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LINEAR COMBINATION, PRODUCT AND RATIO OF NORMAL AND LOGISTIC RANDOM VARIABLES

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The distributions of linear combinations, products and ratios of random variables arise in many areas of engineering. In this note, the exact distributions of $\alpha X+\beta Y$, $|XY|$ and $|X/Y|$ are derived when X and Y are independent normal and logistic random variables. The normal and logistic distributions have been two of the most popular models for measurement errors in engineering.

Keywords: linear combination of random variables, logistic distribution, normal distribution, products of random variables, ratios of random variables

AMS Subject Classification: 62E15

1. INTRODUCTION

The distributions of linear combinations, products and ratios of random variables arise in many areas of engineering. In this note, we study the exact distributions of $\alpha X + \beta Y$, $|XY|$ and $|X/Y|$ when X and Y are independent random variables having the normal and logistic distributions with pdfs

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} \tag{1}$$

and

$$f_Y(y) = \frac{1}{\phi} \exp\left(-\frac{y-\lambda}{\phi}\right) \left\{1+\exp\left(-\frac{y-\lambda}{\phi}\right)\right\}^{-2}, \tag{2}$$

respectively, for $-\infty < x < \infty$, $-\infty < y < \infty$, $-\infty < \mu < \infty$, $-\infty < \lambda < \infty$, $\sigma > 0$ and $\phi > 0$. We assume without loss of generality that $\alpha > 0$. Note that the normal and logistic distributions are two of the most popular models for measurement errors in engineering.

The calculations of this note involve several special functions, including the complementary error function defined by

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} \exp\left(-t^2\right) \mathrm{d}t$$

and the hypergeometric function defined by

$$G(a; b, c, d; x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k (c)_k (d)_k} \frac{x^k}{k!},$$

where $(e)_k = e(e+1)\cdots(e+k-1)$ denotes the ascending factorial. We also need the following important lemmas.

Lemma 1. (Equation (2.8.9.1), Prudnikov et al. [2, Vol. 2].) For $p > 0$,

$$\begin{aligned} & \int_0^{\infty} x^n \exp(-px) \operatorname{erfc}(cx+b)\, dx \\ = \; & (-1)^n \frac{\partial^n}{\partial p^n} \left\{ \frac{1}{p} \operatorname{erfc}(b) - \frac{1}{p} \exp\left(\frac{p^2 + 4pbc}{4c^2} \right) \operatorname{erfc}\left(b + \frac{p}{2c} \right) \right\}. \end{aligned}$$

Lemma 2. (Equation (2.8.5.14), Prudnikov et al. [2, Vol. 2]) For $p > 0$,

$$\begin{aligned} & \int_0^{\infty} x^{\alpha-1} \exp(-p/x) \operatorname{erfc}(cx)\, dx \\ = \; & p^{\alpha}\Gamma(-\alpha) - \frac{2cp^{\alpha+1}}{\sqrt{\pi}} \Gamma(-\alpha-1) G\left(\frac{1}{2}; \frac{3}{2}, \frac{3+\alpha}{2}, 1+\frac{\alpha}{2}; -\frac{c^2p^2}{4} \right) \\ & + \frac{1}{c^{\alpha}\sqrt{\pi}\alpha} \Gamma\left(\frac{\alpha+1}{2} \right) G\left(-\frac{\alpha}{2}; 1-\frac{\alpha}{2}, \frac{1}{2}, \frac{1-\alpha}{2}; -\frac{c^2p^2}{4} \right) \\ & + \frac{p}{c^{\alpha-1}\sqrt{\pi}(1-\alpha)} \Gamma\left(\frac{\alpha}{2} \right) G\left(\frac{1-\alpha}{2}; \frac{3-\alpha}{2}, 1-\frac{\alpha}{2}, \frac{3}{2}; -\frac{c^2p^2}{4} \right). \end{aligned}$$

Further properties of the complementary error function can be found in Prudnikov et al. [2] and Gradshteyn and Ryzhik [1].

2. LINEAR COMBINATION

Theorem 1 derives an explicit expression for the cdf of $\alpha X + \beta Y$ in terms of the complementary error function.

