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ON INVERTIBILITY OF A RANDOM COEFFICIENT MOVING AVERAGE MODEL

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A linear moving average model with random coefficients (RCMA) is proposed as more general alternative to usual linear MA models. The basic properties of this model are obtained. Although some model properties are similar to linear case the RCMA model class is too general to find general invertibility conditions. The invertibility of some special examples of RCMA(1) model are investigated in this paper.

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1. INTRODUCTION

Let $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ be an i.i.d. sequence of random variables. We generalize the linear moving average time series model. We replace the constant coefficient vector by a random one. Assume that $\{\boldsymbol{A}_t\}_{t\in\mathbb{Z}}$, $\boldsymbol{A}_t = (A_{t,0},\ldots,A_{t,p})^T$ is a time series of vector of parameters. Further we assume that for each fixed $k \in \{0,1,\ldots,p\}$ and for each $t \in \mathbb{Z}$ the subsequences $\{A_{t-i,k}\}_{i=0}^{\infty}$ and $\{\varepsilon_{t-k+j}\}_{j=0}^{\infty}$ are independent. This relationship between the series $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ and $\{\boldsymbol{A}_t\}_{t\in\mathbb{Z}}$ we call *future independence condition (FIC)*. Now we introduce a *random coefficient moving average model of order p, RCMA(p)*, of the general form

$$X_t = A_{t,0}\varepsilon_t + A_{t,1}\varepsilon_{t-1} + \cdots + A_{t,p}\varepsilon_{t-p}. \tag{1}$$

The FIC gives the causality connection between the sequences $\{X_t\}$ and $\{\varepsilon_t\}$ and additionally it dictates the inner structure of the model. For example in RCMA(1) form of non-linear moving average model

$$X_t = \varepsilon_t + \alpha\varepsilon_{t-1} + \beta\varepsilon_t\varepsilon_{t-1}$$

is $\boldsymbol{A}_t = (1+\beta\varepsilon_{t-1},\alpha)^T$. Putting $\boldsymbol{A}'_t = (1,\alpha+\beta\varepsilon_t)^T$ we get the same model but the FIC does not hold.

Let us denote $\mathcal{B}_t$, the σ-algebra generated by $\{X_s, s \le t\}$. In this general form, model (1) covers many known models.

- *Linear MA models:* The $\boldsymbol{A}_t$ is a constant vector not depending on t.
- *Time dependent MA models:* The $\boldsymbol{A}_t$ is not random, but it depends on t.
- *Self exciting threshold moving average models:* The $A_{t,k} = b_k^{(J_t)}$, $k = 0, \dots, p$, where J_t is measurable with respect to $\mathcal{B}_{t-1}$ and takes values in $\{1, 2, \dots, l\}$.
- *Strongly subdiagonal bilinear models:* There exist some $q \in \mathbb{N}$ such that each $A_{t,k}$ is linear function of variables $X_{t-k-1}, X_{t-k-2}, \dots, X_{t-k-q}$ with time invariant constant coefficients.
- *Some non-linear moving average models:* The $\boldsymbol{A}_t$ is generally non-linear function of $\varepsilon_{t-1}, \varepsilon_{t-2}, \dots, \varepsilon_{t-q}$, admitting expression in the form (1), where FIC condition holds and $q \in \mathbb{N}$. For example the models that have a finite order Volterra expansion (see e. g. Tong [8]) are included.
- *ARCH models:* The $p = 0$ and $A_{t,0} = \sqrt{\gamma + \sum_{n=1}^{q} \phi_i X_{t-i}^2}$, where $\gamma > 0$, $\phi_i \geq 0$ for all i.
- *Some doubly stochastic models:* The $A_{t,0} = 1$ a. s. for each t and the vector of random coefficients $\boldsymbol{A}_t$ is measurable with respect to $\mathcal{B}_{t-1}$ for each t. A more general doubly stochastic model is introduced by Tjøstheim [7].
- *Product autoregressive models:* Here the $p = 0$, white noise ε_t is positive and $A_{t,0} = X_{t-1}^{\alpha}$. For more details see McKenzie [4].

Some bibliographical notes about mentioned models can be found in Tong [8].

1.1. Stationarity

It is well known that linear MA(p) model is always stationary. The RCMA(p) model may not be stationary as is shown in the Example 1.1.

Example 1.1. Consider $p = 1$, and $\varepsilon_t \sim \mathsf{N}(0, 1)$. Let

$$\boldsymbol{A}_t = \begin{cases} (1, \varepsilon_{t-2})^T & \text{if } t > 4, \\ (1, \varepsilon_{t-3})^T & \text{if } t \leq 4. \end{cases}$$

Thus $\boldsymbol{A}_t$ is the i.i.d. sequence and FIC condition holds. Now X_t is not stationary, because

$$X_5 = \varepsilon_5 + \varepsilon_4 \varepsilon_3, \quad X_4 = \varepsilon_4 + \varepsilon_3 \varepsilon_1, \quad X_3 = \varepsilon_3 + \varepsilon_2 \varepsilon_0,$$

and $\mathsf{E}\, X_3^2 X_4^2 = 6 \neq \mathsf{E}\, X_4^2 X_5^2 = 8$.

