|\theta| = .05$ are

$$\text{PDLM1: } F = \begin{bmatrix} 1.0714 & -0.6920 & .1223 \\ -.3890 & .0416 & -.3921 \end{bmatrix}$$

$$\text{PDLM2: } F = \begin{bmatrix} .5665 & -.5887 & -.0008 \\ -.2899 & -.0298 & -.987 \end{bmatrix}.$$

The maximum uncertainty level while maintaining closed-loop stability with guaranteed cost is equal to $|\theta| = .95$.

Example 4. In this example we consider the linear model of two cooperating DC motors. The problem is to design two PI controllers for a laboratory MIMO system which will guarantee PDQS of the closed-loop uncertain system. The system model is given by (1) and (2) with a time invariant matrix affine type uncertain structure, where

$$A_0 = \begin{bmatrix} 0 & -.2148 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1.014 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -.2605 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -.9107 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -.1639 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -.8137 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -.2279 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -.8251 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 0 & -.025 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -.1395 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -.0938 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -.2911 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .0188 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .0208 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -.0333 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -.1173 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & .0125 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .0594 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .0116 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .0308 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -.0188 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -.0156 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & .0208 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -.0333 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_0 = \begin{bmatrix} .3148 & 0 \\ .0478 & 0 \\ 0 & -.1028 \\ 0 & -.0091 \\ -.0841 & 0 \\ -.0287 & 0 \\ 0 & .3676 \\ 0 & .2448 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad B_1 = \begin{bmatrix} .0625 & 0 \\ -.0798 & 0 \\ 0 & -.0462 \\ 0 & -.0449 \\ .0016 & 0 \\ .0072 & 0 \\ 0 & .077 \\ 0 & -.005 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} -.0094 & 0 \\ .0151 & 0 \\ 0 & .0019 \\ 0 & -.003 \\ -.0121 & 0 \\ -.03 & 0 \\ 0 & -.064 \\ 0 & .0189 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad C^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The number of vertices of polytope systems is equal to 4 and the polytope vertices are computed for two variables θ_1, θ_2 alternatively taken at their maximum $\overline{\theta_i} = 1$ and minimum $\underline{\theta_i} = -1, i = 1, 2$. The decentralized control structure for the two PI controllers can be obtained by the choice of the static output feedback gain matrix F structure. It is given as follows:

$$F = \begin{bmatrix} f_{11} & 0 & f_{13} & 0 \\ 0 & f_{22} & 0 & f_{24} \end{bmatrix}.$$

The results of calculation of a static output feedback gain matrix F are summarized in Table 4.

$r_0 = 75,\ r = 1,\ q = 0.0001$

Table 4. The results of calculation for Example 4.

Methods	θ_m	maxEig
PDLM1	2.3	−.1915
PDLM2	2.3	−.1909
PDLM3	2.1	−.132
IQS	2	−.1141
QS	1.3	−.0824

The static output feedback gain matrices F for $|\theta| = 1$ are as follows

$$\text{PDLM1:}\ F = \begin{bmatrix} -3.4754 & 0 & -.8879 & 0 \\ 0 & -6.0434 & 0 & -1.9952 \end{bmatrix}$$

$$\text{PDLM2:}\ F = \begin{bmatrix} -4.5491 & 0 & -.9967 & 0 \\ 0 & -8.8005 & 0 & -3.6587 \end{bmatrix}.$$

5. CONCLUSION

In this paper, we have proposed two new procedures for the robust output feedback controller design for a polytopic uncertain system with a parameter dependent Lyapunov function. The feasible solutions of the output feedback controller design provide sufficient conditions guaranteeing the PDQS with a guaranteed cost. In the first proposed design method, a feasible solution provides both the guaranteed cost and prescribed D-stability region in the complex plane. The second proposed design method is based on a new robust stability analysis of polytopic systems. The examples show the effectiveness of the proposed methods.

ACKNOWLEDGEMENT

The work has been supported by Grant No. 1/3841/06 of the Slovak Scientific Grant Agency. The author would like to thank the anonymous reviewers for their valuable comments and suggestions which improved this paper.

(Received February 7, 2005.)