Theorem 1. Suppose X and Y are distributed according to (1) and (2), respectively. Then, the cdf of $Z = \alpha X + \beta Y$ can be expressed as

$$F(z) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{k+1} \binom{-2}{k} \left\{ 2\operatorname{erfc}\left(\frac{\beta\lambda + \alpha\mu - z}{\sqrt{2}\alpha\sigma} \right) - G_k(z) - H_k(z) \right\}, \quad (3)$$

where

$$\begin{aligned} G_k(z) = \; & \exp\left\{ \frac{(k+1)^2}{\phi^2} + \frac{2\beta(k+1)(\beta\lambda + \alpha\mu - z)}{\alpha^2\sigma^2\phi} \right\} \\ & \operatorname{erfc}\left(\frac{\beta\lambda + \alpha\mu - z}{\sqrt{2}\alpha\sigma} + \frac{(k+1)\alpha\sigma}{\sqrt{2}\beta\phi} \right) \end{aligned} \quad (4)$$

and

$$H_k(z) = \exp\left\{ \frac{(k+1)^2}{\phi^2} - \frac{2\beta(k+1)(\beta\lambda + \alpha\mu - z)}{\alpha^2\sigma^2\phi} \right\} \operatorname{erfc}\left(\frac{\beta\lambda + \alpha\mu - z}{\sqrt{2}\alpha\sigma} - \frac{(k+1)\alpha\sigma}{\sqrt{2}\beta\phi} \right). \quad (5)$$

Proof. The cdf $F(z) = \Pr(\alpha X + \beta Y \leq z)$ can be expressed as

$$\begin{aligned} F(z) &= \frac{1}{\phi} \int_{-\infty}^{\infty} \exp\left(-\frac{y-\lambda}{\phi} \right) \left\{ 1 + \exp\left(-\frac{y-\lambda}{\phi} \right) \right\}^{-2} \Phi\left(\frac{z - \beta y - \alpha\mu}{\alpha\sigma} \right) dy \\ &= \frac{1}{\phi} \int_{-\infty}^{\infty} \exp\left(-\frac{w}{\phi} \right) \left\{ 1 + \exp\left(-\frac{w}{\phi} \right) \right\}^{-2} \Phi\left(\frac{z - \beta w - \beta\lambda - \alpha\mu}{\alpha\sigma} \right) dw \quad (6) \end{aligned}$$

where $\Phi(\cdot)$ denotes the cdf of the standard normal distribution. Using the series expansion

$$(1+w)^{-2} = \sum_{k=0}^{\infty} \binom{-2}{k} w^k,$$

(6) can be expanded as

$$\begin{aligned} F(z) &= \frac{1}{\phi} \int_{0}^{\infty} \exp\left(-\frac{w}{\phi} \right) \left\{ 1 + \exp\left(-\frac{w}{\phi} \right) \right\}^{-2} \Phi\left(\frac{z - \beta w - \beta\lambda - \alpha\mu}{\alpha\sigma} \right) dw \\ &\quad + \frac{1}{\phi} \int_{-\infty}^{0} \exp\left(\frac{w}{\phi} \right) \left\{ 1 + \exp\left(\frac{w}{\phi} \right) \right\}^{-2} \Phi\left(\frac{z - \beta w - \beta\lambda - \alpha\mu}{\alpha\sigma} \right) dw \\ &= \frac{1}{\phi} \int_{0}^{\infty} \Phi\left(\frac{z - \beta w - \beta\lambda - \alpha\mu}{\alpha\sigma} \right) \sum_{k=0}^{\infty} \binom{-2}{k} \exp\left\{ -\frac{(k+1)w}{\phi} \right\} dw \\ &\quad + \frac{1}{\phi} \int_{-\infty}^{0} \Phi\left(\frac{z - \beta w - \beta\lambda - \alpha\mu}{\alpha\sigma} \right) \sum_{k=0}^{\infty} \binom{-2}{k} \exp\left\{ \frac{(k+1)w}{\phi} \right\} dw \\ &= \frac{1}{\phi} \sum_{k=0}^{\infty} \binom{-2}{k} \int_{0}^{\infty} \exp\left\{ -\frac{(k+1)w}{\phi} \right\} \Phi\left(\frac{z - \beta w - \beta\lambda - \alpha\mu}{\alpha\sigma} \right) dw \\ &\quad + \frac{1}{\phi} \sum_{k=0}^{\infty} \binom{-2}{k} \int_{-\infty}^{0} \exp\left\{ \frac{(k+1)w}{\phi} \right\} \Phi\left(\frac{z - \beta w - \beta\lambda - \alpha\mu}{\alpha\sigma} \right) dw \\ &= \frac{1}{\phi} \sum_{k=0}^{\infty} \binom{-2}{k} \int_{0}^{\infty} \exp\left\{ -\frac{(k+1)w}{\phi} \right\} \Phi\left(\frac{z - \beta w - \beta\lambda - \alpha\mu}{\alpha\sigma} \right) dw \\ &\quad + \frac{1}{\phi} \sum_{k=0}^{\infty} \binom{-2}{k} \int_{-\infty}^{0} \exp\left\{ -\frac{(k+1)w}{\phi} \right\} \Phi\left(\frac{z + \beta w - \beta\lambda - \alpha\mu}{\alpha\sigma} \right) dw. (7) \end{aligned}$$