A sufficient condition that model (1) is stationary is that vector sequence $\{\boldsymbol{\eta}_t\}$, where $\boldsymbol{\eta}_t = (A_{t,0}\varepsilon_t, A_{t,1}\varepsilon_{t-1}, \dots, A_{t,p}\varepsilon_{t-p})^T$, is stationary. Example 1.2 given bellow shows that this condition is not necessary. It means that also non-stationary sequence $\{\boldsymbol{\eta}_t\}$ can produce stationary process $\{X_t\}$. It is also useful to introduce the following notation here. If we have column vectors $\boldsymbol{Y}_1, \dots, \boldsymbol{Y}_n$ we define $\mathrm{Vec}(\boldsymbol{Y}_1, \dots$ $\dots, \boldsymbol{Y}_n) = (\boldsymbol{Y}_1^T, \dots, \boldsymbol{Y}_n^T)^T$.

Example 1.2. Consider $p = 1$, and $\varepsilon_t \sim \mathsf{N}(0,1)$ as in Example 1.1. Let $\lambda > 0$, $t^* \in \mathbb{Z}$,

$$d_t = \begin{cases} 2 & \text{if } t \geq t^*, \\ 3 & \text{if } t < t^*, \end{cases}$$

and

$$X_t = Z_t(\varepsilon_t - \varepsilon_{t-1}), \tag{2}$$

where Z_t is a random sign given by previous white noise exceeding level λ. More exactly

$$Z_t = \begin{cases} +1 & \text{if } \varepsilon_{t-d_t} \geq \lambda, \\ -1 & \text{if } \varepsilon_{t-d_t} < \lambda. \end{cases}$$

It is seen that Z_t's form i.i.d. sequence and $p = \mathsf{P}(Z_t = 1) = 1 - \Phi(\lambda)$, where Φ is distribution function of standard Gaussian $\mathsf{N}(0,1)$ distribution. Thus $\boldsymbol{A}_t = (Z_t, -Z_t)^T$ is i.i.d. sequence and FIC condition clearly holds.

It is not a difficult task to prove that sequence X_t defined in (2) is stationary. For each $n \in \mathbb{N}$ and $t_1 < t_2 < \cdots < t_n$ we denote $\boldsymbol{t}_n = (t_1, \ldots, t_n)^T$ and $\boldsymbol{X}_{\boldsymbol{t}_n} = (X_{t_1}, \ldots, X_{t_n})^T$. First we assume that values $Z_{t_1} = z_{t_1}, \ldots, Z_{t_n} = z_{t_n}$ are known. Conditional distribution of vector $\boldsymbol{X}_{\boldsymbol{t}_n}$ is n-variate normal with zero mean and variance matrix

$$\boldsymbol{V}_{\boldsymbol{t}_n} = \left(v_{i,j}^{\boldsymbol{t}_n}\right)_{i,j=1}^{n}, \tag{3}$$

where $v_{i,i}^{\boldsymbol{t}_n} = 2$ and if $i < j$, $v_{i,j}^{\boldsymbol{t}_n} = -z_{t_i} z_{t_j} \delta(t_i, t_j - 1)$. Symbol $\delta(i,j)$ denotes well known Kronecker function which is equal to one if and only if $i = j$ and zero otherwise.

We have

$$\mathsf{P}\left(Z_{t_1} = z_{t_1}, \ldots, Z_{t_n} = z_{t_n}\right) = p^{N_{\boldsymbol{t}_n}} (1-p)^{n - N_{\boldsymbol{t}_n}},$$

where $N_{\boldsymbol{t}_n} = \mathsf{Card}\{1 \leq i \leq n : \ z_{t_i} = 1\}$. Since ε_t's are i.i.d. the distribution of variable $N_{\boldsymbol{t}_n}$ is independent of the time shift of fixed vector $\boldsymbol{t}_n$. Consequently the distribution of random vector $(Z_{t_1}, \ldots, Z_{t_n})^T$ is independent of the time shift of $\boldsymbol{t}_n$. Employing (3) it follows that unconditional distribution of $\boldsymbol{X}_{\boldsymbol{t}_n}$ is also time shift independent and thus the sequence $\{X_t\}$ is stationary.

Now let us deal with the sequence $\{\boldsymbol{\eta}_t\}$. In this example we have

$$\boldsymbol{\eta}_t = \begin{cases} (\varepsilon_t, -\varepsilon_{t-1})^T & \text{if } \varepsilon_{t-d_t} \geq \lambda, \\ (-\varepsilon_t, \varepsilon_{t-1})^T & \text{if } \varepsilon_{t-d_t} < \lambda. \end{cases}$$

Let us investigate the distribution of vector

$$\boldsymbol{E}_t = \mathsf{Vec}(\boldsymbol{\eta}_t, \boldsymbol{\eta}_{t-1}, \boldsymbol{\eta}_{t-2}) = (Z_t\varepsilon_t, -Z_t\varepsilon_{t-1}, Z_{t-1}\varepsilon_{t-1}, -Z_{t-1}\varepsilon_{t-2}, Z_{t-2}\varepsilon_{t-2}, -Z_{t-2}\varepsilon_{t-3})^T.$$

