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MONOTONICITY AND COMPARISON RESULTS FOR NONNEGATIVE DYNAMIC SYSTEMS
Part I: Discrete-Time Case

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In two subsequent parts, Part I and II, monotonicity and comparison results will be studied, as generalization of the pure stochastic case, for arbitrary dynamic systems governed by nonnegative matrices.

Part I covers the discrete-time and Part II the continuous-time case. The research has initially been motivated by a reliability application contained in Part II.

In the present Part I it is shown that monotonicity and comparison results, as known for Markov chains, do carry over rather smoothly to the general nonnegative case for marginal, total and average reward structures. These results, though straightforward, are not only of theoretical interest by themselves, but also essential for the more practical continuous-time case in Part II (see [9]). An instructive discrete-time random walk example is included.

Keywords: Markov chains, monotonicity, nonnegative matrices

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1. INTRODUCTION

1.1. Motivation: Stochastic case

Markov chains, both in discrete and continuous time, have proven to be most useful for the analysis of stochastic systems in a variety of application fields among which: *queuing, reliability, manufacturing, telecommunications, computer performance evaluation* and *financial analysis.*

In discrete time, the evolutionary characterization of such Markov chains and corresponding performance (or reward) measures can be described in mathematical (or functional) forms by:

$$\begin{cases} \boldsymbol{x}_{k+1} = \boldsymbol{P}\boldsymbol{x}_k, & k = 0, 1, \dots \quad \text{and} \\ \boldsymbol{W}_{k+1} = r + \boldsymbol{P}\boldsymbol{W}_k, & k = 0, 1, \dots \end{cases}$$

where k represents a time parameter, $\boldsymbol{x}_k$ a state vector and $\boldsymbol{W}_k$ a total or cumulative reward vector, while $\boldsymbol{P}$ is a one step transition matrix (kernel). This latter transition matrix (or kernel) thus essentially determines the dynamic behavior of the system.

Unfortunately, as the state space is generally large, the computational efforts can be rather extensive if not prohibitive. In the continuous-time case, as of practical interest for most of the applications, therefore, various results have been reported on the efficient computation of marginal or transient probabilities based on the so-called technique of uniformization (e. g. [11, 14]).

As a different approach, monotonicity and comparison results would be interest to:

- study the time-dependent behaviour of the systems
- investigate the dependence on initial conditions, or
- compare different systems.

Particularly, comparison results for related systems might be highly useful in order to:

- guarantee steady state bounds for time-dependent performance measures (e. g. like the availability of components for performability applications), or to
- facilitate the computation by modifying the system into a simplified system (e. g. a system that is more tractable on analytical basis).

A vast literature on monotonicity and comparison results has therefore appeared over the last decades for the stochastic case, motivated by the pioneering work of Stoyan [17]. Two approaches can roughly be distinguished.

(i) The stochastic comparison approach as based on stochastic monotonicity results for stochastic matrices or sample path comparison (e. g. [12, 13, 19, 20]), in line with the approach followed by Stoyan [17].

(ii) A Markov reward approach, as developed more recently, in order to also deal with reward structures or situations in which stochastic monotonicity cannot be proven (e. g. [1–6], [10]).

A more detailed discussion on the advantages and disadvantages of either approach be be found in Van Dijk [6].

1.2. Motivation: Nonnegative case

A natural mathematical extension of the Markovian structure given above, that is governed by a stochastic matrix $\boldsymbol{P}$ with row sums equal to one (or of the equivalent continuous-time generator), are generalized dynamic systems in which the matrix $\boldsymbol{P}$ is replaced by an arbitrary nonnegative matrix $\boldsymbol{M}$.

A classical application of substantial traditional interest is Leontieff's so-called input-output model in economic analysis. But also other non-stochastic applications are known such as of personnel and population flow models or Markov chains with special non-linear reward structures (cf. [16]).

Last but not least, a substantial subclass of practical interest is the class of substochastic models, with applications in financial analysis, insurance and reliability.

As analytical or computational results are even less available for the strict nonnegative case rather than for the pure stochastic case, the interest in monotonicity comparison results seems even more justified. But such results do not seem to be available.

This leads to the following objectives (for both Parts I and II).

1. To investigate to what extent monotonicity and comparison results for the stochastic case can be generalized to the general nonnegative case,
2. To illustrate and apply the possibly extensions by a special example or application of interest.

In this first part (Part I) the discrete-time case will be dealt with. In a subsequent paper (Part II) the continuous-time case and a practical reliability application that initiated the research will be studied (see [9]). To highlight the analogy but also the differences of the discrete- and continuous-time cases the two parts will have a parallel structure.

1.3. Results (Part I)

The following results will be established in Part I.

- Monotonicity results both in time and initial conditions,
- Comparison results for marginal, total (the transient case) and average rewards (the average case).

The approach that will be followed is a combination of the two approaches mentioned but most closely related to the reward approach as these reward measure are of specific interest. For this discrete-time case it turns out that most results for the stochastic case do carry over rather smoothly. Nevertheless, the results, even though straightforward, seem to be unreported. More precisely, for the marginal reward case (Section 4.1) they can be seen as direct generalization of results for the stochastic case as in [4] and [5]. But for the total and average reward case (Section 4.2 and 4.3) with results under less stronger monotonicity conditions (as specified in more detail in Remark 4.1), comparison results as in Results 4.2, 4.3 and 4.4 have also not been reported in these references for the pure stochastic case, as these rely upon so-called bias-terms as by a reward approach. These discrete-time results are thus of at least theoretical interest by covering the various dynamic (reward) situations and results in a unifying and self-contained manner. Moreover, the discrete-time results in their specific format will be required and used by direct referencing in the more practical continuous-time case in the subsequent Part II.

The technical verification of the conditions as well as the type of results that can be concluded by the general results will be illustrated by a random walk type example.

First in Section 2, the general discrete-time systems of interest and a number of motivational examples will be presented along with some existence results for limiting reward case. Monotonicity results will then be provided in Section 3 and comparison results are established in Section 4. Finally, Section 5 contains the results for an illustrative random walk example.