Using the relationship

$$\Phi(-x) = \frac{1}{2} \operatorname{erfc}\left(\frac{x}{\sqrt{2}} \right),$$

(7) can be further rewritten as

$$\begin{aligned} F(z) &= \frac{1}{2\phi}\sum_{k=0}^{\infty}\binom{-2}{k}\int_0^{\infty}\exp\left\{-\frac{(k+1)w}{\phi}\right\}\operatorname{erfc}\left(\frac{\beta w+\beta\lambda+\alpha\mu-z}{\sqrt{2}\alpha\sigma}\right)dw \\ &\quad +\frac{1}{2\phi}\sum_{k=0}^{\infty}\binom{-2}{k}\int_{-\infty}^{0}\exp\left\{-\frac{(k+1)w}{\phi}\right\} \\ &\quad \operatorname{erfc}\left(\frac{-\beta w+\beta\lambda+\alpha\mu-z}{\sqrt{2}\alpha\sigma}\right)dw. \end{aligned} \tag{8}$$

The two integrals in (8) can be calculated by direct application of Lemma 1. The result follows. □

The following corollaries provide the cdfs for the sum and the difference of the normal and logistic random variables.

Corollary 1. Suppose X and Y are distributed according to (1) and (2), respectively. Then, the cdf of $Z = X + Y$ can be expressed as

$$F(z) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{k+1}\binom{-2}{k}\left\{2\operatorname{erfc}\left(\frac{\lambda+\mu-z}{\sqrt{2}\sigma}\right)-G_k(z)-H_k(z)\right\}, \tag{9}$$

where

$$G_k(z) = \exp\left\{\frac{(k+1)^2}{\phi^2}+\frac{2(k+1)(\lambda+\mu-z)}{\sigma^2\phi}\right\}\operatorname{erfc}\left(\frac{\lambda+\mu-z}{\sqrt{2}\sigma}+\frac{(k+1)\sigma}{\sqrt{2}\phi}\right)$$

and

$$H_k(z) = \exp\left\{\frac{(k+1)^2}{\phi^2}-\frac{2(k+1)(\lambda+\mu-z)}{\sigma^2\phi}\right\}\operatorname{erfc}\left(\frac{\lambda+\mu-z}{\sqrt{2}\sigma}-\frac{(k+1)\sigma}{\sqrt{2}\phi}\right).$$

Proof. Set $\alpha = 1$ and $\beta = 1$ into (3). □

Corollary 2. Suppose X and Y are distributed according to (1) and (2), respectively. Then, the cdf of $Z = X - Y$ can be expressed as

$$F(z) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{k+1}\binom{-2}{k}\left\{2\operatorname{erfc}\left(\frac{-\lambda+\mu-z}{\sqrt{2}\sigma}\right)-G_k(z)-H_k(z)\right\}, \tag{10}$$

where

$$G_k(z) = \exp\left\{\frac{(k+1)^2}{\phi^2}-\frac{2(k+1)(-\lambda+\mu-z)}{\sigma^2\phi}\right\}\operatorname{erfc}\left(\frac{-\lambda+\mu-z}{\sqrt{2}\sigma}-\frac{(k+1)\sigma}{\sqrt{2}\phi}\right)$$

and

$$H_k(z) = \exp\left\{\frac{(k+1)^2}{\phi^2}+\frac{2(k+1)(-\lambda+\mu-z)}{\sigma^2\phi}\right\}\operatorname{erfc}\left(\frac{-\lambda+\mu-z}{\sqrt{2}\sigma}+\frac{(k+1)\sigma}{\sqrt{2}\phi}\right).$$