If $t \geq t^* + 2$, then Z_{t-2}, Z_{t-1} and Z_t depend on ε_{t-4}, ε_{t-3} and ε_{t-2}, respectively. It follows that

$$\mathsf{P}\left(|Z_{t-2}\varepsilon_{t-2}| < \lambda, |Z_{t-2}\varepsilon_{t-3}| < \lambda, -Z_t\varepsilon_{t-1} > 0, Z_{t-1}\varepsilon_{t-1} > 0\right) = 0.$$

If $t < t^* + 2$, then Z_{t-2}, Z_{t-1} and Z_t depend on ε_{t-5}, ε_{t-4} and ε_{t-3}, respectively. In this case it follows that

$$\mathsf{P}\left(|Z_{t-2}\varepsilon_{t-2}| < \lambda, |Z_{t-2}\varepsilon_{t-3}| < \lambda, -Z_t\varepsilon_{t-1} > 0, Z_{t-1}\varepsilon_{t-1} > 0\right) = \frac{1}{2}p(1-2p)^2,$$

and it means that sequence $\{\eta_t\}$ is nonstationary.

Examples given above show that it is very difficult to look for some general condition of stationarity of RCMA time series models. Because of many different kinds of known models that are included it is more practical to investigate special cases which are needed for some situations. For example the RCMA(p) process $\{X_t\}$ given by equation (1) is stationary if it is a finite time shift invariant transformation of white noise, i.e. if $\boldsymbol{A}_t = f_A(\varepsilon_{t-1}, \ldots, \varepsilon_{t-q})$, $q < \infty$, and f_A commutes with time shift operator. More exactly for $j \in \mathbb{Z}$ we denote the pr_j projection to the jth coordinate, $pr_j(\{\varepsilon_t\}) = \varepsilon_j$ and for $k \in \mathbb{Z}$ we denote τ_k the k-step forward translation. It means that $pr_j(\tau_k(\{\varepsilon_t\})) = \varepsilon_{j-k}$, for each j, k. Now the following lemma holds.

Lemma 1.3. Let $\{\varepsilon_t\}$ be i.i.d. sequence of random variables and τ_k the k-step forward translation. Consider transformation $T : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ such that $\{X_t\} = T(\{\varepsilon_t\})$. If T commute with the time shift operator, it means $T \circ \tau_k = \tau_k \circ T$ for each $k \in \mathbb{Z}$, then the sequence $\{X_t\}$ is stationary.

Proof. The proof is based on facts that i.i.d. sequence is stationary and it has time shift invariant distribution. Employing the commutativity with the transformation T we get assertion of the lemma. Complete proof can be found e.g. in Štěpán [6]. □

We meet a more complicated situation if we consider bilinear RCMA(p) model. It is recommended to consult the monograph Granger and Andersen [1] for the stationarity conditions. Similarly if we deal with the product autoregressive RCMA(p) models we refer to McKenzie [4].

1.2. Moments

Consider stationary RCMA(p) time series with stationary parameter sequence $\{\boldsymbol{A}_t\}$. Assuming existence of the moments bellow we denote

$$\mu'_k = \mathsf{E}\,\varepsilon_t^k, \quad \boldsymbol{a} = \mathsf{E}\,\boldsymbol{A}_t, \quad \boldsymbol{V}(k) = \left(v_{i,j}(k)\right)_{i,j=0}^{p} = \mathsf{Cov}\left(\boldsymbol{A}_t, \boldsymbol{A}_{t-k}\right).$$

Additionally we assume $\mu'_1 = 0$ and $\mu'_2 = \sigma_\varepsilon^2$, where $0 < \sigma_\varepsilon^2 < \infty$. Now we can describe the second order moments of the process $\{X_t\}$ using the second moments of sequences $\{\varepsilon_t\}$ and $\{\boldsymbol{A}_t\}$. Clearly $\mathsf{E}\,X_t = 0$ and it is easy to verify that autocovariance function $R(k) = \mathsf{E}\,X_tX_{t-k}$ has the truncation property, $R(k) = 0$, if $|k| > p$. For $|k| \le p$ we have

$$R(k) = \mathsf{E}\,X_tX_{t-k} = \sigma_\varepsilon^2 \sum_{i=k}^{p} \mathsf{E}\,A_{t,i}A_{t-k,i-k} = \sigma_\varepsilon^2 \sum_{i=k}^{p} \left(v_{i,i-k}(k) + a_ia_{i-k}\right).$$

Especially,

$$R(0) = \mathsf{Var}\, X_t = \sigma_\varepsilon^2 \,\mathsf{Tr}\left(V(0)\right) + a^T a.$$

It is useful to note that assumption $\mu_1' = 0$ is important for the truncation covariance property. Counter-example is given in following Example 1.4.

