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NOTES ON FREE LUNCH IN THE LIMIT AND PRICING BY CONJUGATE DUALITY THEORY

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King and Korf [9] introduced, in the framework of a discrete-time dynamic market model on a general probability space, a new concept of arbitrage called *free lunch in the limit* which is slightly weaker than the common free lunch. The definition was motivated by the attempt at proposing the pricing theory based on the theory of conjugate duality in optimization. We show that this concept of arbitrage fails to have a basic property of other common concepts used in pricing theory – it depends on the underlying probability measure more than through its null sets. However, we show that the interesting pricing results obtained by conjugate duality are still valid if it is only assumed that the market admits no free lunch rather than no free lunch in the limit.

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1. INTRODUCTION

A basic result in mathematical finance, called the *fundamental theorem of asset pricing*, states that a financial market admits no arbitrage opportunity if and only if there is an equivalent probability measure for which the price process is a martingale. An *arbitrage opportunity* is a possibility of making a profit on some transaction without the risk of incurring a loss. If such an opportunity existed, then everybody would invest with this trading strategy, affecting the prices of the assets, and this economic model would not be in equilibrium. Therefore the sensible market models must admit no arbitrage opportunities.

The equivalence of no arbitrage with the existence of an equivalent martingale measure is at the basis of the entire theory of 'pricing by arbitrage'. The aim of the theory of arbitrage-free pricing is to assess, in arbitrage-free market, each contingent claim with an initial *fair price* such that the market model augmented by this contingent claim as a new possible investment still admits no risk-free profit. In particular, the problem of fair pricing can then be reduced to taking expected values with respect to the equivalent martingale measures. However, in the case of unattainable contingent claim, the fair price is not determined uniquely but rather as an element of an *arbitrage interval* of all fair prices formed by expected values with respect to all equivalent martingale measures.

The proof of these pricing results requires a separation theorem. In the simplest single-period market model with finitely many states, the mathematical problem in the proof is equivalent to Farkas' Lemma of the alternative and to the basic duality theorem of linear programming (see for instance [3, 12]). In the discrete time multi-period market model with finite horizon and finitely many states, the proof can be still handled with the linear programming duality (see [8, 10]).

Nevertheless, extending these results to more general models in which there are infinitely many states requires some sort of separation theorem for infinite-dimensional spaces. Recently, King and Korf [9] made an attempt to analyze the pricing problems in an infinite-dimensional multistage stochastic programming setting from the perspective of conjugate duality, following [13].

The duality scheme for multistage stochastic problems was introduced in reflexive Banach space setting $\mathcal{L}^p/\mathcal{L}^q$ by Eisner and Olsen [4]. However, the reflexivity restriction $1 < p < +\infty$ is quite artificial and a lot of the subsequent work on duality in multistage stochastic programming has been devoted to problems formulated in non-reflexive Banach space setting, mostly $\mathcal{L}^\infty/\mathcal{L}^1$ and $\mathcal{L}^\infty/(\mathcal{L}^\infty)^*$. Rockafellar and Wets in [15, 16, 17, 14] showed the significance of *relatively complete recourse* condition in obtaining the strong duality results of the form inf (P) = max (D) in $\mathcal{L}^\infty/\mathcal{L}^1$ setting. If the primal problem fails to have the relatively complete recourse, the dual multipliers are to be expected as elements of $(\mathcal{L}^\infty)^*$ rather than $\mathcal{L}^1$.

Following these duality results in $\mathcal{L}^\infty/(\mathcal{L}^\infty)^*$ setting, King and Korf [9] derived the formula for the *writer's price* of an unattainable contingent claim (i. e., the upper bound for all possible fair prices). They showed that the writer's price is actually attained as the expected value of the contingent claim with respect to some martingale pricing measure if the martingale measures are permitted to be finitely-additive. This interesting pricing formula was derived under the assumption of arbitrage-free market where the arbitrage-free condition involved the newly proposed concept of arbitrage called the *free lunch in the limit*.

The purpose of this paper is to investigate the notion of free lunch in the limit and its role in the derivation of the pricing formula. In Section 2, we introduce the convenient market model specification (cf. [5]) and the preliminary results.

Section 3 is devoted to the study of the influence of the underlying probability measure on the free lunch in the limit concept. We show that, unlike other common concepts of arbitrage, the free lunch in the limit depends on the underlying probability measure more than through its null sets. We give an explicit example of a market model and two equivalent probability measures such that the market admits free lunch in the limit with respect to one probability measure and no free lunch in the limit with respect to the other.

In Section 4, we state a fundamental theorem of asset pricing for the common concept of free lunch in the framework of finitely-additive probability measures. For that purpose, an optimization problem (P) is introduced, the dual problem (D) is derived, and the duality result of the type $\inf(\mathsf{P}) = \max(\mathsf{D})$ is established. Finally, this leads to the desired theorem, that the market admits no free lunch if and only if there is an equivalent finitely-additive martingale measure for the price process.

Section 5 establishes the arbitrage-free pricing results for contingent claims. We present two optimization problems associated with each contingent claim: the writer's pricing problem (P_+) (motivated by the problem (P_{wp}) of King and Korf [9]) and the buyer's pricing problem (P_-). Again, the dual problems and duality results follow. Finally, the pricing results extending that of King and Korf [9] are stated.

2. MODEL SPECIFICATION AND PRELIMINARY RESULTS

The underlying *market* is a collection of $J+1$ traded assets indexed by $j = 0, \dots, J$ that are priced at times $t = 0, 1, \dots, T$. The *market price* of the jth asset at time t is modelled as a nonnegative random variable S_t^j on a given probability space $(\Omega, \mathcal{F}, P)$. The random vector $S_t = (S_t^0, \dots, S_t^J)^\top$ is assumed to be measurable with respect to a σ-algebra $\mathcal{F}_t \subset \mathcal{F}$. One should think of $\mathcal{F}_t$ as the class of all events which are observable up to time t. Therefore, we assume that the family $\{\mathcal{F}_t\}_{t=0}^T$ formes a filtration with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$, so that the initial prices at time $t = 0$ are known and are described by a nonnegative vector $S_0 = (S_0^0, \dots, S_0^J)^\top \in \mathbb{R}_+^{J+1}$ and the future prices at times $t = 1, \dots, T$ are described by nonnegative-valued $\mathcal{F}_t$-measurable random vectors $S_t = (S_t^0, \dots, S_t^J)^\top : \Omega \to \mathbb{R}_+^{J+1}$.

It is assumed that the zero asset is *risk-free* in the sense that its market price is always strictly positive ($S_t^0 > 0$ P-a. s., $t = 0, \dots, T$). This assumption allows us to use the zero asset as the *numéraire* and to form the new *discounted price* vectors $Z_t = S_t / S_t^0$. Note that the discounted value of the numéraire satisfies $Z_t^0 = 1$ P-a. s. for all $t = 0, \dots, T$. It is assumed that all other prices and cash flows have been similarly adjusted to reflect this normalization. Prices in the price vector Z_t are assumed to be $\mathcal{F}_t$-measurable and essentially bounded, i. e., $Z_t \in \mathcal{L}_+^\infty(\Omega, \mathcal{F}_t, P; \mathbb{R}^{J+1})$.

A *contingent claim* is a promise to pay $F_t : \Omega \to \mathbb{R}$ at each time $t = 1, \dots, T$. It is assumed that F_t is $\mathcal{F}_t$-measurable and essentially bounded.

An *investor* may hold a *portfolio* of assets $j = 0, \dots, J$, described by a vector $\theta_t = (\theta_t^0, \dots, \theta_t^J)^\top, t = 0, \dots, T$. The investor has some initial wealth to invest, and may change his or her portfolio at each time $t = 0, \dots, T$. The decision of the portfolio arrangement will depend on the market behaviour. A *trading strategy* describes all investment decisions based on all possible outcomes of the market. Therefore, $\theta = (\theta_0, \dots, \theta_T)$ describes a trading strategy, where at time $t = 0$, the market prices are known and θ_0 is described by a vector in $\mathbb{R}^{J+1}$. At times $t = 1, \dots, T$, the market prices are $\mathcal{F}_t$-measurable functions on Ω, so that $\theta_t : \Omega \to \mathbb{R}^{J+1}$ is also $\mathcal{F}_t$-measurable, and describes the portfolio during the trading period between times t and $t+1$. Thus, $Z_t^j \theta_t^j$ is the amount invested into the asset j at time t, while $Z_{t+1}^j \theta_t^j$ is the resulting value at time $t+1$. The total value of the portfolio θ_t at time t is $Z_t^\top \theta_t$, and by time $t+1$ the value of the portfolio has changed to $Z_{t+1}^\top \theta_t$. Note that θ_t is allowed to take on negative values, which corresponds to borrowing or selling short.