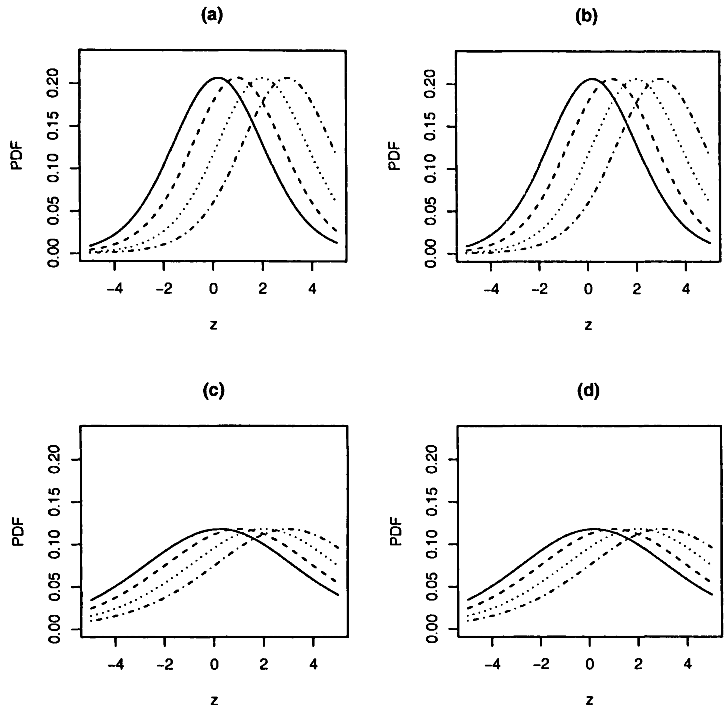

Fig. 1. Plots of the pdf of (3) for $\mu/\sigma = 0.1, 1, 2, 3$, $\lambda/\phi = 0$, $\phi/\sigma = 1$, $\phi = 1$, and (a): $\alpha = 1$ and $\beta = 1$; (b): $\alpha = 1$ and $\beta = -1$; (c): $\alpha = 1$ and $\beta = 2$; and, (d): $\alpha = 1$ and $\beta = -2$. The curves from the left to the right correspond to increasing values of μ/σ.

Proof. Set $\alpha = 1$ and $\beta = -1$ into (3). $\square$

Note that the parameters in (3), (9) and (10) are functions of μ/σ (coefficient of variation for the normal model), λ/ϕ (coefficient of variation for the logistic model), ϕ/σ (ratio of scale parameters), and ϕ. Figures 1 to 3 illustrate possible shapes of the pdf of $\alpha X + \beta Y$ for a range of values of α, β, μ/σ, λ/ϕ and ϕ/σ. Note that μ/σ and λ/ϕ, respectively, control the location and the modality of the distribution, while ϕ/σ largely dictates the scale. The fact that multi-modal shapes are possible is by itself interesting.

3. PRODUCT

Theorem 2 derives an explicit expression for the cdf of $|XY|$ in terms of the hypergeometric function.

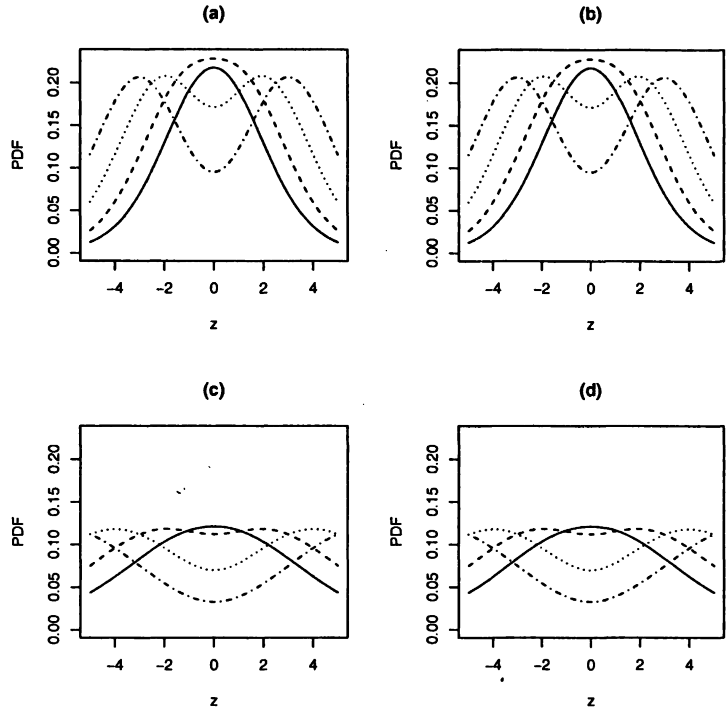

Fig. 2. Plots of the pdf of (3) for $\lambda/\phi = 0.2, 1, 2, 3$, $\mu/\sigma = 0$, $\phi/\sigma = 1$, $\phi = 1$, and (a): $\alpha = 1$ and $\beta = 1$; (b): $\alpha = 1$ and $\beta = -1$; (c): $\alpha = 1$ and $\beta = 2$; and, (d): $\alpha = 1$ and $\beta = -2$. The curves with the lowest to the highest modality correspond to increasing values of λ/ϕ.

Theorem 2. Suppose X and Y are distributed according to (1) and (2), respectively. Then, the cdf of $Z = |XY|$ can be expressed as

$$\begin{aligned} F(z) &= \frac{\sqrt{2}\lambda z}{\sigma} \sum_{k=0}^{\infty} \binom{-2}{k} \left\{ \frac{3C}{\sqrt{\pi}} G\left(\frac{1}{2}; \frac{3}{2}, 1, \frac{1}{2}; -\frac{\lambda^2 (k+1)^2 z^2}{8\sigma^2}\right) \right. \\ &\quad \left. + \frac{\lambda(k+1)z}{\sqrt{2}\sigma} G\left(1; 2, \frac{3}{2}, \frac{3}{2}; -\frac{\lambda^2 (k+1)^2 z^2}{8\sigma^2}\right) \right\}, \end{aligned} \tag{11}$$

where C denotes Euler's constant.