2. DISCRETE–TIME SYSTEMS OF INTEREST

2.1. General form

The general form of the systems of interest is given by the following recursions:
In functional form, for all $k = 0, 1, 2, \ldots$:

$$\begin{cases} \boldsymbol{x}_{k+1} = \boldsymbol{M}\boldsymbol{x}_k, & \boldsymbol{x}_0 = r \\ \boldsymbol{W}_{k+1} = r + \boldsymbol{M}\boldsymbol{W}_k, & \boldsymbol{W}_0 = 0 \end{cases}$$

or more detailed:

$$\begin{cases} \boldsymbol{x}_{k+1}(i) = \sum_j m(i,j)\, \boldsymbol{x}_k(j), \qquad \boldsymbol{x}_0(i) = r(i) & \text{(1a)} \\ \boldsymbol{W}_{k+1}(i) = r(i) + \sum_j m(i,j)\, \boldsymbol{W}_k(j), \qquad \boldsymbol{W}_0(i) = 0 & \text{(1b)} \end{cases}$$

where

$$\begin{cases} \boldsymbol{M} = (m(i,j)) & \text{is an arbitrary transition matrix on a finite or countable state space } S \text{ with strictly nonnegative elements,} \\ \boldsymbol{x}_k \text{ and } \boldsymbol{W}_k & \text{are vectors at } S\text{, for all } k = 0,1,2,\ldots\text{, and} \\ r & \text{represents a reward vector.} \end{cases}$$

Stochastic case

For the pure stochastic case we may read:

$$\boldsymbol{M} = \boldsymbol{P} \quad \text{with} \quad \sum_j \boldsymbol{P}(i,j) = 1 \quad \text{for all} \quad i,$$

where $\boldsymbol{P}$ represents the one-step transition matrix of a discrete-time Markov chain. In this case, it is natural to think of r to represent a reward vector. Accordingly,

$$\boldsymbol{x}_k(i) = \sum_j \boldsymbol{P}_k(i,j) r(j), \qquad \boldsymbol{x}_0(i) = r(i)$$

where $\boldsymbol{P}_k = \boldsymbol{P}^k$ is the transition probability matrix over k steps, which represents the expected reward incurred after k periods when starting in state i at time 0, and

$$\boldsymbol{W}_k(i) = \sum_{s=0}^{k-1} \boldsymbol{x}_k(i) = \sum_{s=0}^{k-1} \boldsymbol{P}^k(i,j) r(j) \qquad \text{with } \boldsymbol{W}_0(i) = 0$$

which represents the expected cumulative reward over k periods when starting in state i at time 0. We will therefore generally refer to the vectors $\boldsymbol{x}_k$ and $\boldsymbol{W}_k$ as:

$\boldsymbol{x}_k(i)$: the marginal (expected) rewards, and
$\boldsymbol{W}_k(i)$: the total cumulative (expected) reward.

Limits

Furthermore, under the condition that the corresponding limit exists, in analogy with the stochastic case, we will also be interested in

$$\boldsymbol{x} = \lim_{k\to\infty} \boldsymbol{x}_k \quad \text{as the } \textit{asymptotic reward} \text{ case,} \tag{2a}$$
$$\boldsymbol{W} = \lim_{k\to\infty} \boldsymbol{W}_k \quad \text{as the } \textit{transient} \text{ (or total reward) case, and} \tag{2b}$$
$$\boldsymbol{G} = \lim_{k\to\infty} k^{-1}\boldsymbol{W}_k \quad \text{as the } \textit{average} \text{ (or Cesaro limit) reward case.} \tag{2c}$$

Here the limits are assumed to exist in componentwise sense. For the stochastic case, the average reward limit $\boldsymbol{G}$ generally exists. For *irreducible* matrices (or chains) it is also known to be constant. In other words, in that case $\boldsymbol{G}(i) = g$ for some value g independent of the initial state i. Similarly, the limit $\boldsymbol{x}$ always exists when the chain is *aperiodic*. In that case, we also have $\boldsymbol{G} = \boldsymbol{x}$.

For the transient case, in contrast, the limit $\boldsymbol{W}$, provided it exists, will generally remain state dependent. For the existence one must typically think of a cumulative reward up to some time (first passage time) of leaving some set B, by $r(i) = 0$ for $i \notin B$.

For the general (nonstochastic) case, in Section 2.3 some technical conditions will be provided for these limits to exists as adopted from literature and [8].

2.2. Motivational examples

Beyond the class of Markov systems with its variety of applications, most notably among which queuing and inventory applications, let us mention some motivational examples which do not belong to the pure stochastic case.

2.2.1. Input–output model

One of the most classical class of models in economic analysis is *Leontieff's input-output* model which in dynamic form is given by:

$$\begin{cases} \boldsymbol{x}_{k+1}(i) = \sum_j a(j,i)\,\boldsymbol{x}_t(j), & \boldsymbol{x}_0(i) = \boldsymbol{x}(i), \quad (\textit{closed case}) \quad (3a) \\ \boldsymbol{x}_{k+1}(i) = d(i) + \sum_j a(j,i)\,\boldsymbol{x}_k(j), & \boldsymbol{x}_0(i) = 0, \quad (\textit{open case}) \quad (3b) \end{cases}$$

for all $i = 1, 2, \ldots, N$ and $k = 0, 1, 2, \ldots$, where

$$\begin{cases} \boldsymbol{x}_t(j) = \text{gross production of an industry (or sector) } j \text{ at time } t, \\ d(j) = \text{net (or final) demand (or output) of industry (sector) } j, \\ a(i,j) = \text{attribution rate of production from industry } i \text{ for industry } j. \end{cases}$$

Here the interpretation is that the production (and consumption) quantities of N industries (or sectors) with internal exchanges (productions for one another) are in full balance as input and output (consumption and production) per industry separately. One may note here that the system has the form with $\boldsymbol{M} = \boldsymbol{A}^T$, the transpose of matrix $\boldsymbol{A}$, and $\boldsymbol{W}_k = \boldsymbol{x}_k$ for the open case. Hence, even though $\boldsymbol{A}$ is stochastic, $\boldsymbol{M}$ will generally be not!