Example 1.4. Consider stationary RCMA(1) model given by formula

$$X_t = \varepsilon_{t-2}\varepsilon_t + \varepsilon_{t-2}\varepsilon_{t-1},$$

where $\mathsf{E}\,\varepsilon_t = \mu \neq 0$. It is easy to calculate that

$$R(2) = \mathsf{Cov}(X_t, X_{t-2}) = 2\mu^2\sigma_\varepsilon^2 \neq 0.$$

In accordance with our expectations if the higher order moments exist they are more complicated and strongly depend on the sequence $\{A_t\}$ and its relationship with the white noise sequence $\{\varepsilon_t\}$. Higher order moments thus can be exploited for identification and estimation of RCMA models.

2. INVERTIBILITY

The invertibility is usually defined as measurability of each variable ε_t with respect to σ-algebra $\mathcal{B}_t$ generated by $\{X_s : s \leq t\}$. The invertibility conditions are well known in linear MA(p) case. The situation in the RCMA case is more complicated, because in a non-linear model the dependence of ε_t on $X_t, X_{t-1}, \ldots$ is more complicated. Thus in this section we deal with RCMA(1) model only instead more general RCMA(p).

Consider RCMA(1) model given by formula

$$X_t = A_{t,0}\varepsilon_t + A_{t,1}\varepsilon_{t-1}. \tag{4}$$

Similarly as in the linear MA(1) case we express

$$\varepsilon_t = \frac{1}{A_{t,0}} X_t - \frac{A_{t,1}}{A_{t,0}}\varepsilon_{t-1}. \tag{5}$$

Iterating equation (5) we get

$$\varepsilon_t = \sum_{j=0}^{n-1} \frac{(-1)^j}{A_{t,0}} \left(\prod_{k=1}^{j} \frac{A_{t-k+1,1}}{A_{t-k,0}} \right) X_{t-j} + (-1)^n \left(\prod_{k=0}^{n-1} \frac{A_{t-k,1}}{A_{t-k,0}} \right) \varepsilon_{t-n}. \tag{6}$$

Denote

$$B_t^{(0)} = \frac{1}{A_{t,0}}, \quad B_t^{(k)} = -\frac{A_{t-k+1,1}}{A_{t-k,0}} B_t^{(k-1)} \quad \text{for } k > 0, \tag{7}$$

and

$$C_t^{(0)} = 1, \quad C_t^{(k)} = -\frac{A_{t-k+1,1}}{A_{t-k+1,0}} C_t^{(k-1)} \qquad \text{for } k > 0.\ . \tag{8}$$

Now equation (6) has the form

$$\varepsilon_t = \sum_{j=0}^{n-1} B_t^{(j)} X_{t-j} + C_t^{(n)} \varepsilon_{t-n}.$$

If all variables $B_t^{(j)}$ are measurable w.r.t. $\mathcal{B}_t$ then it is natural to define a sequence $\{\varepsilon_t^{(n)}\}_{n=1}^{\infty}$ as

$$\varepsilon_t^{(n)} = \sum_{j=0}^{n-1} B_t^{(j)} X_{t-j}. \tag{9}$$

Our next step is to look for conditions ensuring the convergence $\varepsilon_t^{(n)} \to \varepsilon_t$ as $n \to \infty$ for each $t \in \mathbb{Z}$. The next lemma contains some sufficient conditions.

Lemma 2.1. Let $\{X_t\}$ be an RCMA(1) process defined by equation (4). Let $B_t^{(n)}$, $C_t^{(n)}$, and $\varepsilon_t^{(n)}$ be defined by equations (7), (8) and (9), respectively. Let variable $B_t^{(n)}$ be measurable with respect to $\mathcal{B}_t$ for each $t \in \mathbb{Z}$ and $n \in \mathbb{N}$. Assume that at least one of the next conditions holds for each $t \in \mathbb{Z}$:

(i)

$$\left|C_t^{(n)}\right| |\varepsilon_{t-n}| \xrightarrow[n\to\infty]{\mathsf{P}} 0;$$

(ii) there exists $p > 0$ such that

$$\mathsf{E}\left[\left|C_t^{(n)}\right|^p |\varepsilon_{t-n}|^p\right] \xrightarrow[n\to\infty]{} 0;$$

(iii) $\mathsf{P}(|\varepsilon_1| < \infty) = 1$ and there exists a positive sequence $K_n \nearrow \infty$ such that

$$K_n C_t^{(n)} \xrightarrow[n\to\infty]{\mathsf{P}} 0;$$

(iv) there exists $\delta > 0$ such that

$$\mathsf{P}\left(\liminf_{k\in\mathbb{N}_0} \left\{\omega : |A_{t-k,1}(\omega)| \le (1-\delta)|A_{t-k,0}(\omega)|\right\}\right) = 1;$$

(v) $\mathsf{P}(|A_{t,1}| \le |A_{t,0}|) = 1$ and there exists $\delta > 0$ such that

$$\mathsf{P}\left(\limsup_{k\in\mathbb{N}_0} \left\{\omega : |A_{t-k,1}(\omega)| \le (1-\delta)|A_{t-k,0}(\omega)|\right\}\right) = 1;$$

(vi) $\{A_t\}$ is stationary and ergodic process with

$$\mathsf{E}\log|A_{t,1}| < \mathsf{E}\log|A_{t,0}|;$$

(vii) $\{A_t\}$ is stationary and ergodic process and there exists a real constant $R > 1$ such that

$$\mathsf{P}\left(|A_{t,1}| \le R|A_{t,0}|\right) = 1 \quad \text{and} \quad \mathsf{P}\left(R|A_{t,1}| \le |A_{t,0}|\right) > \frac{1}{2}.$$

Then the process $\{X_t\}$ is invertible.