Proof. The cdf $F(z) = \Pr(|XY| \leq z)$ can be expressed as

$$F(z) = \lambda \int_{-\infty}^{\infty} \left\{ \Phi\left(\frac{z}{\sigma|y|}\right) - \Phi\left(-\frac{z}{\sigma|y|}\right) \right\} \frac{\exp(-\lambda y)}{\{1+\exp(-\lambda y)\}^2} \mathrm{d}y$$

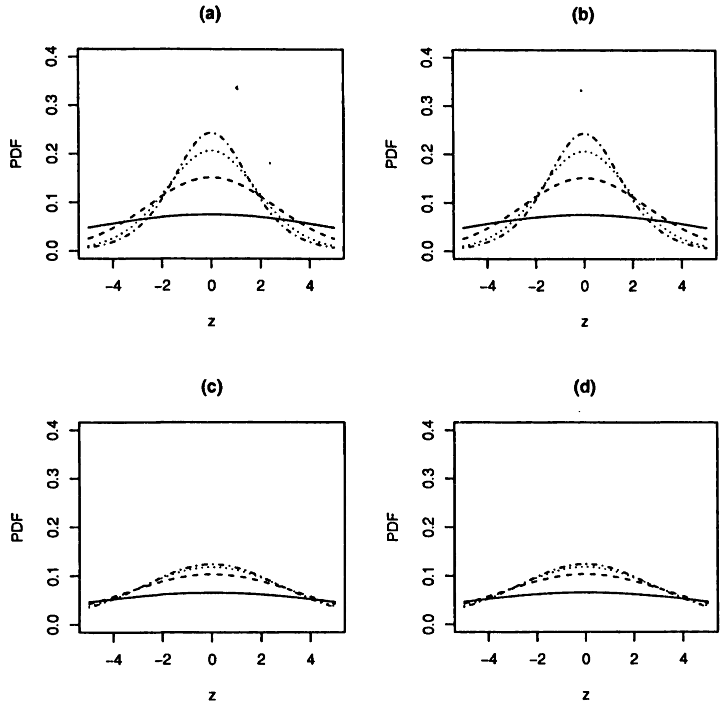

Fig. 3. Plots of the pdf of (3) for $\lambda/\phi = 0$, $\mu/\sigma = 0$, $\phi/\sigma = 0.2, 0.5, 1, 3$, $\phi = 1$, and (a): $\alpha = 1$ and $\beta = 1$; (b): $\alpha = 1$ and $\beta = -1$; (c): $\alpha = 1$ and $\beta = 2$; and, (d): $\alpha = 1$ and $\beta = -2$. The curves from the bottom to the top correspond to increasing values of ϕ/σ.

$$= 2\lambda \int_{-\infty}^{\infty} \Phi\left(\frac{z}{\sigma|y|}\right) \frac{\exp(-\lambda y)}{\{1+\exp(-\lambda y)\}^2} dy - 1, \tag{12}$$

where $\Phi(\cdot)$ denotes the cdf of the standard normal distribution. Using the series expansion

$$(1+w)^{-2} = \sum_{k=0}^{\infty} \binom{-2}{k} w^k,$$

(6) can be expanded as

$$\begin{aligned} F(z) &= 2\lambda \int_{0}^{\infty} \Phi\left(\frac{z}{\sigma|y|}\right) \frac{\exp(-\lambda y)}{\{1+\exp(-\lambda y)\}^2} dy \\ &\quad +2\lambda \int_{-\infty}^{0} \Phi\left(\frac{z}{\sigma|y|}\right) \frac{\exp(\lambda y)}{\{1+\exp(\lambda y)\}^2} dy - 1 \\ &= 2\lambda \int_{0}^{\infty} \Phi\left(\frac{z}{\sigma|y|}\right) \sum_{k=0}^{\infty} \binom{-2}{k} \exp\{-(k+1)\lambda y\} dy \end{aligned}$$

$$
\begin{aligned}
&+2\lambda \int_{-\infty}^{0} \Phi\left(\frac{z}{\sigma |y|}\right) \sum_{k=0}^{\infty} \binom{-2}{k} \exp\{(k+1)\lambda y\} \, dy - 1 \\
= \; & 2\lambda \sum_{k=0}^{\infty} \binom{-2}{k} \int_{0}^{\infty} \Phi\left(\frac{z}{\sigma |y|}\right) \exp\{-(k+1)\lambda y\} \, dy \\
&+2\lambda \sum_{k=0}^{\infty} \binom{-2}{k} \int_{-\infty}^{0} \Phi\left(\frac{z}{\sigma |y|}\right) \exp\{(k+1)\lambda y\} \, dy - 1 \\
= \; & 4\lambda \sum_{k=0}^{\infty} \binom{-2}{k} \int_{0}^{\infty} \Phi\left(\frac{z}{\sigma y}\right) \exp\{-(k+1)\lambda y\} \, dy - 1 \qquad (13)
\end{aligned}
$$