A *self-financing trading strategy* is one in which no new money except the initial investment is required or generated to create it. This is expressed by $Z_t^\top \theta_t = Z_t^\top \theta_{t-1}$ P-a. s. for all $t = 1, \dots, T$. It is convenient to adopt the notation $\Delta\theta_t = \theta_t - \theta_{t-1}$

and write $Z_t^\top \Delta\theta_t = 0$ P-a. s. Obviously, $\Delta\theta_t$ is $\mathcal{F}_t$-measurable.

The class of all possible trading strategies θ is limited to those which are essentially bounded. We denote it briefly by Θ (which stands for $\prod_{t=0}^{T} \mathcal{L}^\infty(\Omega, \mathcal{F}_t, P; \mathbb{R}^{J+1})$). This assumption is necessary in order to exploit duality results for multistage problems. Note that Θ is exactly the space of nonanticipative essentially bounded recourse functions.

For our purpose, we do not explicitly distinguish between an essentially bounded function and its equivalence class and we shall refer to the equivalence class elements of the Banach function space $\mathcal{L}^\infty$ as functions.

The duality results we develop are based on the properties of the space dual to $\mathcal{L}^\infty(\Omega, \mathcal{F}, P)$. The space $(\mathcal{L}^\infty)^*(\Omega, \mathcal{F}, P)$ is isometrically isomorphic to a space of bounded finitely-additive P-absolutely continuous signed measures on Ω and is essentially larger than $\mathcal{L}^1$ (see [7, Theorem 20.35]). We describe a very useful decomposition of $(\mathcal{L}^\infty)^*$ which can be viewed as an analogue to the Lebesgue decomposition for measures. For its proof see [1, Theorem VIII.5].

Every linear functional y in $(\mathcal{L}^\infty)^*(\Omega, \mathcal{F}, P)$ can be expressed uniquely as the sum of an "absolutely continuous" component and a "singular" (or "purely finitely-additive") component

$$y(u) = \int_\Omega u(\omega) y^a(\omega)\, \mathrm{d}P(\omega) + y^s(u), \quad u \in \mathcal{L}^\infty(\Omega, \mathcal{F}, P)$$

where the absolutely continuous component corresponds uniquely to an element $y^a \in \mathcal{L}^1(\Omega, \mathcal{F}, P)$ and the singular component y^s has the property that, for every $\varepsilon > 0$, there is a set $S \in \mathcal{F}$ such that $P(\Omega \backslash S) < \varepsilon$ and $y^s(u) = 0$ for every $u \in \mathcal{L}^\infty(S)$ (i. e., $u \in \mathcal{L}^\infty(\Omega, \mathcal{F}, P)$ vanishing P-a. s. outside of S). Or, equivalently, the singular functionals y^s can be characterized by the property that the underlying space Ω can be expressed as the union of an increasing sequence of measurable sets $\{S_k\}_{k\in\mathbb{N}}$, such that for each $k \in \mathbb{N}$ one has $y^s(u) = 0$ for all the functions $u \in \mathcal{L}^\infty(S_k)$.

Let $\mathcal{S}$ denote the set of all singular functionals y^s in $(\mathcal{L}^\infty)^*(\Omega, \mathcal{F}, P)$ defined above. Then $\mathcal{S}$ is a closed subspace of $(\mathcal{L}^\infty)^*$ which is a complement of the closed subspace that is isomorphic to $\mathcal{L}^1$. The notation $y^s \geq 0$ means that $y^s(u) \geq 0$ for all nonnegative $u \in \mathcal{L}^\infty$.

An *arbitrage opportunity* in the market means that there is a possibility to generate a positive wealth with no risk.

Definition 2.1. A *free lunch* (*FL* for short, with the subscript P indicating the underlying probability measure) is some trading strategy $\theta \in \Theta$ such that

$$\begin{aligned} Z_0^\top \theta_0 &= 0 \\ Z_t^\top \Delta\theta_t &= 0 \ P\text{-a. s.} \quad t = 1, \dots, T \\ Z_T^\top \theta_T &\geq 0 \ P\text{-a. s.} \\ E_P[Z_T^\top \theta_T] &> 0. \end{aligned} \tag{FL_P}$$

.

The classical "fundamental theorem of asset pricing" relates the arbitrage-free market to the existence of an equivalent martingale measure.

Definition 2.2. We say that a probability measure Q on $(\Omega, \mathcal{F})$ is a *martingale measure* for the price process $\{Z_t\}_{t=0}^T$ if $Q \ll P$ (i. e., $Q(A) = 0$ if $P(A) = 0$ for each $A \in \mathcal{F}$) and

$$E_Q[Z_{t+1}|\mathcal{F}_t] = Z_t \ Q\text{-a. s.} \quad t = 0, \dots, T-1.$$

It is an *equivalent martingale measure* if in addition $Q \sim P$ (i. e., $Q(A) = 0$ if and only if $P(A) = 0$ for each $A \in \mathcal{F}$).

Theorem 2.3. ([5, Theorem 5.17]) The market admits no FL_P if and only if there exists an equivalent martingale measure for the price process $\{Z_t\}_{t=0}^T$.

The duality results allow for a limit concept of arbitrage. Slightly weaker than *FL* is the concept of free lunch with vanishing risk that was used in [2] in various asset pricing theorems. The *free lunch with vanishing risk* (*FLVR* for short) is a sequence of self-financing trading strategies $\{\theta^n\}_{n\in\mathbb{N}} \subset \Theta$ with the initial wealths $Z_0^\top \theta_0^n = 0$ and such that the terminal wealths $Z_T^\top \theta_T^n$ are bounded below almost surely by some $H^n \in \mathcal{L}^\infty$ where $\|H^n - H\|_\infty \to 0$ for some $H \in \mathcal{L}_+^\infty$ with $E_P[H] > 0$.

King and Korf in [9] introduced another weaker limit concept of arbitrage closely related to *FLVR* but more intuitive from an investor's perspective.

Definition 2.4. The market is said to admit a *free lunch in the limit* (*FLIL* for short) if there exist $\{\theta^n\}_{n\in\mathbb{N}} \subset \Theta$ and $\{\varepsilon^n\}_{n\in\mathbb{N}} \subset \mathbb{R}$, $\varepsilon^n \searrow 0$ such that

$$\begin{aligned} Z_0^\top \theta_0^n &= 0 \\ Z_t^\top \Delta\theta_t^n &= 0 \ P\text{-a. s.} \quad t = 1, \dots, T \\ Z_T^\top \theta_T^n &\geq -\varepsilon^n \ P\text{-a. s.} \\ \lim_{n\to\infty} E_P[Z_T^\top \theta_T^n] &> 0. \end{aligned} \tag{$FLIL_P$}$$

Note that the sequence $\{\varepsilon^n\}_{n\in\mathbb{N}}$ in the definition of *FLIL* can be without loss of generality taken as $\varepsilon^n = \frac{1}{n}$ and the limit can be replaced by lim sup or lim inf (considering a subsequence of $\{\theta^n\}_{n\in\mathbb{N}}$).

The relations between these concepts and a *FL* were shown in [9].

Theorem 2.5. ([9, Theorem 5.1]) FL_P implies $FLVR_P$ implies $FLIL_P$.

The definition of a finitely-additive martingale measure Q follows. Note that it is in agreement with Definition 2.2 for Q countably-additive.

Definition 2.6. We say that a finitely-additive probability measure Q on $(\Omega, \mathcal{F})$ is a *finitely-additive martingale measure* for the price process $\{Z_t\}_{t=0}^T$ if $Q \ll P$ and

$$E_Q[Z_{t+1}^\top \varphi] = E_Q[Z_t^\top \varphi]$$

for all $\varphi \in \mathcal{L}^\infty(\Omega, \mathcal{F}_t, P; \mathbb{R}^{J+1})$, $t = 0, \dots, T-1$. It is an *equivalent finitely-additive martingale measure* if in addition $Q \sim P$.

We will denote by $\mathcal{Q}$ the set of all finitely-additive martingale measures and by $\mathcal{Q}_e$ the set of all equivalent finitely-additive martingale measures.