Proof. First assume that condition (i) holds. Since

$$\left|\varepsilon_t - \varepsilon_t^{(n)}\right| = \left|C_t^{(n)}\right| |\varepsilon_{t-n}| \xrightarrow[n\to\infty]{\mathsf{P}} 0,$$

we have $\varepsilon_t^{(n)} \xrightarrow[n\to\infty]{\mathsf{P}} \varepsilon_t$. It follows that there exists subsequence $\{\varepsilon_t^{(n_j)}\}_j$ which converges to ε_t almost surely. Considering the $\mathcal{B}_t$-measurability of all variables $B_t^{(j)}$ and following $\mathcal{B}_t$-measurability of $\varepsilon_t^{(n)}$ we have proved the fact that ε_t is a. s. limit of $\mathcal{B}_t$-measurable variables $\varepsilon_t^{(n)}$ and thus also $\mathcal{B}_t$-measurable.

Now we show that condition (ii) implies condition (i). For a given $\delta > 0$ it holds

$$0 \le \mathsf{P}\left(\left|C_t^{(n)}\right| |\varepsilon_{t-n}| \ge \delta\right) \le \delta^{-p}\,\mathsf{E}\left[\left|C_t^{(n)}\right|^p |\varepsilon_{t-n}|^p\right] \xrightarrow[n\to\infty]{} 0.$$

Further we show that also condition (iii) implies condition (i). For a given $0 < \delta < 1$ denote $\delta_n = \delta/K_n$. Condition (iii) gives

$$\mathsf{P}\left(K_n\left|C_t^{(n)}\right| \ge \delta\right) \xrightarrow[n\to\infty]{} 0$$

and also

$$\mathsf{P}\left(|\varepsilon_{t-n}| \ge K_n\right) \xrightarrow[n\to\infty]{} 0.$$

Thus for $m \in \mathbb{N}$ there exists $n_1(m)$ such that for each $n > n_1(m)$

$$\mathsf{P}\left(K_n\left|C_t^{(n)}\right| \ge \delta\right) \le \delta_m.$$

Analogically there exists $n_2(m)$ such that for each $n > n_2(m)$

$$\mathsf{P}\left(|\varepsilon_{t-n}| \ge K_n\right) \le \delta_m.$$

It holds

$$\begin{aligned}&\mathsf{P}\left(|\varepsilon_{t-n}|\left|C_t^{(n)}\right| \ge \delta\right)\\ &= \mathsf{P}\left(|\varepsilon_{t-n}|\left|C_t^{(n)}\right| \ge \delta,\, K_n\left|C_t^{(n)}\right| \ge \delta\right) + \mathsf{P}\left(|\varepsilon_{t-n}|\left|C_t^{(n)}\right| \ge \delta,\, K_n\left|C_t^{(n)}\right| < \delta\right).\end{aligned}$$

For $n > n_1(m)$ we obtain

$$\mathsf{P}\left(|\varepsilon_{t-n}|\left|C_t^{(n)}\right| \ge \delta,\, K_n\left|C_t^{(n)}\right| \ge \delta\right) \le \mathsf{P}\left(K_n\left|C_t^{(n)}\right| \ge \delta\right) \le \delta_m.$$

Additionally

$$\mathsf{P}\left(|\varepsilon_{t-n}|\left|C_t^{(n)}\right| \geq \delta,\, K_n\left|C_t^{(n)}\right| < \delta\right)$$
$$= \mathsf{P}\left(|\varepsilon_{t-n}|\left|C_t^{(n)}\right| \geq \delta,\, K_n\left|C_t^{(n)}\right| < \delta,\, |\varepsilon_{t-n}| \geq K_n\right)$$
$$+ \mathsf{P}\left(|\varepsilon_{t-n}|\left|C_t^{(n)}\right| \geq \delta,\, K_n\left|C_t^{(n)}\right| < \delta,\, |\varepsilon_{t-n}| < K_n\right).$$

It is clear that

$$\mathsf{P}\left(|\varepsilon_{t-n}|\left|C_t^{(n)}\right| \geq \delta,\, K_n\left|C_t^{(n)}\right| < \delta,\, |\varepsilon_{t-n}| < K_n\right) = 0$$

and for $n > n_2(m)$ we have

$$\mathsf{P}\left(|\varepsilon_{t-n}|\left|C_t^{(n)}\right| \geq \delta,\, K_n\left|C_t^{(n)}\right| < \delta,\, |\varepsilon_{t-n}| \geq K_n\right) \leq \mathsf{P}\left(|\varepsilon_{t-n}| \geq K_n\right) \leq \delta_m.$$