Using the relationship

$$
\Phi(-x) = \frac{1}{2} \mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right),
$$

(7) can be rewritten as

$$
\begin{aligned}
F(z) &= 2\lambda \sum_{k=0}^{\infty} \binom{-2}{k} \int_{0}^{\infty} \mathrm{erfc}\left(-\frac{z}{\sqrt{2}\sigma y}\right) \exp\{-(k+1)\lambda y\} \, dy - 1 \\
&= 2\lambda \sum_{k=0}^{\infty} \binom{-2}{k} \int_{0}^{\infty} w^{-2} \mathrm{erfc}\left(-\frac{zw}{\sqrt{2}\sigma}\right) \exp\{-(k+1)\lambda / w\} \, dw - 1. \qquad (14)
\end{aligned}
$$

Direct application of Lemma 2 shows that the integral in (14) can be calculated as

$$
\begin{aligned}
&\int_{0}^{\infty} w^{-2} \mathrm{erfc}\left(-\frac{zw}{\sqrt{2}\sigma}\right) \exp\{-(k+1)\lambda / w\} \, dw \\
= \; & \frac{1}{\lambda(k+1)} + \frac{3Cz}{\sqrt{2\pi}\sigma} G\left(\frac{1}{2}; \frac{3}{2}, 1, \frac{1}{2}; -\frac{\lambda^2 (k+1)^2 z^2}{8\sigma^2}\right) \\
&+ \frac{\lambda(k+1) z^2}{2\sigma^2} G\left(1; 2, \frac{3}{2}, \frac{3}{2}; -\frac{\lambda^2 (k+1)^2 z^2}{8\sigma^2}\right).
\end{aligned}
$$

The result of the theorem follows by using the identity that

$$
\sum_{k=0}^{\infty} \frac{1}{k+1} \binom{-2}{k} = \frac{1}{2}. \qquad \square
$$

Note that the parameters in (11) are functions of λ/σ (ratio of scale parameters). Figure 4 illustrates possible shapes of the pdf of $|XY|$ for a range of values of λ/σ. Note that the shapes are unimodal and that the value of λ/σ largely dictates the behavior of the pdf near $z = 0$.

4. RATIO

Theorem 3 derives an explicit expression for the pdf and the cdf of $|X/Y|$ in terms of the complementary error function.

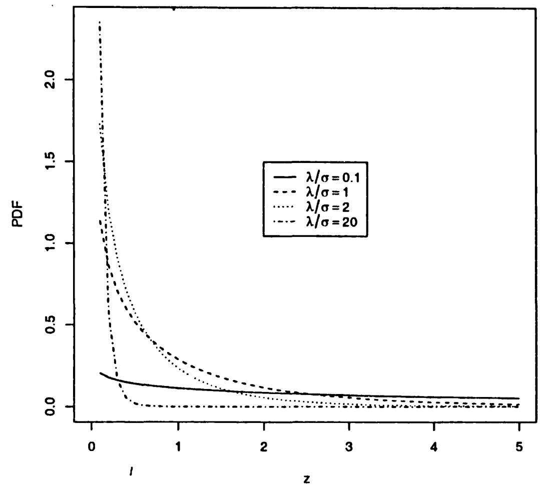

Fig. 4. Plots of the pdf of (11) for $\lambda/\sigma = 0.1, 1, 2, 20$ and $\sigma = 1$.