2.2.2. Personnel example

Let us consider a company whose personnel is divided into N ranks labelled by integers $1,\ldots,N$. Periodically, say at time points $t=0,1,\ldots,$ the status of the personnel is checked. Let $\boldsymbol{x}_k(i)$ be the expected number of personnel of rank i at time k. Then,

$$\boldsymbol{x}_{k+1}(i) = h(i) + \sum_j t(j,i)\,\boldsymbol{x}_k(j), \quad \text{where} \tag{4}$$

$h(i)$: the number of newly hired personnel in rank i within one period

$t(i,j)$: the probability that a person belonging to rank i will promote or demote to rank j within one period

Here we may either assume that $\sum_j t(i,j) = 1$, for all i, which we could consider as a growing nonstationary case, or that $\sum_j t(i,j) < 1$ for at least one i to reflect departures. Again note that the system has the form (1) if we set $\boldsymbol{M} = \boldsymbol{T}^T$, the transpose of the transfer matrix $\boldsymbol{T}$ (so that we do not have $\sum_j m(i,j) = 1$).

2.2.3. Substochastic models

Another substantial class of interest is the class of substochastic models, that is with

$$\begin{aligned} \sum_j m(i,j) &\leq 1 \qquad \text{for all } i, \text{ and} \\ \sum_j m(i,j) &< 1 \qquad \text{for at least one } i. \end{aligned}$$

This may reflect for instance that the system may have been left of stopped. For example for the personnel example it may reflect that a personnel member is retired or dropped out. Even though such situations could be modelled by a special state 0, for the results that will follow such states cannot be included, e. g. as it violates irreducibility such as in Proposition 2.1.

2.2.4. A random walk example

An example that resembles a random walk but which is not stochastic, consider the nonnegative matrix $\boldsymbol{M} = (m(i,j))$:

$$\boldsymbol{M} = \begin{bmatrix} \nu & 2\lambda & 0 & 0 & 0 & \ldots \\ \frac{1}{2}\nu & 0 & \frac{3}{2}\lambda & 0 & 0 & \ldots \\ 0 & \frac{2}{3}\nu & 0 & \frac{4}{3}\lambda & 0 & \ldots \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

more precisely, with

$$m(i,j) = \begin{cases} \lambda\,(i+2)/(i+1), & \text{for } j = i+1 \\ \nu(i)/(i+1), & \text{for } j = i-1 > 0 \\ \nu, & \text{for } j = i = 0. \end{cases}$$

This example will be analyzed more detailed in Section 5.

2.3. Some existence results

The following two propositions were proven in [8] (Propositions 3.1 and 3.2) as relying upon results from literature. Herein, we let u be the right eigenvector with eigenvalue ρ where ρ is the spectral radius, i. e. the largest eigenvalue of $\boldsymbol{M}$. Hence,

$$\rho u = \boldsymbol{M}u.$$

Proposition 2.1. $\rho = 1$. Assume that $\boldsymbol{M}$ is irreducible and that there exist left and right eigenvectors u and z, both with eigenvalue 1, i. e. with

$$u(i) = \sum_j m(i,j)u(j), \qquad \forall\, i \tag{5a}$$

$$z(j) = \sum_i m(i,j)z(i), \qquad \forall\, j \tag{5b}$$

$$\text{and scaled such that } \; z^T u = 1 \;\text{ or }\; \textstyle\sum_i z(i)u(i) = 1. \tag{5c}$$

Then,

$$\left.\begin{array}{l} \boldsymbol{G} \;\text{ exists,} \\ \frac{G(i)}{u(i)} = \sum_j z(j)r(j) \quad \text{(constant)} \quad \text{for all } \; i, \\ \text{and when } \boldsymbol{M} \text{ is aperiodic, also } \boldsymbol{x} \text{ exists and } \boldsymbol{x} = \boldsymbol{G}. \end{array}\right\} \tag{5d}$$

While Proposition 2.1 is appropriate for a notion of an average reward provided $\rho = 1$, when $\rho < 1$ or $\rho > 1$ the notion of an average reward is no longer meaningful. For $\rho < 1$, it will generally become 0, while for $\rho > 1$ its existence can no longer be concluded in general (only for special examples). For $\rho < 1$ or $\rho = 1$ the notion of a total cumulative reward (the transient case) is valid as according to the following proposition.

Proposition 2.2. (Transient case) The limit $\boldsymbol{W}$ exists and

$$\boldsymbol{W} = r + \boldsymbol{M}\boldsymbol{W}$$

under either of the following conditions

(i) $r = 0$

(ii) $\rho < 1$

(iii) $\rho = 1$

$r \geq 0$ and $\boldsymbol{G} = 0$

$\boldsymbol{M}$ reducible and $\lim_{n\to\infty} \boldsymbol{M}^n = \boldsymbol{M}^\star$ exists.

3. MONOTONICITY RESULTS

In this section we will focus our attention to monotonicity results for values

$$(\boldsymbol{x}_k|y) := \sum_i \boldsymbol{x}_k(i)y(i),$$

as extension of marginal expectations in the stochastic case. Here y represents an given initial condition $\boldsymbol{x}_0 = y$ at time 0, where y is some given vector $y = \{y(i)\}$.

More precisely, with $\boldsymbol{M}^k$ the kth power of $\boldsymbol{M}$, where $\boldsymbol{M}^0$ is the identity matrix, and given reward function r we can write

$$\boldsymbol{x}_k(i) = \sum_j \boldsymbol{M}^k(i,j)r(j) \tag{6}$$

so that

$$(\boldsymbol{x}_k \,|\, y) = \sum_i y(i)\boldsymbol{M}^k r(i) = \sum_i y(i) \sum_j \boldsymbol{M}^k(i,j)r(j). \tag{7}$$

Then we will be interested in the monotone behavior of $(\boldsymbol{x}_k \,|\, y)$ with respect to k (the time parameter) and y (the initial condition).

First, let us introduce a useful notion of a *monotonicity class* $\mathcal{M}$ with respect to the one-step transition matrix $\boldsymbol{M}$. In words, this monotonicity class $\mathcal{M}$ is required to remain closed (or invariant) under $\boldsymbol{M}$.

Condition 3.1. Let $\mathcal{M}$ be some class of functions $f : S \to \mathcal{R}$ under the condition that

$$f \in \mathcal{M} \;\Rightarrow\; \boldsymbol{M}f \in \mathcal{M}. \tag{8}$$

$\mathcal{M}$ will be referred to as *'monotonicity'* class as it will appear to preserve monotonicity properties in time and initial condition. Furthermore, in applications, as will be illustrated in Section 5.1 from this Part I as well as Section 5.1 from Part II (the reliability application) (see [9]), $\mathcal{M}$ will typically represent a class of functions with natural monotonicity properties such as increasing (or decreasing) in component numbers.