King and Korf [9] made an attempt to find a characterization of a market with no *FLIL*. They claimed that the market admits no *FLIL* if and only if there exists an equivalent finitely-additive martingale measure. However, the proof of this claim has serious gaps. The corrected version of this claim was shown in [6].

Theorem 2.7. ([6, Theorem 5.4]) The market admits no free lunch in the limit if and only if there exist an equivalent finitely-additive martingale measure Q for the price process $\{Z_t\}_{t=0}^T$ with the representation $(q^a, q^s) \in (\mathcal{L}^\infty)^*(\Omega, \mathcal{F}, P)$ and a real constant $\delta > 0$ such that

$$q^a \geq \delta \ \ P\text{-a.s.} \tag{1}$$

Remark. The condition (1) can be equivalently stated as

$$Q(E) \geq \delta P(E) \qquad \text{for all } E \in \mathcal{F}. \tag{2}$$

Indeed, if $q^a \geq \delta$ P-a.s., we have $Q(E) \geq \int_E q^a \, \mathrm{d}P \geq \delta P(E)$ for each $E \in \mathcal{F}$. On the other hand, let $A \in \mathcal{F}$ be such that $P(A) > 0$ and $q^a(\omega) < \delta$ for every $\omega \in A$. By the definition of $q^s \in \mathcal{S}$, there is $S \in \mathcal{F}$ such that $P(\Omega \setminus S) < \frac{P(A)}{2}$ and $\langle \eta, q^s \rangle = 0$ for each $\eta \in \mathcal{L}^\infty(S)$. Then $Q(A \cap S) = \int_{A \cap S} q^a \, \mathrm{d}P + \langle \chi_{A \cap S}, q^s \rangle = \int_{A \cap S} q^a \, \mathrm{d}P < \delta P(A \cap S)$, contrary to (2).

Proof of Theorem 2.7. We only give a sketch of the proof (for details see [6]).

Fix some contingent claim $\{F_t\}_{t=1}^T$ with the initial price $F_0 > \operatorname{ess\,inf} \sum_{t=1}^T F_t$ such that

$$\begin{aligned} Z_0^\top \tilde{\theta}_0 &\leq F_0 - \varepsilon \\ Z_t^\top \Delta \tilde{\theta}_t &\leq -F_t - \varepsilon \quad P\text{-a.s.} \quad t = 1, \dots, T, \\ Z_T^\top \tilde{\theta}_T &\geq 0 \quad P\text{-a.s.} \end{aligned}$$

for some $\varepsilon > 0$ and $\tilde{\theta} \in \Theta$. Consider the stochastic optimization problem

$$\begin{aligned} \text{maximize } E_P[Z_T^\top \theta_T] \quad \text{over all} \quad & \theta \in \Theta \\ \text{subject to} \quad & Z_0^\top \theta_0 \leq F_0 \\ & Z_t^\top \Delta \theta_t \leq -F_t \quad P\text{-a.s.} \quad t = 1, \dots, T \\ & Z_T^\top \theta_T \geq 0 \quad P\text{-a.s.} \end{aligned} \tag{P_w}$$

It follows by the same method as in Section 4 of this paper, that the dual problem to (P_w) is

$$\begin{aligned} &\text{minimize } \langle F_0, y_0\rangle_0 - \sum_{t=1}^{T} \langle F_t, y_t\rangle_t && (\mathsf{D}_w)\\ &\text{over all } y \in \prod_{t=0}^{T} (\mathcal{L}^\infty)^*(\Omega, \mathcal{F}_t, P)\\ &\text{subject to } \langle Z_t^\top \eta_t, y_t\rangle_t = \langle Z_{t+1}^\top \eta_t, y_{t+1}\rangle_{t+1} \text{ for all } \eta_t \in \mathcal{L}^\infty_{J+1}(\Omega, \mathcal{F}_t, P),\\ &\qquad t = 0, \dots, T-1\\ &\qquad y \geq 0\\ &\qquad y_T^a \geq 1 \ P\text{-a.s.} \end{aligned}$$

(The dualization of more general primal problem involving a writer's utility function was studied in [9].) By [9, Theorem 5.2], no $FLIL_P$ is equivalent to the boundedness of (P_w). By Theorems 17 and 18(i) in [13], we have $\sup(\mathsf{P}_w) = \min(\mathsf{D}_w)$. Hence the boundedness of (P_w) is equivalent to the feasibility of (D_w). The set of feasible solutions of (D_w) corresponds to the set of elements of $\mathcal{Q}$ satisfying (1) through the identities

$$\delta = \frac{1}{y_0} = \frac{1}{E_P[y_T^a] + \langle \chi_\Omega, y_t^s\rangle_T},$$
$$q_t = \frac{1}{y_0} y_t \quad t = 0, \dots, T$$

where q_t is the representation of $Q|_{\mathcal{F}_t}$ in $(\mathcal{L}^\infty)^*(\Omega, \mathcal{F}_t, P)$. □

Finally, under the assumption of no *FLIL*, King and Korf [9] proposed a formula for a *writer's price* of a contingent claim $\{F_t\}_{t=1}^T$, that is the lowest price such that the writer of the contingent claim will be able to invest his or her earnings from the sale in the market to almost surely cover the cash flow.

Theorem 2.8. ([9, Theorem 7.3]) Suppose the market admits no free lunch in the limit. Then the writer's price of the contingent claim is

$$\max\Big\{\sum_{t=1}^{T} E_Q[F_t] \Big| Q \in \mathcal{Q}\Big\}.$$

3. FREE LUNCH IN THE LIMIT AND EQUIVALENT MEASURES

The underlying probability measure P enters the definitions of FL_P and $FLVR_P$ only through its null sets. Therefore, for every two probability measures P_1 and P_2 on $(\Omega, \mathcal{F})$ such that $P_1 \sim P_2$, there is FL_{P_1} if and only if there is FL_{P_2}, and there is $FLVR_{P_1}$ if and only if there is $FLVR_{P_2}$. However, the concept of *FLIL* fails to have such a property except in the static case $T = 1$.

Note that $P_1 \sim P_2$ implies that some statement holds P_1-a. s. if and only if it holds P_2-a. s. Therefore, in the sequel, we shall write only P-a. s.

Theorem 3.1. Let P_1 and P_2 be equivalent probability measures on $(\Omega, \mathcal{F})$ and $T = 1$. Then there is $FLIL_{P_1}$ if and only if there is $FLIL_{P_2}$. In this case, there exists FL_{P_1}, say $\tilde{\theta}$, such that the constant sequence $\theta^n = \tilde{\theta}$ for $n \in \mathbb{N}$ is $FLIL_{P_1}$ as well as $FLIL_{P_2}$.

Proof. Since $Z_t^0 = 1$ P-a. s. for $t = 0, 1$, there is $FLIL_P$ if and only if there exist $\{\theta_0^{j,n}\}_{n\in\mathbb{N}} \subset \mathbb{R}, j = 1, \dots, J$ such that

$$\begin{aligned} \sum_{j=1}^{J} (Z_1^j - Z_0^j)\theta_0^{j,n} &\geq -\frac{1}{n} \; P\text{-a. s.} \\ \lim_{n\to\infty} \sum_{j=1}^{J} (E_P[Z_1^j] - Z_0^j)\theta_0^{j,n} &> 0. \end{aligned} \tag{3}$$

There is no loss of generality in assuming that the functions $Z_1^j - Z_0^j, j = 1, \dots, J$ are linearly independent in the sense: for $c_j \in \mathbb{R}$, $j = 1, \dots, J$,

$$\sum_{j=1}^{J} c_j(Z_1^j - Z_0^j) = 0 \; P\text{-a. s.} \quad \text{implies} \quad c_j = 0, \; j = 1, \dots, J.$$

Indeed, if $Z_1^J - Z_0^J = \sum_{j=1}^{J-1} c_j(Z_1^j - Z_0^j)$ P-a. s. then $\bar{\theta}_0^{j,n} = \theta_0^{j,n} + c_j\theta_0^{J,n}$ for $j = 1, \dots, J-1$ satisfies (3) with $Z_1^j - Z_0^j$, $j = 1, \dots, J-1$, if and only if $\theta_0^{j,n}$, $j = 1, \dots, J$, satisfies (3).