Theorem 3. Suppose X and Y are distributed according to (1) and (2), respectively. Then, the cdf of $Z = |X/Y|$ can be expressed as

$$F(z) = \sum_{k=0}^{\infty} \frac{1}{k+1} \binom{-2}{k} \left\{ G_k(z) - G_k(-z) \right\}, \tag{15}$$

where

$$G_k(z) = \exp\left(\frac{\lambda^2 (k+1)^2 \sigma^2 + 2\lambda (k+1) \mu z}{2z^2} \right) \operatorname{erfc}\left(\frac{\mu}{\sqrt{2}\sigma} + \frac{\lambda (k+1) \sigma}{\sqrt{2} z} \right). \tag{16}$$

The corresponding pdf is

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{k+1} \binom{-2}{k} \left\{ g_k(z) + g_k(-z) \right\}, \tag{17}$$

where g_k is the derivative of G_k given by

$$\begin{aligned} g_k(z) = {} & \frac{\lambda}{\sqrt{\pi} z^3} \exp\left(\frac{\lambda^2 (k+1)^2 \sigma^2 + 2\lambda (k+1) \mu z}{2z^2} \right) \\ & \left[\sqrt{2} \sigma z \exp\left\{ - \left(\frac{\mu}{\sqrt{2} z} + \frac{\lambda (k+1) \sigma}{\sqrt{2} z} \right)^2 \right\} \right. \\ & - \sqrt{\pi} \lambda (k+1) \sigma^2 \operatorname{erfc}\left(\frac{\mu}{\sqrt{2} z} + \frac{\lambda (k+1) \sigma}{\sqrt{2} z} \right) \\ & \left. - \sqrt{\pi} \mu z \operatorname{erfc}\left(\frac{\mu}{\sqrt{2} z} + \frac{\lambda (k+1) \sigma}{\sqrt{2} z} \right) \right]. \end{aligned} \tag{18}$$

Proof. The cdf $F(z) = \Pr(|X/Y| \le z)$ can be expressed as

$$F(z) = \lambda \int_{-\infty}^{\infty} \left\{ \Phi\left(\frac{\mu + z|y|}{\sigma}\right) - \Phi\left(\frac{\mu - z|y|}{\sigma}\right) \right\} \frac{\exp(-\lambda y)}{\{1+\exp(-\lambda y)\}^2} dy, \quad (19)$$

where $\Phi(\cdot)$ denotes the cdf of the standard normal distribution. Using the series expansion

$$(1+w)^{-2} = \sum_{k=0}^{\infty} \binom{-2}{k} w^k,$$

(19) can be expanded as

$$\begin{aligned}
F(z) &= \lambda \Bigg[\int_0^{\infty} \left\{ \Phi\left(\frac{\mu + z|y|}{\sigma}\right) - \Phi\left(\frac{\mu - z|y|}{\sigma}\right) \right\} \frac{\exp(-\lambda y)}{\{1+\exp(-\lambda y)\}^2} dy \\
&\quad + \int_{-\infty}^{0} \left\{ \Phi\left(\frac{\mu + z|y|}{\sigma}\right) - \Phi\left(\frac{\mu - z|y|}{\sigma}\right) \right\} \frac{\exp(\lambda y)}{\{1+\exp(\lambda y)\}^2} dy \Bigg] \\
&= \lambda \Bigg[\int_0^{\infty} \left\{ \Phi\left(\frac{\mu + z|y|}{\sigma}\right) - \Phi\left(\frac{\mu - z|y|}{\sigma}\right) \right\} \sum_{k=0}^{\infty} \binom{-2}{k} \exp\{-(k+1)\lambda y\} dy \\
&\quad + \int_{-\infty}^{0} \left\{ \Phi\left(\frac{\mu + z|y|}{\sigma}\right) - \Phi\left(\frac{\mu - z|y|}{\sigma}\right) \right\} \sum_{k=0}^{\infty} \binom{-2}{k} \exp\{(k+1)\lambda y\} dy \Bigg] \\
&= \lambda \Bigg[\sum_{k=0}^{\infty} \binom{-2}{k} \int_0^{\infty} \left\{ \Phi\left(\frac{\mu + z|y|}{\sigma}\right) - \Phi\left(\frac{\mu - z|y|}{\sigma}\right) \right\} \exp\{-(k+1)\lambda y\} dy \\
&\quad + \sum_{k=0}^{\infty} \binom{-2}{k} \int_{-\infty}^{0} \left\{ \Phi\left(\frac{\mu + z|y|}{\sigma}\right) - \Phi\left(\frac{\mu - z|y|}{\sigma}\right) \right\} \exp\{(k+1)\lambda y\} dy \Bigg] \\
&= 2\lambda \sum_{k=0}^{\infty} \binom{-2}{k} \int_0^{\infty} \left\{ \Phi\left(\frac{\mu + z|y|}{\sigma}\right) - \Phi\left(\frac{\mu - z|y|}{\sigma}\right) \right\} \exp\{-(k+1)\lambda y\} dy. \quad (20)
\end{aligned}$$

Using the relationship

$$\Phi(-x) = \frac{1}{2} \mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right),$$