With this definition and restricted to this monotonicity class intuitively appealing results can now be established. The first result shows that an initial ordering is preserved by the nonnegative system governed by $\boldsymbol{M}$ for functions restricted to the monotonicity class $\mathcal{M}$. The next more practical result shows that an increase (or a decrease) in a step *single* step by the transition matrix $\boldsymbol{M}$ also leads to an increase (or decrease) throughout all time.

Lemma 3.1. (Monotonicity in initial condition) Let $\overline{y}$ and y be such that:

$$(f \,|\, \overline{y}) \geq (f \,|\, y), \qquad \text{for all} \;\; f \in \mathcal{M}. \tag{9}$$

Then also

$$(\boldsymbol{x}_k \,|\, \overline{y}) \geq (\boldsymbol{x}_k \,|\, y), \qquad \text{for all } \ k > 0 \ \text{ and any } r \in \mathcal{M}. \tag{10}$$

P r o o f. This follows immediately by induction in k, by noting that (10) holds for $k = 0$ by condition (9), by recalling that $\boldsymbol{M}r \in \mathcal{M}$ for $r \in \mathcal{M}$, and by writing

$$(\boldsymbol{M}^{k+1} r \,|\, \overline{y}) - (\boldsymbol{M}^{k+1} r \,|\, y) = (\boldsymbol{M}^{k}(\boldsymbol{M}r) \,|\, \overline{y}) - (\boldsymbol{M}^{k}(\boldsymbol{M}r) \,|\, y). \qquad \square$$

Lemma 3.2. (Monotonicity in time) Let y be such that

$$(\boldsymbol{M}f \,|\, y) \geq (f \,|\, y), \qquad \text{for all } \ f \in \mathcal{M}. \tag{11}$$

Then,

$$(\boldsymbol{M}^{k+1} f \,|\, y) \geq (\boldsymbol{M}^{k} f \,|\, y) \qquad \text{for all } k \text{ and } \ f \in \mathcal{M}. \tag{12}$$

P r o o f. Again this follows by induction in k, using (11), noting that $\boldsymbol{M}r \in \mathcal{M}$ for $r \in \mathcal{M}$, and writing

$$(\boldsymbol{M}^{m+1} r \,|\, y) - (\boldsymbol{M}^{m} r \,|\, y) = (\boldsymbol{M}^{m}(\boldsymbol{M}r) \,|\, y) - (\boldsymbol{M}^{m-1}(\boldsymbol{M}r) \,|\, y). \qquad \square$$

As a corollary monotone convergence to an asymptotic reward can so be concluded given that this asymptotic value exists for the given initial condition y.

Corollary 3.1. (Monotone convergence) Suppose that condition (11) holds with $\geq$ or $\leq$ sign for some vector y as well as that for some $r \in \mathcal{M}$ and value r^{∞}

$$(\boldsymbol{x}_k \,|\, y) \to r^{\infty}. \tag{13}$$

Then

$$(\boldsymbol{x}_k \,|\, y) \uparrow r^{\infty} \qquad \text{or} \qquad (\boldsymbol{x}_k \,|\, y) \downarrow r^{\infty}. \tag{14}$$

Remark 3.1. Note that condition (9) is required for all $f \in \mathcal{M}$. A natural situation for condition (9) to hold is when:

- the state space is ordered,
- f is monotone at S, and where
- $\overline{y}$ puts more weight on highly ordered states than y.

Remark 3.2. Corollary 3.1 is of practical interest as an asymptotic reward (or average value) is generally far more natural or easier to obtain (analytically or numerically) as a single value such as by solving the system $\boldsymbol{x} = \boldsymbol{M}\boldsymbol{x}$, than the series of marginal values $(\boldsymbol{x}_k|y)$. By Corollary 3.1 a secure upper or lower bound for these marginal values is then provided. In Part II for the continuous-time case a secure upper bound for the availability of a reliability system will be developed based upon Corollary 3.1.

4. COMPARISON RESULTS

As indicated in the introduction there can be a number of practical reasons for which it would be useful to compare a reward measure for two different systems, most notably:

(i) For what-if analysis with changing parameters,

(ii) For computational simplification.

While comparison results have received considerable attention for the stochastic case, for the more general case such results seem less available. Particularly, not in the setting of generalized reward structures. In this section, therefore, we will collect such results even though the steps are straightforward as direct extensions of the stochastic case, following the Markov reward approach (see [10]). We will distinguish

- the marginal reward case (Section 4.1)
- the total reward case (Section 4.2)
- the average case (Section 4.3)

To this end, let $\boldsymbol{M}$ and $\overline{\boldsymbol{M}}$ be the matrices for two nonnegative systems. Let the functions $\boldsymbol{x}_k$ and $\boldsymbol{W}_k$, as defined in Section 2, be denoted accordingly by $\overline{\boldsymbol{x}}_k$ and $\overline{\boldsymbol{W}}_k$ and assume that:

$$\boldsymbol{M} f \in \mathcal{M} \quad \text{and} \quad \overline{\boldsymbol{M}} f \in \mathcal{M}, \quad \text{for all} \quad f \in \mathcal{M}. \tag{15}$$

Furthermore, ordering for vectors is assumed to be componentwise, i. e. $u \geq v$ if and only if $u(i) \geq v(i)$ for all i.

4.1. Marginal reward case and related results

First, we will show that a one-step ordering with respect to the one-step matrix $\boldsymbol{M}$ is preserved for any number of steps provided we restrict ourselves to the monotonicity class $\mathcal{M}$.