Set

$$\begin{aligned} M_{E_{P_1}} &= \Big\{ \big(\theta_0^1, \theta_0^2, \dots, \theta_0^J\big)^\top \in \mathbb{R}^J \Big| \sum_{j=1}^{J} (E_{P_1}[Z_1^j] - Z_0^j)\theta_0^j = 0 \Big\}, \\ M_{E_{P_2}} &= \Big\{ \big(\theta_0^1, \theta_0^2, \dots, \theta_0^J\big)^\top \in \mathbb{R}^J \Big| \sum_{j=1}^{J} (E_{P_2}[Z_1^j] - Z_0^j)\theta_0^j = 0 \Big\}, \\ M &= \Big\{ \big(\theta_0^1, \theta_0^2, \dots, \theta_0^J\big)^\top \in \mathbb{R}^J \Big| \sum_{j=1}^{J} (Z_1^j - Z_0^j)\theta_0^j \geq 0 \; P\text{-a. s.} \Big\}, \\ M_n &= \Big\{ \big(\theta_0^1, \theta_0^2, \dots, \theta_0^J\big)^\top \in \mathbb{R}^J \Big| \sum_{j=1}^{J} (Z_1^j - Z_0^j)\theta_0^j \geq -\frac{1}{n} \; P\text{-a. s.} \Big\}. \end{aligned}$$

Let us show that $M \subset M_{E_{P_1}}$ if and only if $M \subset M_{E_{P_2}}$. Suppose that $M \subset M_{E_{P_1}}$, i. e., $\sum_{j=1}^{J} (Z_1^j - Z_0^j)\theta_0^j \geq 0$ P-a. s. implies $\sum_{j=1}^{J} (E_{P_1}[Z_1^j] - Z_0^j)\theta_0^j = 0$. Then $\sum_{j=1}^{J} (Z_1^j - Z_0^j)\theta_0^j \geq 0$ P-a. s. implies $\sum_{j=1}^{J} (Z_1^j - Z_0^j)\theta_0^j = 0$ P-a. s. On the contrary, suppose that there exists $A \subset \Omega$, $P(A) > 0$, such that $\sum_{j=1}^{J} (Z_1^j(\omega) - Z_0^j)\theta_0^j > 0$ for all $\omega \in A$. Hence $E_{P_1}\big[\sum_{j=1}^{J} (Z_1^j - Z_0^j)\theta_0^j\big] > 0$, a contradiction. From this we conclude that

$$M = \Big\{ \big(\theta_0^1, \theta_0^2, \dots, \theta_0^J\big)^\top \in \mathbb{R}^J \Big| \sum_{j=1}^{J} (Z_1^j - Z_0^j)\theta_0^j = 0 \; P\text{-a. s.} \Big\} \subset M_{E_{P_2}}.$$

We have proved that $M \subset M_{E_{P_1}}$ if and only if $M \subset M_{E_{P_2}}$. Furthermore, if $M \subset M_{E_{P_1}}$ then $M = \{(\theta_0^1, \theta_0^2, \dots, \theta_0^J)^\top \in \mathbb{R}^J \big| \sum_{j=1}^J (Z_1^j - Z_0^j)\theta_0^j = 0 \ P\text{-a.s.}\}$ which under assumption about linear independence of $Z_1^j - Z_0^j$ leads to $M = \{0\}$.

From what has already been proved, it follows that there are only two cases that can occur: either $M \not\subset M_{E_{P_1}}$ and $M \not\subset M_{E_{P_2}}$, or $M \subset M_{E_{P_1}} \cap M_{E_{P_2}}$.

Let us first suppose that $M \not\subset M_{E_{P_1}}$ and $M \not\subset M_{E_{P_2}}$. Thus there exists $\tilde{\theta} \in M$ such that $\sum_{j=1}^J (Z_1^j - Z_0^j)\tilde{\theta}_0^j > 0$ on some $A \in \mathcal{F}$, $P(A) > 0$. Setting $\theta_0^{j,n} = \tilde{\theta}_0^j$ we have $FLIL_{P_1}$ and $FLIL_{P_2}$.

Now suppose that $M \subset M_{E_{P_1}} \cap M_{E_{P_2}}$. Clearly $M \subset M_{n+1} \subset M_n$ for all $n \in \mathbb{N}$. We show that $M_n \underset{n\to\infty}{\longrightarrow} \{0\}$ in the following sense: $\forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N} \ \forall n > n_0$ $M_n \subset \mathbb{B}_\varepsilon(0)$. On the contrary, suppose that there exists $\varepsilon > 0$ such that for all $n_0 \in \mathbb{N}$ there are $n > n_0$ and $x_n \in M_n$ such that $x_n \notin \mathbb{B}_\varepsilon(0)$ (which means $\|x_n\| \geq \varepsilon$). Since M_n is a convex subset of $\mathbb{R}^J$ and $0 \in M_n$, we have $y_n = x_n \frac{\varepsilon}{\|x_n\|} \in M_n$, $\|y_n\| = \varepsilon$. Since the set $\{\, y \in \mathbb{R}^J \big| \, \|y\| = \varepsilon\}$ is compact, we can find a convergent subsequence $z_n \to z$ such that $\|z\| = \varepsilon$. Since $z_n \in M_n$, $\sum_{j=1}^J (Z_1^j - Z_0^j) z_n^j \geq -\frac{1}{n}$ P-a.s. Letting $n \to \infty$ we get $\sum_{j=1}^J (Z_1^j - Z_0^j) z^j \geq 0$ P-a.s. Hence $z \in M$ and $\|z\| = \varepsilon$, a contradiction.

Set $C_1 = \max_{j=1,\dots,J} |E_{P_1}[Z_1^j] - Z_0^j|$ and $C_2 = \max_{j=1,\dots,J} |E_{P_2}[Z_1^j] - Z_0^j|$. For every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$ we have $\max_{j=1,\dots,J} |\theta_0^{j,n}| < \varepsilon$ for every $\theta_0^n \in M_n$. Hence we get $\forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N} \ \forall n > n_0 \ \forall \theta_0^n \in M_n$

$$\Big|\sum_{j=1}^J (E_{P_1}[Z_1^j] - Z_0^j)\theta_0^{j,n}\Big| \leq \sum_{j=1}^J \big|E_{P_1}[Z_1^j] - Z_0^j\big| |\theta_0^{j,n}| \leq J\varepsilon C_1$$

and

$$\Big|\sum_{j=1}^J (E_{P_2}[Z_1^j] - Z_0^j)\theta_0^{j,n}\Big| \leq \sum_{j=1}^J \big|E_{P_2}[Z_1^j] - Z_0^j\big| |\theta_0^{j,n}| \leq J\varepsilon C_2.$$

That means that there is no $FLIL_{P_1}$ and no $FLIL_{P_2}$. □

We will show that the statement of Theorem 3.1 cannot be extended to cover the case $T > 1$. For $T \in \mathbb{N}$, there is $FLIL_P$ if and only if there exist $\{\theta_t^{j,n}\}_{n\in\mathbb{N}} \subset \mathcal{L}^\infty(\Omega, \mathcal{F}_t, P), j = 1, \dots, J, t = 0, \dots, T-1$ such that

$$\sum_{t=1}^T \sum_{j=1}^J (Z_t^j - Z_{t-1}^j)\theta_{t-1}^{j,n} \geq -\frac{1}{n} \ P\text{-a.s.}$$

$$\lim_{n\to\infty} E_P\Big[\sum_{t=1}^T \sum_{j=1}^J (Z_t^j - Z_{t-1}^j)\theta_{t-1}^{j,n}\Big] > 0.$$

In Example 3.2, we give an explicit example of two equivalent probability measures P_1, P_2 and a market that admits $FLIL_{P_1}$ and no $FLIL_{P_2}$.

Example 3.2. Let the underlying probability space be $(\Omega, \mathcal{F}, P_1)$ where $\Omega = [-1,1]$, $\mathcal{F} = \mathcal{B}_{[-1,1]}$ (the Borel σ-algebra on $[-1,1]$), and $P_1 = \frac{1}{2}\lambda_{[-1,1]}$ (where $\lambda_{[-1,1]}$ stands for the Lebesgue measure on $[-1,1]$). There is an equivalent probability measure $P_2 \sim P_1$ on $(\Omega, \mathcal{F})$ and a σ-algebra $\mathcal{F}_1 \subset \mathcal{F}$ such that there exists a market model with 2 assets and the horizon $T = 2$ that admits $FLIL_{P_1}$ but does not admit $FLIL_{P_2}$.