(20) can be rewritten as

$$\begin{aligned}
F(z) &= \lambda \sum_{k=0}^{\infty} \binom{-2}{k} \int_0^{\infty} \left\{ \mathrm{erfc}\left(\frac{\mu - z|y|}{\sqrt{2}\sigma}\right) - \mathrm{erfc}\left(\frac{\mu + z|y|}{\sqrt{2}\sigma}\right) \right\} \exp\{-(k+1)\lambda y\} dy \\
&= \lambda \sum_{k=0}^{\infty} \binom{-2}{k} \Bigg[\int_0^{\infty} \mathrm{erfc}\left(\frac{\mu - zy}{\sqrt{2}\sigma}\right) \exp\{-(k+1)\lambda y\} dy \\
&\quad - \int_0^{\infty} \mathrm{erfc}\left(\frac{\mu + zy}{\sqrt{2}\sigma}\right) \exp\{-(k+1)\lambda y\} dy \Bigg]. \quad (21)
\end{aligned}$$

The two integrals in (21) can be calculated by direct application of Lemma 1. The result follows. □

Note that the parameters in both (16) and (18) are functions of μ/σ (coefficient of variation) and $\lambda\sigma$ (ratio of scale parameters). The following corollary shows that (16) and (18) reduce to simpler forms when the coefficient of variation approaches zero.

Corollary 3. Suppose X and Y are distributed according to (1) and (2), respectively. If $\mu/\sigma \to 0$ then the cdf of $Z = |X/Y|$ takes the form (15), where

$$G_k(z) = \exp\left(\frac{\lambda^2(k+1)^2\sigma^2}{2z^2}\right)\operatorname{erfc}\left(\frac{\lambda(k+1)\sigma}{\sqrt{2}z}\right).$$

The corresponding pdf takes the form (17), where g_k is the derivative of G_k given by

$$\begin{aligned} g_k(z) &= \frac{\lambda(k+1)\sigma}{\sqrt{\pi}z^3}\exp\left(\frac{\lambda^2(k+1)^2\sigma^2}{2z^2}\right)\left\{\sqrt{2}z\exp\left(-\frac{\lambda^2(k+1)^2\sigma^2}{2z^2}\right)\right.\\ &\quad \left.-\sqrt{\pi}\lambda(k+1)\sigma\operatorname{erfc}\left(\frac{\lambda(k+1)\sigma}{\sqrt{2}z}\right)\right\}. \end{aligned}$$

Proof. The proof follows by limiting $\mu/\sigma \to 0$ in (16). □

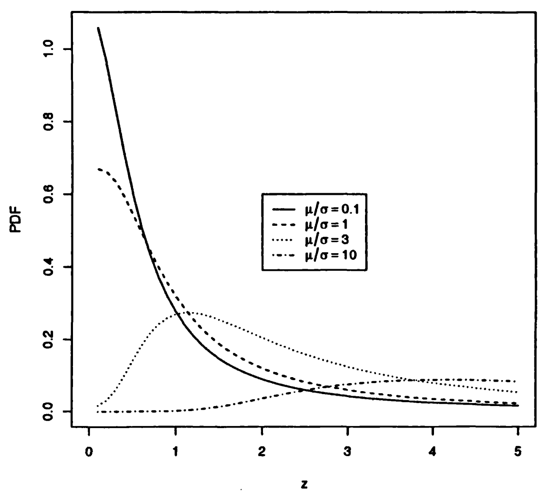

Fig. 5. Plots of the pdf (17) for $\mu/\sigma = 0.1, 1, 3, 10$, $\sigma = 1$ and $\lambda = 1$.

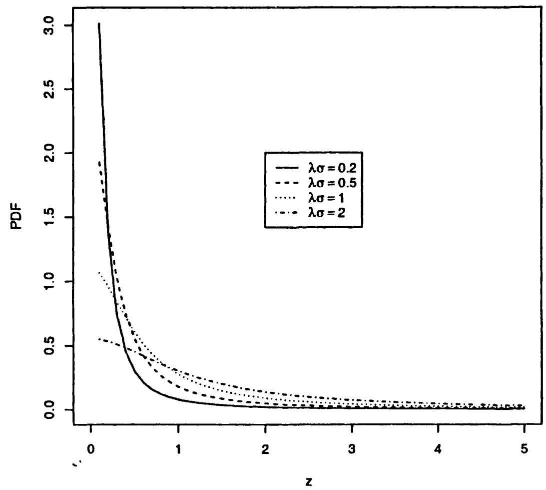

Fig. 6. Plots of the pdf (17) for $\lambda\sigma = 0.2, 0.5, 1, 2$, $\sigma = 1$ and $\mu = 1$.

Figures 5 and 6 illustrate possible shapes of the pdf of $|X/Y|$ for a range of values of μ/σ and $\lambda\sigma$. Note that the shape of the distribution is largely controlled by the value of μ/σ.

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