Result 4.1. If

$$\overline{\boldsymbol{M}} f \geq \boldsymbol{M} f \qquad \text{for all} \quad f \in \mathcal{M} \tag{16}$$

then for any $r \in \mathcal{M}$:

$$\overline{\boldsymbol{x}}_k \geq \boldsymbol{x}_k \quad \text{for all } k \tag{17}$$

and, under the condition that $\overline{\boldsymbol{x}} = \lim_{k\to\infty} \overline{\boldsymbol{x}}_k$ and $\boldsymbol{x} = \lim_{k\to\infty} \boldsymbol{x}_k$ exist:

$$\overline{\boldsymbol{x}} \geq \boldsymbol{x}. \tag{18}$$

Proof. Again this will follow by induction in k. (17) holds for $k = 1$ by (16). Assume that (17) holds for $k \leq m$. Also note that $\boldsymbol{M}^m r \in \mathcal{M}$ as $\mathcal{M}$ is closed for $\boldsymbol{M}$ and that $\overline{\boldsymbol{M}} v \geq 0$ for any $v \geq 0$. From these facts, we can conclude:

$$(\overline{\boldsymbol{M}}^{m+1} r) - (\boldsymbol{M}^{m+1} r) = \overline{\boldsymbol{M}}(\overline{\boldsymbol{M}}^m r - \boldsymbol{M}^m r) + (\overline{\boldsymbol{M}} - \boldsymbol{M})(\boldsymbol{M}^m r) \geq 0$$

by which relation (17) follows also for $k = m+1$. The induction concludes the proof of (17) with with (18) as an immediate consequence. □

By Result 4.1 also the following corollary for the total reward, the transient reward and average reward case is concluded immediately by the observation:

$$\boldsymbol{W}_k = \sum_{s=0}^{k-1} \boldsymbol{M}^s r = \sum_{s=0}^{k-1} \boldsymbol{x}_s. \tag{19}$$

Corollary 4.1. For any $r \in \mathcal{M}$ and under condition (16)

$$\overline{\boldsymbol{W}}_k \geq \boldsymbol{W}_k \qquad \forall\, k$$

and if, the limits from (2) exist,

$$\begin{aligned} \overline{\boldsymbol{W}} &\geq \boldsymbol{W} \\ \overline{\boldsymbol{G}} &\geq \boldsymbol{G}. \end{aligned}$$

Remark 4.1. In the next two sections, the transient and average case will also be dealt with separately for a number of reasons:

1. Condition (16) is generally quite strong (as corresponding to stochastic ordering conditions) and can be violated.
2. Condition (16) can be hard to verify as it requires ordering for all $f \in \mathcal{M}$. In fact, it may even be difficult to determine $\mathcal{M}$ itself.
3. Comparison results can be of interest for one specific (type) reward function r.
4. Average comparison results might be established for functions $r \notin \mathcal{M}$. That is, without underlying monotonicity properties. (See for instance [10] for the stochastic case.)
5. As also the reward function r itself may differ for the systems to be compared, i. e. $\overline{r} \neq r$.

Remark 4.2. Clearly, Result 4.1 and Corollary 4.1 also hold for reward functions $r \in \mathcal{R}$ such that $\mathcal{R} \subset \mathcal{M}$ is closed under $\boldsymbol{M}$, i. e. $\boldsymbol{M}r \subset \mathcal{R}$ for any $r \in \mathcal{R}$, and such that (16) holds for all $f \in \mathcal{R}$. In other words, we do not necessarily have to determine $\mathcal{M}$ but just a sufficient subclass. (This will be used in Section 5.)

4.2. Total reward case

In contrast with the preceding section for marginal rewards, which requires a (strong) comparison for all $f \in \mathcal{M}$ (as related to stochastic comparison), for a comparison of a total reward $\boldsymbol{W}_k$, only a comparison for the specific reward function r may be sufficient.

To this end, consider equation (1) for:

$$\begin{array}{ll} \boldsymbol{W}_k & \text{with} \quad (r, \boldsymbol{M}) \\ \overline{\boldsymbol{W}}_k & \text{with} \quad (\overline{r}, \overline{\boldsymbol{M}}). \end{array}$$

A direct comparison result is the following. Though the result and its proof are straightforward, we include it for completeness as well as for its use in Part II (see Part II, Results 3.3 and 3.4).

Result 4.2.

$$\overline{\boldsymbol{W}}_k \geq \boldsymbol{W}_k, \tag{20}$$

if for all $\ell \leq k$,

$$(\overline{r} - r) + (\overline{\boldsymbol{M}} - \boldsymbol{M})\boldsymbol{W}_\ell \geq 0. \tag{21}$$

Proof. By virtue of (1) we can write

$$\begin{aligned} \overline{\boldsymbol{W}}_k - \boldsymbol{W}_k &= (\overline{r} - r) + \overline{\boldsymbol{M}}\,\overline{\boldsymbol{W}}_{k-1} - \boldsymbol{M}\boldsymbol{W}_{k-1} \\ &= (\overline{r} - r) + (\overline{\boldsymbol{M}} - \boldsymbol{M})\boldsymbol{W}_{k-1} + \overline{\boldsymbol{M}}(\overline{\boldsymbol{W}}_{k-1} - \boldsymbol{W}_{k-1}). \end{aligned}$$

Hence, by iteration

$$(\overline{\boldsymbol{W}}_k - \boldsymbol{W}_k) = \sum_{s=0}^{k-1} \overline{\boldsymbol{M}}^s [(\overline{r} - r) + (\overline{\boldsymbol{M}} - \boldsymbol{M})\boldsymbol{W}_{k-1-s}] + \overline{\boldsymbol{M}}^k (\overline{\boldsymbol{W}}_0 - \boldsymbol{W}_0). \tag{22}$$

By using that $\overline{\boldsymbol{W}}_0 = \boldsymbol{W}_0 = 0$ and noting that $\overline{\boldsymbol{M}}$ (and hence $\overline{\boldsymbol{M}}^s$) is a monotone operator (i. e. $\overline{\boldsymbol{M}}g \geq 0$ if $g \geq 0$), the proof of (20) is completed by combining (21) and (22). □

Condition (21) may still be hard to verify as negative values may arise in $(\overline{\boldsymbol{M}} - \boldsymbol{M})(i, j)$. In analogy with the stochastic case (see [5]), the following result can therefore be more convenient.

