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EIGENSTRUCTURE ASSIGNMENT BY PROPORTIONAL–PLUS–DERIVATIVE FEEDBACK FOR SECOND–ORDER LINEAR CONTROL SYSTEMS

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This paper introduces a complete parametric approach for solving the eigenstructure assignment problem using proportional-plus-derivative feedback for second-order linear control systems. In this work, necessary and sufficient conditions that ensure the solvability for the second-order system are derived. A parametric solution to the feedback gain matrix is introduced that describes the available degrees of freedom offered by the proportional-plus-derivative feedback in selecting the associated eigenvectors from an admissible class. These freedoms can be utilized to improve robustness of the closed-loop system. The main advantage of the described approach is that the problem is tackled directly in the second-order form without transformation into the first-order form and without mass matrix inversion and the computation is numerically stable as it uses only the singular value decomposition and simple matrix transformation. Numerical examples are included to show the effectiveness of the proposed approach.

Keywords: eigenstructure assignment, second-order systems, proportional-plus-derivative feedback, feedback stabilization, parameterization

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1. INTRODUCTION

Many physical systems can generally be described by systems of second-order differential equations. The second-order system arises naturally in a wide variety of applications, including, control of large flexible space structures, spacecraft control, control of mechanical multibody systems, robotics control, vibration control in structural dynamics and earthquake engineering. In recent years, control design for the second-order system has gained much attention (e. g. [1–9]). The research of the second-order systems has gradually developed techniques that utilize the special structure and properties of such systems.

In [9] the criteria for the determination of controllability and observability for linear second-order systems have been discussed. The dynamic characteristics of certain types of mechanical systems can be improved effectively by directly assigning desired eigenvalues and associated eigenvectors that is called eigenstructure assignment. Concerning the eigenstructure assignment of second-order systems, only little research has been done under proportional and derivative feedbacks ([6, 8]).

In [8] it was proposed an algorithm of eigenstructure assignment for second-order systems using proportional and derivative feedbacks. In this approach, a set of target eigenvectors have to be prescribed, and the aim is to assign a set of closed-loop eigenvectors which are as close as possible to the prescribed target eigenvectors in a least-squares sense. The approach does not provide any design degrees of freedom and the closed-loop eigenvalues are restricted to be different from the open-loop ones. Moreover, the mass, damping and stiffness matrices are all restricted to be positive or semi-definite symmetric.

Recently, in [6] it was presented a complete parametric approach of eigenstructure assignment problem for second-order linear systems using proportional-plus-derivative controller. Complete parametric expressions for both the closed-loop feedback gains and the eigenvector matrices are established in terms of the closed-loop eigenvalues and a group of free vectors. However, the approach needs the inversion of the mass matrix. As a consequence, if the mass matrix is ill-conditioned, then the eigenvalues and eigenvectors will not be computed accurately. Furthermore, some properties offered by the system matrices such as definiteness, sparseness, bandedness, etc., are completely destroyed. In addition, the solution involves two sets of singular value decompositions, which needs a lot of computations.

The dynamic characteristics of mechanical systems can be improved effectively by direct assigning desired eigenvalues and associated eigenvectors that is called eigenstructure assignment (ESA). Assigning eigenvalues allows one to alter the stability characteristics while assigning eigenvectors alters the transient response of the system. ESA for the second-order system have developed the design method for a wide class of systems under proportional and derivative feedbacks [6, 8]. However, only the proposed ESA solution in this paper is straightforward and fully utilizes the properties of the second-order system. It involves only one set of singular value decomposition and the inversion of the mass matrix is not needed. Consequently, this solution is more accurate. To design a control system for such a dynamic model, the second-order equations have been usually rearranged into the first-order (state-space) form. However, for large structural systems, the resulting model suffers from the increased dimension. Additionally, the sparseness of the matrices is destroyed by matrix inversion. As a result, computational efficiency and physical insight are lost.

The other method is recently available for stabilizing second-order models in [7]. It is introduced a new, different approach to robust stabilization of the second-order systems with proportional-derivative compensators. A sufficient linear matrix inequality condition for robust stabilizability is obtained.