Thus for $n > \max\{n_1(m), n_2(m)\}$ we obtain

$$\mathsf{P}\left(|\varepsilon_{t-n}|\left|C_t^{(n)}\right| \geq \delta\right) \leq 2\delta_m \xrightarrow[m\to\infty]{} 0.$$

The conditions (iv) and (v) ensure a. s. convergence of $C_t^{(n)}$ to zero and so we have $\varepsilon_t^{(n)} \xrightarrow{\mathsf{P}} \varepsilon_t$. Assuming that condition (vi) holds we obtain

$$\begin{aligned}\frac{1}{n}\log\left|\varepsilon_t - \varepsilon_t^{(n)}\right| &= \frac{1}{n}\log\left|C_t^{(n)}\right| + \frac{1}{n}\log|\varepsilon_{t-n}| \\ &= \frac{1}{n}\sum_{k=1}^{n}\log|A_{t-k+1,1}| - \frac{1}{n}\sum_{k=1}^{n}\log|A_{t-k+1,0}| + \frac{1}{n}\log|\varepsilon_{t-n}| \\ &\xrightarrow[n\to\infty]{\text{a. s.}} \mathsf{E}\log|A_{t,1}| - \mathsf{E}\log|A_{t,0}| < 0.\end{aligned}$$

Thus $\left|\varepsilon_t - \varepsilon_t^{(n)}\right| \xrightarrow[n\to\infty]{\text{a. s.}} 0$.

In the last step of the proof we show that condition (vii) implies condition (vi). Denote $p = \mathsf{P}\left(R|A_{t,1}| \leq |A_{t,0}|\right)$. A direct calculation gives

$$\mathsf{E}\log|A_{t,1}| - \mathsf{E}\log|A_{t,0}| = \mathsf{E}\log\left|\frac{A_{t,1}}{A_{t,0}}\right| \leq -p\log R + (1-p)\log R < 0. \qquad \square$$

In the RCMA(1) case it is difficult to find some general necessary and sufficient condition for invertibility. In the next section we give some special examples of invertible RCMA(1) processes.

2.1. Examples

Simple bilinear RCMA(1) model

Consider a simple bilinear model

$$X_t = \varepsilon_t + (a + \theta X_{t-2})\varepsilon_{t-1}, \tag{10}$$

where ε_t's have zero mean and variance σ_ε^2 and a, θ are real valued parameters. It is well known (e. g. Granger and Andersen [1]) that the model is stationary iff $\theta^2\sigma_\varepsilon^2 < 1$. Using formula (6) we obtain

$$\varepsilon_t = \sum_{j=0}^{n-1}(-1)^j X_{t-j}\prod_{k=1}^{j}(a + \theta X_{t-1-k}) + (-1)^n \varepsilon_{t-n}\prod_{k=0}^{n}(a + \theta X_{t-1-k}).$$

It follows that

$$C_t^{(n)} = (-1)^n \prod_{k=0}^{n}(a + \theta X_{t-1-k}).$$

Assuming $\{X_t\}$ is stationary and ergodic sequence the Lemma 2.1 (vi) gives a sufficient condition for the model invertibility as $\mathsf{E}\log|a+\theta X_t| < 0$. It is very difficult to verify this condition as the distribution of X_t is not known. Using Jensen's inequality

$$\mathsf{E}\log|a + \theta X_t|^2 \le \log \mathsf{E}\,|a + \theta X_t|^2$$

we obtain weaker sufficient invertibility condition $\mathsf{E}(a+\theta X_t)^2 < 1$. Calculating expectation in equation (10) we have $\mathsf{E}\,X_t = 0$. Squaring (10) and calculating expectation we obtain

$$\mathsf{E}\,X_t^2 = \frac{\sigma_\varepsilon^2(1 + a^2)}{1 - \theta^2\sigma_\varepsilon^2}.$$

Thus

$$\theta^2 < \frac{1 - a^2}{2\sigma_\varepsilon^2}$$

is a sufficient condition for invertibility.

The RCMA(1) model with uniformly distributed parameter

Consider the RCMA(1) model given by formula

$$X_t = \varepsilon_t + (a + \theta Y_{t-2})\varepsilon_{t-1}, \tag{11}$$

where a, and $\theta > 0$ are real valued parameters, $\{Y_t\}$ is stationary and ergodic sequence and each variable Y_t is $\mathcal{B}_t$-measurable with the uniform distribution on the interval $(-1, 1)$.

Using notation introduced above we have $A_{t,0} = 1$ and $A_{t,1} = a+\theta Y_{t-2}$. To verify condition (vi) of Lemma 2.1 we calculate $\mathsf{E}\log|A_{t,1}|$. Using the fact that $A_{t,1}$ has the uniform distribution on the interval $(a-\theta, a+\theta)$ we obtain

$$\begin{aligned}\mathsf{E}\log|A_{t,1}| &= \frac{1}{2\theta}\int_{a-\theta}^{a+\theta}\log|x|\,\mathrm{d}x\\ &= \frac{1}{2\theta}\Big[(a+\theta)\log|a+\theta| - (a-\theta)\log|a-\theta| - 2\theta\Big]. \qquad (12)\end{aligned}$$