Result 4.3. Suppose that for all i:

$$\sum_j \overline{m}(i, j) = \sum_j m(i, j). \tag{23}$$

Then (20) also holds if for all $\ell \leq k$ and all i

$$(\overline{r} - r)(i) + \sum_j [\overline{m}(i, j) - m(i, j)][\boldsymbol{W}_\ell(j) - \boldsymbol{W}_\ell(i)] \geq 0. \tag{24}$$

Proof. Immediate by Result 4.2 and writing

$$[\overline{\boldsymbol{M}} - \boldsymbol{M}]\boldsymbol{W}_\ell(i) = \sum_j [\overline{m}(i, j) - m(i, j)]\boldsymbol{W}_\ell(j) = \sum_j [\overline{m}(i, j) - m(i, j)][\boldsymbol{W}_\ell(j) - \boldsymbol{W}_\ell(i)].$$

□

Remark 4.3. The advantage of (24) over (21) arises for example when negative values can be involved for either $\overline{r} - r$ or $[\overline{\boldsymbol{M}} - \boldsymbol{M}]$. For the stochastic case, most notably for queuing network applications, comparison results could also be concluded for specific reward or comparison structures while the underlying stochastic processes were not stochastically ordered (e. g. see [10]). In such situations it can be sufficient to show that

$$[\boldsymbol{W}_k(j) - \boldsymbol{W}_k(i)] \geq 0 \qquad \text{for all} \;\; j, i \;\; \text{such that} \;\; [\overline{m}(i,j) - m(i,j)] > 0.$$

This in turn might be proved by induction based upon the recursive relation (1).

4.3. Average reward case

Also for the average reward case, similar to the total reward case, three types of results can be provided.

(i) Clearly, a first result, provided the average reward limits exist for the systems to be compared, has already been provided by Corollary 4.1; that is, by limits of the marginal reward case. However, this result requires the strong condition (16) and the same reward functions $\overline{r} = r$.

(ii) As a second result, which does allow different reward functions $\overline{r}$ and r, Result 4.2 can be adopted directly with $\overline{\boldsymbol{W}}_k$ and $\boldsymbol{W}_k$ replaced by $\overline{\boldsymbol{G}}$ and $\boldsymbol{G}$, provided (21) holds for all $\ell > 0$. However, while Result 4.2 seems appropriate for the transient case as it often concerns substochastic cases, it seems less so for the average case. More precisely, for the same reason as also mentioned under Result 4.2, ($[\overline{\boldsymbol{M}}(i,j) - \boldsymbol{M}(i,j)]$ may lead to both negative and positive values) condition (16) can be too restrictive.

(iii) As a third result, therefore, an extension of the Markov reward case may be thought of similar to Result 4.3. To this end, difference (or so-called bias) terms of total reward functions are used. These difference terms may enable one to verify monotonicity conditions in an analytic and iterative manner as will be illustrated in the proof of Lemma 5.2.

As extension of the stochastic case, however, row sum of the matrices $\overline{\boldsymbol{M}}$ and $\boldsymbol{M}$ will have to be scaled as these are not necessarily equal. To this end, in correspondence with the so-called Balanced Normalization in [8] (see assumption 4.2), we make the following condition. This condition seems to be natural in situations when the average case is well defined (that is, when the limits $\boldsymbol{G}$ and $\overline{\boldsymbol{G}}$ exist).

Assumption 4.1. (Balanced Normalization) There exist a function $\mu(i)$ such that for all i:

$$\sum_{j \in S} m(i,j)\, \mu(j) = \mu(i) \tag{25a}$$

$$\sum_{j \in S} \overline{m}(i,j)\, \mu(j) = \mu(i). \tag{25b}$$

The following generalized form of Result 4.3 can then be proven in which the vectors $\boldsymbol{s}$, $\boldsymbol{V}_k$ and the matrix $\boldsymbol{T}$ (and similarly $\overline{\boldsymbol{s}}$, $\overline{\boldsymbol{V}}_k$ and $\overline{\boldsymbol{T}}$) are defined by

$$\boldsymbol{s}(i) := r(i)/\mu(i), \tag{26a}$$

$$\boldsymbol{V}_k(i) := \boldsymbol{W}_k(i)/\mu(i), \tag{26b}$$

$$\boldsymbol{T}(i,j) := m(i,j)\mu(j)/\mu(i). \tag{26c}$$

Result 4.4.

$$\overline{\boldsymbol{G}} \geq \boldsymbol{G} \qquad \text{(provided these limits exists)} \tag{27}$$

if for all i and k:

$$(\overline{\boldsymbol{s}} - \boldsymbol{s})(i) + \sum_j [\overline{\boldsymbol{T}}(i,j) - \boldsymbol{T}(i,j)][\boldsymbol{V}_k(j) - \boldsymbol{V}_k(i)] \geq 0. \tag{28}$$

P r o o f . First note that:

$$(\overline{\boldsymbol{V}}_{k+1} - \boldsymbol{V}_{k+1}) = (\overline{\boldsymbol{s}} - \boldsymbol{s}) + \overline{\boldsymbol{T}}\,\overline{\boldsymbol{V}}_k - \boldsymbol{T}\boldsymbol{V}_k \tag{29}$$

so that, as in the proof of Result 4.2, by iteration

$$(\overline{\boldsymbol{V}}_k - \boldsymbol{V}_k) = \sum_{\ell=0}^{k-1} \overline{\boldsymbol{T}}^{\ell}[(\overline{\boldsymbol{s}} - \boldsymbol{s}) + (\overline{\boldsymbol{T}} - \boldsymbol{T})\boldsymbol{V}_{k-1-\ell}]. \tag{30}$$

Now recall that $\overline{\boldsymbol{T}}$ and thus also $\overline{\boldsymbol{T}}^{\ell}$ are nonnegative and therefore monotone operators. Furthermore observe that for any t and i:

$$\begin{aligned}(\overline{\boldsymbol{T}} - \boldsymbol{T})\boldsymbol{V}_t(i) &= \sum_j [\overline{\boldsymbol{T}}(i,j) - \boldsymbol{T}(i,j)]\boldsymbol{V}_t(j) \\ &= \sum_j [\overline{\boldsymbol{T}}(i,j) - \boldsymbol{T}(i,j)][\boldsymbol{V}_t(j) - \boldsymbol{V}_t(i)] \end{aligned} \tag{31}$$

as by the balanced normalization condition (25). By (30) and (31) we now easily conclude: $\overline{\boldsymbol{V}}_k \geq \boldsymbol{V}_k$ for all k. By recalling (26b), dividing by k, and taking the limit for $k \to \infty$, the proof is then completed. □

In the next section we will illustrate how Result 4.4 can be applied, particularly, how condition (28) can be verified in a concrete example.