The purpose of this paper is to present a simple numerical technique to solve ESA for the second-order system dynamic controller. The feedback control design based on a combination of proportional and velocity feedbacks. Complete parametric expressions for both the closed-loop eigenvector matrices and the controller feedback gains are established in terms of the closed-loop eigenvalues and a group of free parameter vectors. The main computation involves only the singular value decomposition (SVD) that is stable in nature or a series of simple elementary matrix transformations if the desired eigenvalues not known *a priori*. The approach utilizes directly the original system data. The main advantages of the algorithm, besides being simple and numerically stable, are that it avoids complex arithmetics and it is easy to be implemented on a computer.

This paper is organized as follows. In the next section, the ESA problem formulation using proportional-plus-derivative feedback is described. Moreover, a complete parametric solution to this problem is presented. In Section 3, the illustrative examples are presented to show the effectiveness of the proposed technique. Finally, conclusions are discussed in Section 4.

2. EIGENSTRUCTURE ASSIGNMENT BY PROPORTIONAL–PLUS–DERIVATIVE FEEDBACK FOR SECOND–ORDER LINEAR SYSTEMS

In this section, we present an explicit parametric approach for solving the ESA problem for the second-order systems using proportional-plus-derivative feedback.

2.1. Eigenstructure assignment problem formulation

Consider a second-order linear, time-invariant, system equation in the form

$$\boldsymbol{M}\ddot{\boldsymbol{x}}(t) + \boldsymbol{D}\dot{\boldsymbol{x}}(t) + \boldsymbol{K}\boldsymbol{x}(t) = \boldsymbol{B}\boldsymbol{u}(t), \quad \boldsymbol{x}_0, \dot{\boldsymbol{x}}_0 \quad \text{given} \tag{1}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the vector of internal generalized coordinates, $\boldsymbol{u}(t) \in \mathbb{R}^m$ is the control vector, $\boldsymbol{M}, \boldsymbol{D}, \boldsymbol{K} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{B} \in \mathbb{R}^{n\times m}$ are, respectively, the mass, damping, stiffness and control matrices, and an overdot represents a differential with respect to time. The fundamental assumption imposed on the system is that the system is completely controllable [9]. The corresponding quadratic pencil (*characteristic polynomial matrix*) is

$$\boldsymbol{P}(\lambda) = \lambda^2\boldsymbol{M} + \lambda\boldsymbol{D} + \boldsymbol{K} \tag{2}$$

and system (1) is *regular* if and only if $\det(\boldsymbol{P}(\lambda))$ does not vanish identically. In this work we restrict ourselves to the regular quadratic pencils. The roots of $\det(\boldsymbol{P}(\lambda) = = 0$ are known as the *eigenfrequencies* of the system and play an important role in system stability. Stability of the system implies that these zeros must lie in the open left half plane.

The objective is to stabilize the system by means of a linear proportional and derivative feedback controller of the form

$$\boldsymbol{u}(t) = \boldsymbol{F}_1\boldsymbol{x}(t) - \boldsymbol{F}_2\dot{\boldsymbol{x}}(t), \quad \boldsymbol{F} = [\boldsymbol{F}_1, \boldsymbol{F}_2] \tag{3}$$

where $\boldsymbol{F}_1$ and $\boldsymbol{F}_2 \in \mathbb{R}^{m\times n}$ are, respectively, constant proportional and velocity feedback matrices, which assigns prescribed closed-loop eigenvalues and eigenvectors that stabilize the system and achieve the desired performance. By the substitution of (3) into (1), we obtain the closed-loop system

$$\boldsymbol{M}\ddot{\boldsymbol{x}}(t) + \boldsymbol{D}\dot{\boldsymbol{x}}(t) + \boldsymbol{K}\boldsymbol{x}(t) = -\boldsymbol{B}(\boldsymbol{F}_1\boldsymbol{x}(t) + \boldsymbol{F}_2\dot{\boldsymbol{x}}(t)). \tag{4}$$

Then the problem is to find the matrices F_1 and F_2 such that the eigenvalues and eigenvectors of the associated closed-loop quadratic pencil

$$P_c(\lambda) = \lambda^2 M + \lambda(D + BF_2)) + (K + BF_1) \tag{5}$$

can be altered as required to ensure and improve the stability of the system. The problem of finding F_1 and F_2 such that the closed-loop quadratic pencil $P_c(\lambda)$ has a desired set of eigenvalues and eigenvectors is called the ESA problem for the system (1).