REFERENCES

[1] P. Apkarian, E. Feron, and P. Gahinet: Parameter-dependent Lyapunov functions for robust control of systems with real parametric uncertainty. In: Proc. 3rd EEC, Rome 1995, pp. 2275–2280.

[2] O. Bachelier, J. Bernussou, M. C. de Oliveira, and J. C. Geromel: Parameter dependent Lyapunov control design: Numerical evaluation. In: Proc. 38th Conference on DC, Phoenix 1999, Arizona, pp. 293–297.

[3] R. J. Benton Jr. and D. Smith: A non iterative LMI based algorithm for robust static output feedback stabilization. Internat. J. Control *72* (1999), 14, 1322–1330.

[4] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan: Linear Matrix Inequalities in System and Control Theory. SIAM, Philadelphia 1994.

[5] M. Chilali, P. Gahinet, and P. Apkarian: Robust pole placement in LMI region. IEEE Trans. Automat. Control *44* (1999), 12, 2257–2267.

[6] M. Dettori and W. C. Scherer: New robust stability and performance conditions based on parameter dependent multiplier. In: Proc. 39th IEEE Conference on Decision and Control, Sydney 2000, CD-ROM.

[7] P. Gahinet, P. Apkarian, and M. Chilali: Affine parameter-dependent Lyapunov functions and real parametric uncertainty. IEEE Trans. Circuits and Systems *41* (1996), 3, 436–442.

[8] L. Grman, D. Rosinová, A. Kozáková, and V. Veselý: Robust stability conditions for polytopic systems. Internat. J. System Sci. *36* (2005), 15, 961–973.

[9] J. Han and R. E. Skelton: An LMI optimization approach for structured linear controllers. In: Proc. 42th IEEE Conference on Decision and Control, Maui 2003, pp. 5143–5148.

[10] D. Henrion, D. Alzelier, and D. Peaucelle: Positive polynomial matrices and improved robustness conditions. In: 15th Triennial World Congress of the International Federation of Automatic Control, Barcelona 2002, CD-ROM.

[11] D. Henrion, Kenji Sagimoto, and M. Šebek: Rank-one LMI approach to robust stability of polynomial matrices. Kybernetika *38* (2002), 5, 643–656.

[12] V. Kučera and C. E. De Souza: A necessary and sufficient conditions for output feedback stabilizability. Automatica *31* (1995), 9, 1357–1359.

[13] M. C. de Oliveira, J. Bernussou, and J. C. Geromel: A new discrete–time robust stability condition. Systems Control Lett. *37* (1999), 261–265.

[14] M. C. de Oliveira, J. F. Camino, and R. E. Skelton: A convexifying algorithm for the design of structured linear controllers. In: Proc. 39th IEEE Conference on Decision and Control, Sydney 2000, pp. 2781–2786.

[15] D. Peaucelle, D. Arzelier, O. Bachelier, J. Bernussou: A new robust D-stability condition for real convex polytopic uncertainty. Systems Control Lett. *40* (2000), 21–30.

[16] D. Rosinová and V. Veselý: Robust output feedback design of linear discrete–time systems: LMI Approach. In: 4th IFAC Symposium on Robust Control Design, Milano 2003, CD-ROM.

[17] R. E. Skelton, T. Iwasaki, and K. Grigoriadis: A Unified Algebraic Approach to Linear Control Design. Taylor and Francis, London 1998.

[18] R. H. C. Takahashi, D. C. W. Ramos, and P. L. D. Peres: Robust control synthesis via a genetic algorithm and LMI's. In: 15th Triennial World Congress of the International Federation of Automatic Control, Barcelona 2002, CD-ROM.

[19] V. Veselý: Static output feedback controller design. Kybernetika *37* (2001), 2, 205–221.

[20] V. Veselý: Robust output feedback control synthesis: LMI approach. In: 3rd IFAC Conference on Control System Design, Bratislava 2003, CD-ROM.

[21] Yong Yan Cao and You Xian Sun: Static Output feedback simultaneous stabilization. LMI approach. Internat. J. Control *70* (1998), 5, 803–814.

Vojtech Veselý, Department of Automatic Control Systems, Faculty of Electrical Engineering and Information Technology, Slovak University of Technology, Ilkovičova 2, 812 19 Bratislava. Slovak Republic.

e-mail: vojtech.vesely@stuba.sk