Proof. Suppose $T = 2, J = 1$ and $\mathcal{F}_1 = \{A_1 \in \mathcal{F} | A_1 = A \cup (-A), A \in \mathcal{B}_{[0,1]}\}$. Note that a function on $[-1,1]$ is $\mathcal{F}_1$-measurable if and only if it is a Borel-measurable even function. Set

$$Z_0^1 = 1,$$
$$Z_1^1 = \chi_\Omega,$$
$$Z_2^1(\omega) = \begin{cases} (\omega + \frac{1}{2})^3 + 1 & \text{if } \omega \in [-1,0), \\ \omega + \frac{1}{2} & \text{if } \omega \in (0,1]. \end{cases}$$

$$\text{Thus } \sum_{t=1}^{2} (Z_t^1(\omega) - Z_{t-1}^1(\omega))\theta_{t-1}^1(\omega) = \begin{cases} (\omega + \frac{1}{2})^3 \theta_1^1(\omega) & \text{if } \omega \in [-1,0), \\ (\omega - \frac{1}{2})\theta_1^1(\omega) & \text{if } \omega \in (0,1]. \end{cases}$$

A sequence $\{\theta_1^{1,n}(\omega)\}_{n\in\mathbb{N}}$ fulfills the condition $\sum_{t=1}^{2}(Z_t^1 - Z_{t-1}^1)\theta_{t-1}^{1,n} \geq -\frac{1}{n}$ P-a.s. if and only if

$$\Big(\omega + \frac{1}{2}\Big)^3 \theta_1^{1,n}(\omega) \geq -\frac{1}{n} \ P\text{-a.s. on } [-1,0),$$
$$\Big(\omega - \frac{1}{2}\Big)\theta_1^{1,n}(\omega) \geq -\frac{1}{n} \ P\text{-a.s. on } (0,1].$$

Since $\theta_1^{1,n}$ is $\mathcal{F}_1$-measurable, $\theta_1^{1,n}(-\omega) = \theta_1^{1,n}(\omega)$ on $[-1,1]$. Hence

$$\Big(-\omega - \frac{1}{2}\Big)\theta_1^{1,n}(\omega) \geq -\frac{1}{n} \ P\text{-a.s. on } [-1,0),$$
$$\Big(-\omega + \frac{1}{2}\Big)^3 \theta_1^{1,n}(\omega) \geq -\frac{1}{n} \ P\text{-a.s. on } (0,1].$$

It follows that

$$|\theta_1^{1,n}(\omega)| \leq \begin{cases} \frac{1}{n}|\omega + \frac{1}{2}|^{-3} P\text{-a.s.} & \text{if } \omega \in [-1,0), \\ \frac{1}{n}|\omega - \frac{1}{2}|^{-3} P\text{-a.s.} & \text{if } \omega \in (0,1] \end{cases}$$

for any sequence $\{\theta_1^{1,n}\}_{n\in\mathbb{N}}$ satisfying $\sum_{t=1}^{2}(Z_t^1 - Z_{t-1}^1)\theta_{t-1}^{1,n} \geq -\frac{1}{n}$ P-a.s.

Let P_2 be a probability measure on $[-1,1]$ having the following density with respect to Lebesgue measure

$$\frac{\mathrm{d}P_2(\omega)}{\mathrm{d}\lambda_{[-1,1]}} = \begin{cases} 6(\omega + \frac{1}{2})^2 & \text{if } \omega \in [-1,0], \\ 6(\omega - \frac{1}{2})^2 & \text{if } \omega \in (0,1]. \end{cases}$$

We have $P_1 \sim P_2$. We show that there is $FLIL_{P_1}$ and no $FLIL_{P_2}$.

Let $\{\theta_1^{1,n}\}_{n\in\mathbb{N}}$ be given by $\theta_1^{1,n} = n^2\chi_{(-\frac{1}{2}-\frac{1}{n},-\frac{1}{2})\cup(\frac{1}{2},\frac{1}{2}+\frac{1}{n})}$. We have $\sum_{t=1}^{2}(Z_t^1 - Z_{t-1}^1)\theta_{t-1}^{1,n} \geq -\frac{1}{n}$ P-a. s. since

$$\Big(\omega+\frac{1}{2}\Big)^3\theta_1^{1,n}(\omega) \geq \inf_{\omega\in(-\frac{1}{2}-\frac{1}{n},-\frac{1}{2})} n^2\Big(\omega+\frac{1}{2}\Big)^3 = -\frac{1}{n} \text{ for } P\text{-almost all } \omega\in[-1,0),$$
$$\Big(\omega-\frac{1}{2}\Big)\theta_1^{1,n}(\omega) \geq \inf_{\omega\in(\frac{1}{2},\frac{1}{2}+\frac{1}{n})} n^2\Big(\omega-\frac{1}{2}\Big) = 0 \text{ for } P\text{-almost all } \omega\in(0,1].$$

By an easy computation,

$$\begin{aligned} &E_{P_1}\Big[\sum_{t=1}^{2}(Z_t^1 - Z_{t-1}^1)\theta_{t-1}^{1,n}\Big] \\ &= \int_{-\frac{1}{2}-\frac{1}{n}}^{-\frac{1}{2}} n^2\left(\omega+\frac{1}{2}\right)^3\frac{1}{2}\,\mathrm{d}\omega + \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} n^2\left(\omega-\frac{1}{2}\right)\frac{1}{2}\,\mathrm{d}\omega \\ &= \frac{1}{4}-\frac{1}{8n^2} \underset{n\to\infty}{\longrightarrow} \frac{1}{4}. \end{aligned}$$

Setting $\theta_0^{0,n}=0$, $\theta_0^{1,n}=0$, $\theta_1^{0,n}=-\theta_1^{1,n}$ P-a. s., $\theta_2^{0,n}=-\theta_1^{1,n}$ P-a. s., and $\theta_2^{1,n}=\theta_1^{1,n}$ P-a. s., we have

$$Z_2^\top\theta_2^n = -\theta_1^{1,n}+Z_2^1\theta_1^{1,n} = n^2\left(\omega+\frac{1}{2}\right)^3\chi_{(-\frac{1}{2}-\frac{1}{n},-\frac{1}{2})}(\omega)+n^2\left(\omega-\frac{1}{2}\right)\chi_{(\frac{1}{2},\frac{1}{2}+\frac{1}{n})}(\omega).$$

Hence

$$\begin{aligned} Z_0^\top\theta_0^n &= 0 \\ Z_t^\top\Delta\theta_t^n &= 0 \;P\text{-a. s.} \quad t=1,2 \\ Z_2^\top\theta_2^n &\geq -\frac{1}{n}\;P\text{-a. s.} \\ \lim_{n\to\infty} E_{P_1}[Z_2^\top\theta_2^n] &> 0 \end{aligned}$$

and we have found a $FLIL_{P_1}$.

On the other hand, for arbitrary $\{\theta_1^{1,n}\}_{n\in\mathbb{N}}$ satisfying $\sum_{t=1}^{2}(Z_t^1 - Z_{t-1}^1)\theta_{t-1}^{1,n} \geq -\frac{1}{n}$ P-a. s. we have

$$\begin{aligned} &\left|E_{P_2}\Big[\sum_{t=1}^{2}(Z_t^1 - Z_{t-1}^1)\theta_{t-1}^{1,n}\Big]\right| \\ &\leq \int_{-1}^{0} 6\left|\omega+\frac{1}{2}\right|^5|\theta_1^{1,n}(\omega)|\,\mathrm{d}\omega + \int_0^1 6\left|\omega-\frac{1}{2}\right|^3|\theta_1^{1,n}(\omega)|\,\mathrm{d}\omega \\ &\leq \int_{-1}^{0}\frac{6}{n}\left(\omega+\frac{1}{2}\right)^2\mathrm{d}\omega + \int_0^1\frac{6}{n}\,\mathrm{d}\omega = \frac{13}{2n} \underset{n\to\infty}{\longrightarrow} 0. \end{aligned}$$

Hence there is no $FLIL_{P_2}$. □

Finally there is some strengthening of an assumption that probability measures are equivalent under which the concept of *FLIL* is stable.

Theorem 3.3. Let $P_1 \sim P_2$ be probability measures on $(\Omega, \mathcal{F})$ such that

$$\alpha \cdot P_1(A) \le P_2(A) \le \beta \cdot P_1(A)$$

for any $A \in \mathcal{F}$ where $\alpha, \beta \in \mathbb{R}$ are some constants satisfying $0 < \alpha \le \beta$. Then there is $FLIL_{P_1}$ if and only if there is $FLIL_{P_2}$.