For analysis and design purposes, the system dynamics are usually transformed to the standard first-order state form by introducing the $2n\times 1$ dimensional state vector $z(t)^{\mathrm{T}} \equiv [x(t)^{\mathrm{T}}, \dot{x}^{\mathrm{T}}]$ as follows

$$\dot{z}(t) = Az(t) + \tilde{B}u(t) \tag{6}$$

where

$$A = \begin{pmatrix} 0 & I_n \\ -M^{-1}K & -M^{-1}D \end{pmatrix} \quad \text{and} \quad \tilde{B} = \begin{pmatrix} 0 \\ M^{-1}B \end{pmatrix} \tag{7}$$

where I_n is the $n \times n$ identity matrix. Throughout this paper, M is assumed to be invertible.

The state controller is of the form

$$u(t) = -Fz(t), \quad F = [F_1, F_2]. \tag{8}$$

Then the closed-loop system dynamics becomes

$$\dot{z}(t) = A_c z(t) \tag{9}$$

where

$$A_c = (A - \tilde{B}F) = \begin{pmatrix} 0 & I_n \\ -M^{-1}(K + BF_1) & -M^{-1}(D + BF_2) \end{pmatrix}. \tag{10}$$

In the majority of methods that have been proposed for solving this problem, the second-order system (1) is rewritten as a first-order system (6) and the techniques for treating the linear feedback design problem can be applied. However, several difficulties arise and retaining the model in the matrix second-order form has many advantages as follows:

1. Physical insight of the original problem is preserved;
2. It is computationally efficient as the dimension of the system is lower than that of the first-order form; and
3. System matrix sparsity and any other special nature of the original matrices are preserved, which is useful in analysis and design.

The above concerns favor tackling ESA problem in second-order form directly.

Let $\Gamma = \{\lambda_i \in \mathbb{C},\ i = 1, \ldots, r,\ 1 \le r \le 2n\}$ be a set of desired self-conjugate eigenvalues, where r is the number of distinct eigenvalues, and denote the algebraic and geometric multiplicity of the ith eigenvalue λ_i by m_i and q_i, respectively, $(1 \le$

$q_i \le m_i$). The length of q_i chains of generalized eigenvectors with λ_i are denoted by p_{ij}, ($j = 1, \ldots, q_i$). Then in the Jordan canonical form of the closed-loop matrix, there are q_i blocks associated with the ith eigenvalue λ_i of orders p_{ij}. It satisfies

$$\sum_{i=1}^{s}\sum_{j=1}^{q_i} p_{ij} = 2n.$$

Let the right eigenvector and generalized eigenvectors of the closed-loop matrix with λ_i be denoted by $v_{ij,k} \in \mathbb{C}^{2n}$, $i = 1, \ldots, r$, $j = 1, \ldots, q_i$, $k = 1, \ldots, p_{ij}$. According to the definition of the right eigenvector and generalized eigenvectors for a multiple eigenvalue, then

$$\boldsymbol{A}_c v_{ij,k} = \lambda_i v_{ij,k} + v_{ij,k-1}, \quad v_{ij,0} = \boldsymbol{0}, \quad \forall i,\, j,\, k. \tag{11}$$

This equation demonstrates the relation of assignable right eigenvector and generalized eigenvectors with the associated eigenvalue. The notations of the set $v_{ij,k}$ are defined as

$$\boldsymbol{V} \equiv (\boldsymbol{V}_1, \ldots, \boldsymbol{V}_r) \equiv \begin{pmatrix} \boldsymbol{V}^1 \\ \boldsymbol{V}^2 \end{pmatrix} \equiv \begin{pmatrix} \boldsymbol{V}_1^1, & \ldots, & \boldsymbol{V}_r^1 \\ \boldsymbol{V}_1^2, & \ldots, & \boldsymbol{V}_r^2 \end{pmatrix} \in \mathbb{C}^{2n\times 2n},$$

$$\boldsymbol{V}_{ij}^l \equiv [v_{ij,1}^l, \cdots, v_{ij,p_{ij}}^l] \in \mathbb{C}^{n\times p_{ij}} \tag{12}$$

where $l = 1, 2$, and $\boldsymbol{V}_i \in \mathbb{C}^{2n\times m}$ contains all right eigenvector and generalized eigenvectors associated with the eigenvalue λ_i, and $\det(\boldsymbol{V}) \neq 0$.