5. A RANDOM WALK TYPE EXAMPLE

For illustrative purposes let us reconsider the random walk type example from Section 2.2.4. That is with

$$m(i,j) = \begin{cases} \lambda\,(i+2)/(i+1), & \text{for } j = i+1 \\ \nu(i)/(i+1), & \text{for } j = i-1 > 0 \\ \nu, & \text{for } j = i = 0 \end{cases}$$

where without restriction of generality we have assumed that $[\lambda + \mu] = 1$.
Then by straightforward calculus an important observation to make is that

$$\boldsymbol{M} r \notin \mathcal{N} \quad \text{for any} \quad r \in \mathcal{N} \quad \text{with} \quad \mathcal{N} = \{r : \mathcal{N} \to \mathcal{R} \ | r(i+1) \geq r(i) \text{ for all } i\}.$$

In other words, nondecreasing functions are not preserved under $\boldsymbol{M}$. This also implies that the cumulative reward functions $\boldsymbol{W}_k$ are not necessarily nondecreasing ($\boldsymbol{W}_k \notin \mathcal{N}$) for reward functions $r \in \mathcal{N}$.

As a consequence, the monotonicity and comparison results from Sections 3 and 4, particularly Corollary 4.1, cannot be applied for nondecreasing functions $r \in \mathcal{N}$. (One may read again through the proof of Result 4.1 to see why this is needed.) In Lemma 5.1 we therefore first establish a suitable monotonicity class for $\boldsymbol{M}$. Herein, let μ be given by

$$\mu(i) = (i+1)^{-1}. \tag{32}$$

Lemma 5.1. $\mathcal{M}$ is closed under $\boldsymbol{M}$ with $\mathcal{M}$ given by

$$\mathcal{M} = \left\{ f \middle| \frac{f}{\mu} \in \mathcal{N} \text{ and } f \geq 0 \right\} \tag{33}$$

P r o o f. We need to show that $\boldsymbol{M} f \in \mathcal{M}$ for $f \in \mathcal{M}$. This follows by

$$\begin{aligned} &(i+2)(\boldsymbol{M} f)(i+1) - (i+1)(\boldsymbol{M} f)(i) \\ &= \left[(i+2)\frac{(i+3)}{(i+2)}\lambda f(i+2) + (i+2)\frac{(i+1)}{(i+2)}\nu f(i) \right] \\ &- \left[(i+1)\frac{(i+2)}{(i+1)}\lambda f(i+1) + (i+1)\frac{i}{(i+1)}\nu f(i-1) \right] \\ &= \lambda[(i+3)f(i+2) - (i+2)f(i+1)] + \nu[(i+1)f(i) - i f(i-1)] \geq 0 \end{aligned}$$

where the last inequality follows by the assumption that f/μ is nondecreasing. □

Now that we have established a monotonicity class for $\boldsymbol{M}$ we can apply the results from Sections 3 and 4. For the purpose of illustration we restrict the presentation to the results of most interest while leaving other to the reader.

5.1. Monotonicity

5.1.1. Initial condition

Clearly, with $\mathcal{M}$ as defined above, we can directly apply Lemma 3.1 as for $f \in \mathcal{M}$

$$(f \,|\, \overline{y}) \geq (f \,|\, y) \tag{34}$$

when $\overline{y}$ as compared to y puts more weight on highly ordered states, up to scaling by the function $\mu(i) = (i+1)^{-1}$. For example, with

$$y = 1_{\{i=0\}}, \qquad \overline{y} = 3 \cdot 1_{\{i=2\}}$$

we verify (34) by:

$$(f \,|\, \overline{y}) = 3f(3) \geq 3 \cdot \frac{2}{3} \cdot f(1) \geq 3 \cdot \frac{2}{3} \cdot \frac{1}{2} \cdot f(0) = f(0) = (f \,|\, y).$$

5.1.2. Monotonicity in time

Result 5.1. For $y = 1_{\{i=0\}}$:

$$(\boldsymbol{M}^k f \,|\, y) \uparrow, \qquad \forall\, f \in \mathcal{M}.$$

Proof. By Lemma 3.2 it suffices to verify condition (11), i. e.:

$$(\boldsymbol{M} f \,|\, y) \geq (f \,|\, y), \qquad \forall\, f \in \mathcal{M}.$$

This follows by

$$(\boldsymbol{M} f \,|\, y) - (f \,|\, y) = \boldsymbol{M} f(0) - f(0)$$
$$= \left[\lambda \frac{2}{1} f(1) + \nu f(0)\right] - [\lambda + \nu]\, f(0) = \lambda[2f(1) - f(0)] \geq 0.$$

Here we have used the scaling assumption $[\lambda + \mu] = 1$ and the fact that $f \in \mathcal{M}$. □

This results are particularly of interest if there exists an asymptotic reward for the given initial condition $y = 1_{\{i=0\}}$. This will be concluded below for two example functions $r(i)$.

5.1.3. Monotone convergence

Result 5.2. Let $y = 1_{\{i=0\}}$:
Then

$$(\boldsymbol{M}^k r \,|\, y) = (\boldsymbol{x}_k \,|\, y) \uparrow r^\infty$$

for

$$\begin{cases} \frac{r(i)}{\mu(i)} = i & \text{with } \; r^\infty = \frac{\lambda}{\nu - \lambda} \\ \frac{r(i)}{\mu(i)} = 1_{\{i \geq k\}} & \text{with } \; r^\infty = \left(\frac{\lambda}{\nu}\right)^k. \end{cases}$$

Proof. The proof is a direct consequence of Corollary 3.1 and Result 5.1 provided the asymptotic convergence to or rather existence of the average reward r^∞ has been shown. This, in turn, can be verified by Proposition 3.1 and its Corollary 3.1 from Part I, as also summarized by Proposition 2.1 in Section 2, by choosing:

$$\begin{cases} u(i) = \mu(i) = (i+1)^{-1} & \text{and} \\ z(i) = (i+1)\left(\frac{\lambda}{\nu}\right)^i \left(1-\frac{\lambda}{\nu}\right)^{-1}. \end{cases}$$

To this end (see Proposition 2.1) we need to verify (5a), (5b) and (5c).