It follows from Lemma 2.1 that model (11) is invertible if $\mathsf{E}\log|A_{t,1}| < 0$. We can also compare this result with the result which follows from condition (vii) of Lemma 2.1. The constant R must be greater or equal to $|a|+\theta$. If $R = |a|+\theta$, then

$$\left[\mathsf{P}(|A_{t,1}| \le R^{-1}) > \frac{1}{2}\right] \Leftrightarrow \begin{cases} R^{-1} > |a| & \text{if } \theta > 2\min\{|a-\theta|, |a+\theta|\},\\ 2R^{-1} > \theta & \text{if } \theta \le 2\min\{|a-\theta|, |a+\theta|\}.\end{cases} \qquad (13)$$

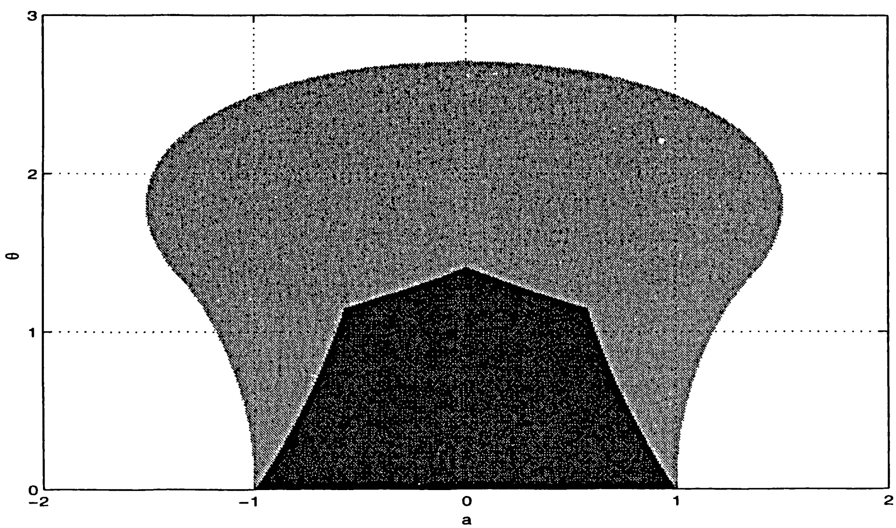

Fig. 1. Parameter regions ensuring invertibility of model (11). The bright-gray region is based on formula (12) and represents pairs (a,θ) that give negative value of $\mathsf{E}\log|A_{t,1}|$. The dark-gray region represents pairs (a,θ) that satisfy condition (13).

Especially if $a = 0$ substituting to equation (12) we get

$$\mathsf{E}\log|A_{t,1}| = \log\theta - 1,$$

and thus the model is invertible if $\theta < \mathrm{e}$. Substituting $a = 0$ to formula (13) we see that the constant R must be greater or equal to θ and R^{-1} must be greater than $\theta/2$. It follows that

$$\frac{2}{\theta} > \theta,$$

and thus the model is invertible if $\theta < \sqrt{2}$. The regions of invertibility of the model (11) are illustrated in Figure 1.

The RCMA(1) model with Gaussian distribution of parameter

Consider RCMA(1) model given by formula (11) where $Y_t \sim \mathsf{N}(0,1)$. In this case we have $A_{t,1} \sim \mathsf{N}(a, \theta^2)$ and

$$\mathsf{E}\log|A_{t,1}| = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\theta^2}} \exp\left\{-\frac{(x-a)^2}{2\theta^2}\right\} \log|x| \, dx.$$

Using substitution $x = \theta y + a$ we obtain

$$\begin{aligned} \mathsf{E}\log|A_{t,1}| &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{y^2}{2}\right\} \log|\theta y + a| \, dy \\ &= \log\theta + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{y^2}{2}\right\} \log\left|y + \frac{a}{\theta}\right| \, dy. \end{aligned} \tag{14}$$

The region of invertibility of the model (11) with $Y_t \sim \mathsf{N}(0,1)$ is illustrated in Figure 2.

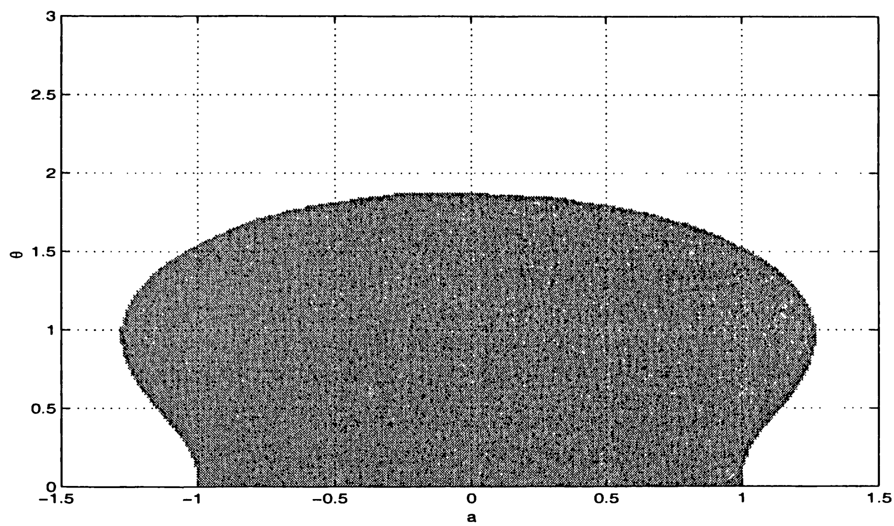

Fig. 2. Parameter region ensuring invertibility of model (11) with $Y_t \sim \mathsf{N}(0,1)$. The gray region is based on formula (14) and represents pairs (a,θ) that give negative value of $\mathsf{E}\log|A_{t,1}|$.