Proof. By Theorem 2.7, there is no $FLIL_{P_1}$ if and only if there exists $Q \in \mathcal{Q}_e$ with the representation $(q_1^a, q_1^s) \in (\mathcal{L}^\infty)^*(\Omega, \mathcal{F}, P_1)$ such that $q_1^a \ge \delta$ P-a.s. for some $\delta > 0$. Since $P_1 \sim P_2$, Q is represented in $(\mathcal{L}^\infty)^*(\Omega, \mathcal{F}, P_2)$ by $q_2^a = q_1^a \cdot \frac{dP_1}{dP_2}$, $q_2^s = q_1^s$. Since $\frac{1}{\beta} \le \frac{dP_1}{dP_2} \le \frac{1}{\alpha}$ P-a.s., we have $q_2^a \ge \frac{\delta}{\beta}$ P-a.s., so that there is no $FLIL_{P_2}$. $\square$

4. FUNDAMENTAL THEOREM OF ASSET PRICING

In order to state a characterization of no free lunch in the framework of finitely-additive martingale measures, we present the following multistage stochastic optimization problem.

$$\begin{aligned} \text{minimize } Z_0^\top \theta_0 \quad \text{over all} \quad & \theta \in \Theta \\ \text{subject to} \quad & Z_t^\top \Delta\theta_t \le 0 \quad P\text{-a.s.} \quad t = 1, \dots, T \\ & Z_T^\top \theta_T \ge 0 \quad P\text{-a.s.} \\ & E_P[Z_T^\top \theta_T] = 1. \end{aligned} \tag{P}$$

We derive a problem dual to (P) using the abstract conjugate duality theory by Rockafellar [13]. In our computation we will need the following lemma.

Lemma 4.1. Let $y = (y^a, y^s) \in (\mathcal{L}^\infty)^*(\Omega, \mathcal{F}, P)$ be such that $y \ge 0$. Then

$$\inf\{y(\eta) | \eta \in \mathcal{L}_+^\infty(\Omega, \mathcal{F}, P),\ E_P[\eta] = 1\} = \operatorname{ess\,inf} y^a.$$

Proof. Since $y(\eta) = E_P[\eta y^a] + y^s(\eta)$ and $E_P[\eta y^a] \ge \operatorname{ess\,inf} y^a$ for each $\eta \in \mathcal{L}_+^\infty$ such that $E_P[\eta] = 1$, we have

$$\inf\{y(\eta) | \eta \in \mathcal{L}_+^\infty(\Omega, \mathcal{F}, P),\ E_P[\eta] = 1\} \ge \operatorname{ess\,inf} y^a.$$

We show that for each $\varepsilon > 0$ there is $\tilde{\eta} \in \mathcal{L}_+^\infty$ such that $E_P[\tilde{\eta}] = 1$ and

$$y(\tilde{\eta}) < \operatorname{ess\,inf} y^a + \varepsilon.$$

Set $A = \{\omega \in \Omega | y^a(\omega) < \operatorname{ess\,inf} y^a + \varepsilon\}$. Then $A \in \mathcal{F}$ and $P(A) > 0$. By the definition of $y^s \in \mathcal{S}$, there is $S \in \mathcal{F}$ such that $P(\Omega \setminus S) < \frac{P(A)}{2}$ and $y^s(\eta) = 0$ for each $\eta \in \mathcal{L}^\infty(S)$. Set $B = A \cap S$. Then $P(B) > \frac{P(A)}{2} > 0$. Set $\tilde{\eta} = \frac{1}{P(B)} \chi_B$. We have $\tilde{\eta} \in \mathcal{L}_+^\infty$ such that $E_P[\tilde{\eta}] = 1$. Moreover, $y^s(\tilde{\eta}) = 0$ and

$$E_P[\tilde{\eta} y^a] = \frac{1}{P(B)} \int_B y^a \mathrm{d}P < \frac{1}{P(B)} \int_B (\operatorname{ess\,inf} y^a + \varepsilon) \,\mathrm{d}P = \operatorname{ess\,inf} y^a + \varepsilon.$$

Therefore $y(\tilde{\eta}) < \operatorname{ess\,inf} y^a + \varepsilon$, which completes the proof. $\square$

Taking problem (P) as an abstract primal problem of [13], we set the perturbation space

$$U = \{(u_1, \dots, u_T) | u_t \in \mathcal{L}^\infty(\Omega, \mathcal{F}_t, P),\ t = 1, \dots, T\} = \prod_{t=1}^{T} \mathcal{L}^\infty(\Omega, \mathcal{F}_t, P).$$

The space U, equipped with the strong product topology, is then paired with

$$\begin{aligned} Y =& \{(y_1, \dots, y_T) | y_t = (y_t^a, y_t^s) \in (\mathcal{L}^\infty)^*(\Omega, \mathcal{F}_t, P),\ t = 1, \dots, T\} \\ =& \prod_{t=1}^{T} (\mathcal{L}^\infty)^*(\Omega, \mathcal{F}_t, P), \end{aligned}$$

equipped with the weak* product topology, by the pairing $\langle ., . \rangle$ on $U \times Y$ given by

$$\langle u, y \rangle = \sum_{t=1}^{T} \langle u_t, y_t \rangle_t.$$

Here y_t^a and y_t^s denote the absolutely continuous and the singular component of y_t, respectively, and $\langle ., . \rangle_t$ denotes the natural pairing of $\mathcal{L}^\infty(\Omega, \mathcal{F}_t, P)$ and $(\mathcal{L}^\infty)^*(\Omega, \mathcal{F}_t, P)$, i. e., $\langle u_t, y_t \rangle_t = y_t(u_t)$, for $t = 1, \dots, T$.

Consider the perturbation function F on $\Theta \times U$ defined by

$$F(\theta, u) = \begin{cases} Z_0^\top \theta_0 & \text{if } Z_t^\top \Delta\theta_t \le u_t \ P\text{-a. s.},\ t = 1, \dots, T, \\ & \quad Z_T^\top \theta_T \ge 0 \ P\text{-a. s.}, \\ & \quad E_P[Z_T^\top \theta_T] = 1; \\ +\infty & \text{otherwise.} \end{cases}$$

Clearly, $\inf(\mathrm{P}) = \inf\{F(\theta, 0) | \theta \in \Theta\}$. To shorten notation, we let ψ_C stand for an indicator function of a set C, i. e.,

$$\psi_C(x) = \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{otherwise.} \end{cases}$$

The Lagrangian function on $\Theta \times Y$ is given by

$$\begin{aligned} L(\theta, y) &= \inf\{F(\theta, u) + \langle u, y \rangle | u \in U\} \\ &= Z_0^\top \theta_0 + \psi_{\{Z_T^\top \theta_T \ge 0\ P\text{-a. s.},\ E_P[Z_T^\top \theta_T] = 1\}}(\theta) + \sum_{t=1}^{T} \langle Z_t^\top \Delta\theta_t, y_t \rangle_t - \psi_{\{y \ge 0\}}(y) \\ &= \begin{cases} Z_0^\top \theta_0 - \langle Z_1^\top \theta_0, y_1 \rangle_1 + \displaystyle\sum_{t=1}^{T-1} (\langle Z_t^\top \theta_t, y_t \rangle_t - \langle Z_{t+1}^\top \theta_t, y_{t+1} \rangle_{t+1}) & \text{if } y \ge 0, \\ \qquad + \langle Z_T^\top \theta_T, y_T \rangle_T + \psi_{\{Z_T^\top \theta_T \ge 0\ P\text{-a. s.},\ E_P[Z_T^\top \theta_T] = 1\}}(\theta_T) & \\ -\infty & \text{otherwise.} \end{cases} \end{aligned}$$

Clearly,

$$\inf\{Z_0^\top \theta_0 - \langle Z_1^\top \theta_0, y_1\rangle_1 | \theta_0 \in \mathbb{R}^{J+1}\}$$

equals 0 if $Z_0^\top \theta_0 = \langle Z_1^\top \theta_0, y_1\rangle_1$ for all $\theta_0 \in \mathbb{R}^{J+1}$ and is $-\infty$ otherwise, for $t = 1, \ldots, T-1$,

$$\inf\{\langle Z_t^\top \theta_t, y_t\rangle_t - \langle Z_{t+1}^\top \theta_t, y_{t+1}\rangle_{t+1} | \theta_t \in \mathcal{L}^\infty_{J+1}(\Omega, \mathcal{F}_t, P)\}$$

equals 0 if

$$\langle Z_t^\top \theta_t, y_t\rangle_t = \langle Z_{t+1}^\top \theta_t, y_{t+1}\rangle_{t+1} \text{ for all } \theta_t \in \mathcal{L}^\infty_{J+1}(\Omega, \mathcal{F}_t, P)$$

and is $-\infty$ otherwise, and, for $y \geq 0$,

$$\inf\{\langle Z_T^\top \theta_T, y_T\rangle_T | \theta_T \in \mathcal{L}^\infty_{J+1},\ Z_T^\top \theta_T \geq 0\ P\text{-a.s.},\ E_P[Z_T^\top \theta_T] = 1\} = \operatorname{ess\,inf} y_T^a,$$

the last equality being a consequence of Lemma 4.1.