(5a) is checked by:

$$\begin{cases} (i+1)\left[\lambda \cdot \frac{i+2}{i+1} \cdot \frac{1}{i+2}\nu \cdot \frac{i}{i+1} \cdot \frac{1}{i}\right] = 1 & \text{for } i \geq 0 \\ 2 \cdot \left[\lambda \cdot 2 \cdot \frac{1}{2} + \nu\right] = 1 & \text{for } i = 0. \end{cases}$$

(5b) by:

$$\sum_i m(i,j)z(i) = j\lambda\left(\frac{j+1}{j}\right)\left(\frac{\lambda}{\nu}\right)^{j-1}\left(1-\frac{\lambda}{\nu}\right) + (j+2)\nu\left(\frac{j+1}{j+2}\right)\left(\frac{\lambda}{\nu}\right)^{j+1}\left(1-\frac{\lambda}{\nu}\right)$$

$$= \begin{cases} (j+1)\left(\frac{\lambda}{\nu}\right)^j\left(1-\frac{\lambda}{\nu}\right)[\lambda+\nu] = z(j)[\lambda+\nu] = z(j) & \text{for } j > 0 \\ \nu\frac{1}{2}2\frac{\lambda}{\nu}\left(1-\frac{\lambda}{\nu}\right) + \nu\left(1-\frac{\lambda}{\nu}\right) = \left(1-\frac{\lambda}{\nu}\right) = z(0) & \text{for } j = 0. \end{cases}$$

(5c) by:

$$\sum_i z(i)u(i) = \sum_i \left(\frac{\lambda}{\nu}\right)^i \cdot \left(1-\frac{\lambda}{\nu}\right) = 1.$$

With $r^\infty = \boldsymbol{G}(0)$ in Proposition 2.1, the values r^∞ are now obtained by

$$r^\infty = \sum_j r(j)(j+1)\left(\frac{\lambda}{\nu}\right)^j\left[1-\frac{\lambda}{\nu}\right] = \left[1-\frac{\lambda}{\nu}\right]\sum_j r(j)\left[\frac{\lambda}{\nu}\right]^j. \qquad \square$$

5.2. Comparison results

Let us consider the effect of an increase in λ, i.e. with

$$\overline{\lambda} \geq \lambda.$$

Then condition (16) is readily verified by

$$(\overline{\boldsymbol{M}}f - \boldsymbol{M}f)(i) = [\overline{\lambda} - \lambda]\left[\frac{i+2}{i+1}\right] f(i+1) \geq 0 \qquad \text{for } f \in \mathcal{M}.$$

Result 5.3. For any $r \geq 0$ with $r \in \mathcal{M}$ and all k:

$$\overline{\boldsymbol{x}}_k \geq \boldsymbol{x}_k \quad \text{and} \quad \overline{\boldsymbol{W}}_k \geq \boldsymbol{W}_k$$

and, as far as the limits exist:

$$\overline{\boldsymbol{G}} \geq \boldsymbol{G} \quad \text{or} \quad \overline{\boldsymbol{W}} \geq \boldsymbol{W}.$$

P r o o f . This follows as a direct consequence of Lemma 5.1, Result 4.1 and Corollary 4.1, where we also refer to Remark 4.2. □

In order to illustrate how we can also apply Result 4.4, which in addition allows us to use different reward functions $\overline{r}$ and r, first note that the balanced normalization condition (25) is verified with μ given by (32).

$$\sum_j m(i,j)\mu(j) = \mu(i) \qquad \sum_j \overline{m}(i,j)\mu(j) = \mu(i).$$

Furthermore, with T as given by (26c):

$$T(i,j) = \begin{cases} \lambda & \text{for } j = i+1 \\ \nu, & \text{for } j = i-1 > 0 \\ \nu, & \text{for } j = i = 0 \end{cases} \qquad \text{or} \qquad \boldsymbol{T} = \begin{bmatrix} \nu & \lambda & 0 & 0 & \dots \\ \nu & 0 & \lambda & 0 & \dots \\ 0 & \nu & 0 & \lambda & \dots \\ 0 & 0 & \nu & 0 & \lambda \end{bmatrix}$$

and similarly for $\overline{\boldsymbol{T}}$ with $\overline{\lambda}$. With this transformation and $\boldsymbol{V}_k = \boldsymbol{W}_k/\mu$ we can now prove.

Lemma 5.2. For any $r \in \mathcal{M}$; that is r with r/λ nondecreasing and positive:

$$\boldsymbol{V}_k \in \mathcal{N} \qquad (k \geq 0). \tag{35}$$

P r o o f . The proof will be given by induction in k. Suppose that (35) holds for $t \leq k$. Then for $t = k+1$ and by the form of $\boldsymbol{T}$ we obtain

$$\boldsymbol{V}_{k+1}(i+1) - \boldsymbol{V}_{k+1}(i)$$

$$= [(i+2)r(i+1) - (i+1)r(i)]$$

$$+\lambda[\boldsymbol{V}_k(i+2) - \boldsymbol{V}_k(i+1)] + \nu 1_{\{i>0\}}[\boldsymbol{V}_k(i) - \boldsymbol{V}_k(i-1)] \geq 0. \tag{36}$$

Here the last inequality is concluded from the induction hypothesis (35) for $t = k$ and $r \in \mathcal{M}$. The induction completes the proof. □

Result 5.4. For any $\overline{r}, r \in \mathcal{M}$ with $\overline{r} \geq r$:

$$\overline{\boldsymbol{G}}(i) \geq \boldsymbol{G}(i) \qquad \text{for all} \quad i. \tag{37}$$

P r o o f. To apply Result 4.4, the left hand side of (28) reduces to

$$\left(\frac{\overline{r}}{\mu} - \frac{r}{\mu}\right)(i) + (\overline{\lambda} - \lambda)[\boldsymbol{V}_k(i+1) - \boldsymbol{V}_k(i)].$$

By the condition $\frac{\overline{r}}{\mu} \geq \frac{r}{\mu}$ and Lemma 5.2, condition (28) is thus verified. Result 4.4 with $\overline{\mu} = \mu$ completes the proof. □

Example. As a special example to support Results 5.3 and 5.4, let

$$r(i) = 1_{\{i>0\}}.$$

In that case, based on Proposition 2.1 and by choosing $u = \mu$, it can be shown that

$$(i+1)\boldsymbol{G}(i) = \frac{\lambda}{\nu},$$

from which (37) follows immediately with $\overline{\lambda} \geq \lambda$, as could also have been concluded directly by Results 5.3 or 5.4.

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