Simple non-linear moving average RCMA(1) model

Consider non-linear moving average model

$$X_t = (a + \theta\varepsilon_{t-1})\varepsilon_t + \alpha\varepsilon_{t-1}, \tag{15}$$

where $a \neq 0$, $\alpha \neq 0$, and $\theta > 0$ are real valued parameters. This model has been mentioned by Tong [8] and Robinson [5]. The invertibility of model (15) is investigated in Marek [3]. Additionally we assume that the $|\varepsilon_t| < 1$ almost surely. Using (15) we get formula

$$\varepsilon_t = \frac{X_t - \alpha\varepsilon_{t-1}}{a + \theta\varepsilon_{t-1}}. \tag{16}$$

Thus for each t we construct sequence $\{\varepsilon_t^{(n)}\}_{n=0}^{\infty}$, such that $\varepsilon_t^{(0)} = 0$, and for $n \geq 1$

$$\varepsilon_t^{(n)} = \frac{X_t - \alpha\varepsilon_{t-1}^{(n-1)}}{a + \theta\varepsilon_{t-1}^{(n-1)}}.$$

Each variable $\varepsilon_t^{(n)}$ is measurable w.r.t. $\mathcal{B}_t$. Thus provided $\varepsilon_t^{(n)} \xrightarrow[n\to\infty]{\mathsf{P}} \varepsilon_t$ the model (15) is invertible. Denote $d_t^{(n)} = \left|\varepsilon_t^{(n)} - \varepsilon_t\right|$. Using formula (16) we obtain

$$d_t^{(n)} = \frac{|a\alpha + \theta X_t| d_{t-1}^{(n-1)}}{|a + \theta\varepsilon_{t-1}| \left|a + \theta\varepsilon_{t-1}^{(n-1)}\right|}.$$

Using formula (15) we express

$$d_t^{(n)} = \frac{|\alpha + \theta\varepsilon_t| d_{t-1}^{(n-1)}}{\left|a + \theta\varepsilon_{t-1}^{(n-1)}\right|}. \tag{17}$$

Iterating (17) we obtain

$$d_t^{(n)} = |\varepsilon_{t-n}| \prod_{k=0}^{n-1} \frac{|\alpha + \theta\varepsilon_{t-k}|}{\left|a + \theta\varepsilon_{t-k-1}^{(n-k-1)}\right|}.$$

If inequality $|\alpha + \theta\varepsilon_{t-k}| < \left|a + \theta\varepsilon_{t-k-1}^{(n-k-1)}\right|$ holds then the $\{d_t^{(n)}\}$ sequence decreases. We get

$$\left|a + \theta\varepsilon_{t-1}^{(n-1)}\right| \geq |a| - \theta\left|\varepsilon_{t-1}^{(n-1)} - \varepsilon_{t-1} + \varepsilon_{t-1}\right| \geq |a| - 2\theta$$

and

$$|\alpha + \theta\varepsilon_t| \leq |\alpha| + \theta|\varepsilon_t| \leq |\alpha| + \theta.$$

Thus using more restrictive inequality

$$|a| - 2\theta > |\alpha| + \theta$$

we obtain a sufficient invertibility condition in the form

$$|\alpha| < |a| \quad \text{and} \quad \theta < \frac{|a| - |\alpha|}{3}. \tag{18}$$

Note that condition (18) also ensures that the singular point of transformation (16) is outside of the interval $(-1, 1)$.

Granger and Andersen [2] give the example of similar fashioned non-linear moving average model given by equation

$$X_t = \varepsilon_t + \theta \varepsilon_{t-2} \varepsilon_{t-1}.$$

The authors suggest that this model is never invertible with respect to value of parameter $\theta \neq 0$.

3. CONCLUSIONS

We proposed the RCMA model as a special non-linear generalization of linear moving average process with similar autocovariance structure. This class of models includes many well known time series models as well as models which have not been investigated yet. In linear moving average case the invertibility condition is used to select one of many alternative models which have the same autocovariances. The invertibility condition can play the similar role also in RCMA models case. However, the importance of the invertibility is also that a non-invertible models cannot be used to forecast because of it is often necessary to estimate white noise terms using the finite number of observations X_t and some initial constants only. Generally, to find an invertibility condition of RCMA(1) model is very difficult. While some RCMA models (e. g. ARCH) are always invertible or always non-invertible (e. g. see Granger and Andersen [2]) and while there had been found sufficient invertibility condition in some special cases like bilinear models, many unexplored time series models remain.

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