Therefore, the dual objective function is

$$\begin{aligned} g(y) &= \inf\{L(\theta, y) | \theta \in \Theta\} \\ &= \begin{cases} \operatorname{ess\,inf} y_T^a & \text{if } y \in Y_m,\ y \geq 0, \\ -\infty & \text{otherwise,} \end{cases} \end{aligned}$$

with the notation

$$\begin{aligned} Y_m = \{y \in Y | Z_0^\top \theta_0 = \langle Z_1^\top \theta_0, y_1\rangle_1 &\text{ for all } \theta_0 \in \mathbb{R}^{J+1}, \\ \langle Z_t^\top \theta_t, y_t\rangle_t = \langle Z_{t+1}^\top \theta_t, y_{t+1}\rangle_{t+1} &\text{ for all } \theta_t \in \mathcal{L}^\infty_{J+1}(\Omega, \mathcal{F}_t, P),\ t = 1, \ldots, T-1\}. \end{aligned}$$

Hence the dual problem to (P) is

$$\begin{aligned} \text{maximize } \operatorname{ess\,inf} y_T^a \quad &\text{over all } y \in Y \\ \text{subject to } &y \in Y_m \\ &y \geq 0. \end{aligned} \tag{D}$$

Note that the set of feasible solutions to (D) corresponds to the set of finitely-additive martingale measures $\mathcal{Q}$ through the identity

$$\langle \chi_E, y_T\rangle_T = Q(E) \text{ for all } E \in \mathcal{F}.$$

Indeed, the set function Q defined above is a nonnegative finitely-additive measure on $(\Omega, \mathcal{F})$ such that $Q \ll P$. It follows from $y \in Y_m$ that $Q(\Omega) = 1$ and Q is a martingale measure.

Theorem 4.2. We have $\inf(\mathrm{P}) = \max(\mathrm{D})$.

Proof. Set $\tilde{\theta}_t = (T+1-t, 0, \ldots, 0)^\top$ for $t = 0, \ldots, T$. Then

$$\begin{aligned} Z_t^\top \Delta\tilde{\theta}_t &= -1 \quad t = 1, \ldots, T \\ Z_T^\top \tilde{\theta}_T &= 1. \end{aligned}$$

Therefore the function $u \mapsto F(\tilde{\theta}, u)$ on U is bounded above on a norm-neighbourhood of $0 \in U$. Since F is convex, Theorem 18(i) in [13] shows that the optimal value function $\varphi(u) = \inf\{F(\theta, u) | \theta \in \Theta\}$ is bounded above on a norm-neighbourhood of 0. The conclusion follows from [13, Theorem 17]. □

We show that the ordinary concept of FL can be characterized by duality results for (P) and (D).

Lemma 4.3. Suppose that $\inf(\mathrm{P}) = \max(\mathrm{D}) = 0$ and let $\bar{\theta} \in \Theta$ be some trading strategy feasible for (P). Then the following conditions are equivalent:

(i) $\bar{\theta}$ is optimal for (P),

(ii) $E_Q[Z_T^\top \bar{\theta}_T] = 0$ and $E_Q[\sum_{t=1}^T Z_t^\top \Delta\bar{\theta}_t] = 0$ for all $Q \in \mathcal{Q}$,

(iii) $E_Q[Z_T^\top \bar{\theta}_T] = 0$ and $E_Q[\sum_{t=1}^T Z_t^\top \Delta\bar{\theta}_t] = 0$ for some $Q \in \mathcal{Q}$.

P r o o f. We have

$$\langle Z_t^\top \bar{\theta}_t, y_t \rangle_t = \langle Z_t^\top \Delta\bar{\theta}_t, y_t \rangle_t + \langle Z_t^\top \bar{\theta}_{t-1}, y_t \rangle_t = \langle Z_t^\top \Delta\bar{\theta}_t, y_t \rangle_t + \langle Z_{t-1}^\top \bar{\theta}_{t-1}, y_{t-1} \rangle_{t-1}$$

for all $t = 1, \dots, T$ and all $y \in Y_m$. Hence

$$\langle Z_T^\top \bar{\theta}_T, y_T \rangle_T = \sum_{t=1}^T \langle Z_t^\top \Delta\bar{\theta}_t, y_t \rangle_t + Z_0^\top \bar{\theta}_0$$

for all y feasible to (D). By assumption, $\bar{\theta}$ is optimal if and only if $Z_0^\top \bar{\theta}_0 = 0$, which is equivalent to

$$\langle Z_T^\top \bar{\theta}_T, y_T \rangle_T = \sum_{t=1}^T \langle Z_t^\top \Delta\bar{\theta}_t, y_t \rangle_t = 0 \qquad \text{for all } y \text{ feasible to (D)},$$

or, equivalently,

$$\langle Z_T^\top \bar{\theta}_T, y_T \rangle_T = \sum_{t=1}^T \langle Z_t^\top \Delta\bar{\theta}_t, y_t \rangle_t = 0 \qquad \text{for some } y \text{ feasible to (D)}.$$

Since the set of feasible solutions to (D) corresponds to $\mathcal{Q}$, we have $\bar{\theta}$ optimal if and only if

$$E_Q[Z_T^\top \bar{\theta}_T] = E_Q\Big[\sum_{t=1}^T Z_t^\top \Delta\bar{\theta}_t\Big] = 0 \text{ for all } Q \in \mathcal{Q}$$

which is equivalent to

$$E_Q[Z_T^\top \bar{\theta}_T] = E_Q\Big[\sum_{t=1}^T Z_t^\top \Delta\bar{\theta}_t\Big] = 0 \text{ for some } Q \in \mathcal{Q}.$$

□

Lemma 4.4. Suppose that $\inf(\mathrm{P}) = \max(\mathrm{D}) = 0$. Then $\inf(\mathrm{P})$ is attained if and only if $\mathcal{Q}_e = \emptyset$.

Proof. Suppose that there is an optimal solution $\bar{\theta}$ to (P). By Lemma 4.3, $E_Q[Z_T^\top \bar{\theta}_T] = 0$ for all $Q \in \mathcal{Q}$. Fix $\varepsilon > 0$ such that $P(Z_T^\top \bar{\theta}_T \geq \varepsilon) > 0$ and set $A = \{\omega \in \Omega | Z_T(\omega)^\top \bar{\theta}_T(\omega) \geq \varepsilon\}$. Then $P(A) > 0$ and

$$Q(A) \leq \frac{1}{\varepsilon} E_Q[Z_T^\top \bar{\theta}_T \chi_A] \leq \frac{1}{\varepsilon} E_Q[Z_T^\top \bar{\theta}_T] = 0$$

for each $Q \in \mathcal{Q}$. Therefore $\mathcal{Q}_e = \emptyset$.

On the other hand, suppose that the infimum in (P) is not attained. Then there is no FL_P. By Theorem 2.3, there is an equivalent martingale measure Q^a for the price process. This measure can be regarded as the absolutely-continuous component of an equivalent finitely-additive martingale measure with the zero singular component. Therefore $\mathcal{Q}_e \neq \emptyset$. □

Theorem 4.5. The market admits no free lunch if and only if there exists an equivalent finitely-additive martingale measure for the price process.

Proof. Let us explore the duality results for the pair of problems (P) and (D). By Theorem 4.2, $\inf(\mathrm{P}) = \max(\mathrm{D})$. There are four possible outcomes:

(i) $\inf(\mathrm{P}) < 0$,

(ii) $\min(\mathrm{P}) = 0$,

(iii) $\inf(\mathrm{P}) = 0$ is not attained,

(iv) $\inf(\mathrm{P}) > 0$.

By the structure of (D):

(i) appears if and only if (D) has no feasible solution, which is equivalent to $\mathcal{Q} = \emptyset$.

(iv) appears if and only if (D) has a feasible solution such that $\operatorname{ess\,inf} y_T^a > 0$. That is equivalent to the existence of $Q \in \mathcal{Q}_e$ such that $q^a \geq \delta$ for some $\delta > 0$, a characterization of no $FLIL$ by Theorem 2.7.

(ii) or (iii) appears if and only if (D) has a feasible solution but all of them such that $\operatorname{ess\,inf} y_T^a = 0$. Moreover, in such case, (iii) appears if and only if $\mathcal{Q}_e \neq \emptyset$ by Lemma 4.4.

Indeed, there is no free lunch if and only if (iii) or (iv) appears. From what we have already shown, (iii) or (iv) appears if and only if $\mathcal{Q}_e \neq \emptyset$. □

This theorem is an extension of Theorem 2.3 in the framework of finitely-additive probability measures. Though its proof is based on duality results for (P) and (D), in the proof of Lemma 4.4 we have used the implication of Theorem 2.3 proved by some different separation argument.

5. ARBITRAGE–FREE PRICING

Now we state the arbitrage-free pricing results for contingent claims. We present two optimization problems associated with a contingent claim $\{F_t\}_{t=1}^T$, the *writer's pricing problem* (P_+) in which the writer of a contingent claim determines the price of the contingent claim as the lowest price such that he or she will be able to invest his or her earnings from the sale in the market to cover the cash flow by the investment, and the *buyer's pricing problem* (P_-) in which the buyer determines the price of a contingent claim as the highest price such that he or she will be able to cover his or her purchase by an investment on the market:

$$\begin{aligned} &\text{minimize } Z_0^\top \theta_0 \quad \text{over all} \quad \theta \in \Theta \\ &\qquad \text{subject to } Z_t^\top \Delta\theta_t \le -F_t \quad P\text{-a.s.} \quad t = 1, \dots, T \\ &\qquad Z_T^\top \theta_T \ge 0 \quad P\text{-a.s.} \end{aligned} \tag{P_+}$$

$$\begin{aligned} &\text{maximize } -Z_0^\top \theta_0 \quad \text{over all} \quad \theta \in \Theta \\ &\qquad \text{subject to } Z_t^\top \Delta\theta_t \le F_t \quad P\text{-a.s.} \quad t = 1, \dots, T \\ &\qquad Z_T^\top \theta_T \ge 0 \quad P\text{-a.s.} \end{aligned} \tag{P_-}$$

Analysis similar to that in Section 4 shows that the dual problem to (P_+) is

$$\begin{aligned} &\text{maximize } \sum_{t=1}^T \langle F_t, y_t\rangle_t \quad \text{over all } y \in Y \\ &\qquad \text{subject to } y \in Y_m \\ &\qquad y \ge 0 \end{aligned} \tag{D_+}$$

and the dual problem to (P_-) is

$$\begin{aligned} &\text{minimize } \sum_{t=1}^T \langle F_t, y_t\rangle_t \quad \text{over all } y \in Y \\ &\qquad \text{subject to } y \in Y_m \\ &\qquad y \ge 0. \end{aligned} \tag{D_-}$$

Theorem 5.1. We have $\inf(\mathsf{P}_+) = \max(\mathsf{D}_+)$ and $\sup(\mathsf{P}_-) = \min(\mathsf{D}_-)$.

Proof. Fix $\varepsilon > 0$ such that $\varepsilon \ge \operatorname{ess\,sup}(\sum_{t=1}^T F_t)$. Set

$$\tilde\theta_t = \Big((T+1-t)\varepsilon - \sum_{\tau=1}^t F_\tau, 0, \dots, 0\Big)^\top$$

for $t = 0, \dots, T$. Then

$$Z_t^\top \Delta\tilde{\theta}_t = -\varepsilon - F_t \quad t = 1, \dots, T$$

$$Z_T^\top \tilde{\theta}_T = \varepsilon - \sum_{t=1}^{T} F_t \geq 0 \ P\text{-a. s.}$$

From this it follows that the function $u \mapsto \tilde{F}(\tilde{\theta}, u)$ on U, where $\tilde{F}$ is the convenient perturbation function of problem (P_+), is bounded above on a norm-neighbourhood of $0 \in U$. Since $\tilde{F}$ is convex, Theorems 18(i) and 17 in [13] yield the former equation. The same reasoning, with F_t replaced by $-F_t$, applies to the latter case. □

Definition 5.2. A contingent claim $\{F_t\}_{t=1}^T$ is *attainable* if there is some trading strategy $\theta \in \Theta$, called *replicating strategy*, such that

$$\begin{aligned} Z_t^\top \theta_t &= Z_t^\top \theta_{t-1} - F_t \ P\text{-a. s.} \quad t = 1, \dots, T \\ Z_T^\top \theta_T &= 0 \ P\text{-a. s.} \end{aligned}$$

Clearly, if there is no free lunch and the contingent claim $\{F_t\}_{t=1}^T$ is attainable, then the optimal values of (P_+) and (P_-) coincide and are equal to $Z_0^\top \tilde{\theta}_0$ for each replicating strategy $\tilde{\theta}$. This value is the unique fair price of the attainable contingent claim.

Finally we present pricing results for a general contingent claim. The following theorem is an extension of Theorem 2.8.

Theorem 5.3. Suppose that the market admits no free lunch. Then the fair price of the contingent claim $\{F_t\}_{t=1}^T$ lays in the interval $[F_0^-, F_0^+] \subset \mathbb{R}$ where

$$F_0^+ = \max\Big\{\sum_{t=1}^{T} E_Q[F_t] \Big| Q \in \mathcal{Q}\Big\} = \sup\Big\{\sum_{t=1}^{T} E_Q[F_t] \Big| Q \in \mathcal{Q}_e\Big\}$$

$$F_0^- = \min\Big\{\sum_{t=1}^{T} E_Q[F_t] \Big| Q \in \mathcal{Q}\Big\} = \inf\Big\{\sum_{t=1}^{T} E_Q[F_t] \Big| Q \in \mathcal{Q}_e\Big\}$$

are the writer's and the buyer's price, respectively.

P r o o f. The fair price must not be greater that $F_0^+ = \inf(\mathsf{P}_+)$ or less than $F_0^- = \sup(\mathsf{P}_-)$, for otherwise the price would allow a risk-free profit in the market. By Theorem 5.1, we have $\inf(\mathsf{P}_+) = \max(\mathsf{D}_+)$ and $\sup(\mathsf{P}_-) = \min(\mathsf{D}_-)$. The set of feasible solutions to (D_+) coincides with that of (D_-) and corresponds to $\mathcal{Q}$, while the objective function corresponds to $\sum_{t=1}^T E_Q[F_t]$. The set of feasible solutions to (D_+) is nonempty since $\mathcal{Q} \supset \mathcal{Q}_e \neq \emptyset$ by Theorem 4.5. From this we obtain

$$-\infty < F_0^- = \min\Big\{\sum_{t=1}^{T} E_Q[F_t] \Big| Q \in \mathcal{Q}\Big\} \leq F_0^+ = \max\Big\{\sum_{t=1}^{T} E_Q[F_t] \Big| Q \in \mathcal{Q}\Big\} < +\infty.$$

It is not difficult to see that

$$F_0^+ = \sup\Big\{\sum_{t=1}^{T} E_Q[F_t]\Big| Q \in \mathcal{Q}_e\Big\} \text{ and } F_0^- = \inf\Big\{\sum_{t=1}^{T} E_Q[F_t]\Big| Q \in \mathcal{Q}_e\Big\},$$

since $\mathcal{Q}$ is a convex set and every non-trivial convex combination of an element of $\mathcal{Q}$ and that of $\mathcal{Q}_e$ is in $\mathcal{Q}_e$. This completes the proof. □

The theorem gains in interest if we realize that the endpoints of the arbitrage interval $[F_0^-, F_0^+]$ are actually attained as expected values of the contingent claim with respect to some finitely-additive martingale measures.

For an unattainable contingent claim $\{F_t\}_{t=1}^T$, if $F_0^- < F_0^+$ then any price F_0 such that $F_0^- < F_0 < F_0^+$ is a fair price for the contingent claim by the definition of problems (P_+) and (P_-). One may ask whether the writer's and the buyer's prices F_0^+ and F_0^- are themselves fair prices or not. In the market with finitely many states, it was shown in [11, Theorem 2] that F_0^+ and F_0^- are not fair prices unless the contingent claim is attainable. However, in the market model on general probability space, the question is unanswered.

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