Then the ESA problem for second-order linear systems using proportional and derivative feedback can be formulated as follows:

ESA Problem. Given the real matrices $\boldsymbol{M}, \boldsymbol{D}, \boldsymbol{K}$ and $\boldsymbol{B}$, and the desired $2n$ self-conjugate set Γ, find real proportional and derivative feedback gain matrices $\boldsymbol{F}_1$ and $\boldsymbol{F}_2 \in \mathbb{R}^{m\times n}$ such that the spectrum of the closed-loop quadratic pencil $\boldsymbol{P}_c(\lambda)$ has admissible eigenvalues and associated eigenvectors.

The aim now is to develop a simple algorithm, which manipulates the system data, and solve the above problem.

2.2. Solution to eigenstructure assignment problem

In this subsection, the solution to ESA problem for second-order system is introduced.

Utilizing (10), then equation (11) can be rewritten as

$$\begin{pmatrix} \boldsymbol{0} & \boldsymbol{I}_n \\ -\boldsymbol{M}^{-1}(\boldsymbol{K}+\boldsymbol{B}\boldsymbol{F}_1) & -\boldsymbol{M}^{-1}(\boldsymbol{D}+\boldsymbol{B}\boldsymbol{F}_2) \end{pmatrix} \begin{pmatrix} v_{ij,k}^1 \\ v_{ij,k}^2 \end{pmatrix}$$

$$= \lambda_i \begin{pmatrix} v_{ij,k}^1 \\ v_{ij,k}^2 \end{pmatrix} + \begin{pmatrix} v_{ij,k-1}^1 \\ v_{ij,k-1}^2 \end{pmatrix}, \quad v_{ij,0} = \boldsymbol{0}, \quad \forall i,\, j,\, k. \tag{13}$$

The above equation can be decomposed into

$$v^2_{ij,k} = \lambda_i v^1_{ij,k} + v^1_{ij,k-1}, \quad v^1_{ij,0} = \mathbf{0}, \quad \forall i,\ j,\ k \tag{14}$$

and

$$-\boldsymbol{M}^{-1}((\boldsymbol{K} + \boldsymbol{B}\boldsymbol{F}_1)v^1_{ij,k} + (\boldsymbol{D} + \boldsymbol{B}\boldsymbol{F}_2)v^2_{ij,k}) = \lambda_i v^2_{ij,k} + v^2_{ij,k-1},\ v^2_{ij,0} = \mathbf{0},\ \forall i,\ j,\ k. \tag{15}$$

Then the above equation can be rewritten as

$$(\boldsymbol{K} + \boldsymbol{B}\boldsymbol{F}_1)v^1_{ij,k} + (\boldsymbol{D} + \boldsymbol{B}\boldsymbol{F}_2)v^2_{ij,k} = -\lambda_i \boldsymbol{M} v^2_{ij,k} - \boldsymbol{M} v^2_{ij,k-1}, \quad \forall i,\ j,\ k. \tag{16}$$

Utilizing (14) and substitute in (16), this leads to

$$\boldsymbol{K}v^1_{ij,k} + (\lambda_i\boldsymbol{M} + \boldsymbol{D})(\lambda_i v^1_{ij,k} + v^1_{ij,k-1}) + \boldsymbol{B}(\boldsymbol{F}_1 v^1_{ij,k} + \boldsymbol{F}_2 v^2_{ij,k}) = -\boldsymbol{M}v^2_{ij,k-1}, \forall i,\ j,\ k. \tag{17}$$

Collecting similar terms in (17), we can get

$$(\lambda_i^2\boldsymbol{M} + \lambda_i\boldsymbol{D} + \boldsymbol{K})v^1_{ij,k} + \boldsymbol{B}(\boldsymbol{F}_1 v^1_{ij,k} + \boldsymbol{F}_2 v^2_{ij,k}) = -(\lambda_i\boldsymbol{M} + \boldsymbol{D})v^1_{ij,k-1} - \boldsymbol{M}v^2_{ij,k-1}, \forall i,\ j,\ k. \tag{18}$$

Let the auxiliary vectors

$$w_{ij,k} = \boldsymbol{F}_1 v^1_{ij,k} + \boldsymbol{F}_2 v^2_{ij,k} \in \mathbb{C}^m, i = 1, \dots, r, \quad j = 1, \dots, q_i, \quad k = 1, \dots, p_{ij}, \tag{19}$$

are introduced. The set of $w_{ij,k}$ is defined in a similar manner to the set of $v_{ij,k}$ as

$$\boldsymbol{W} \equiv [\boldsymbol{W}_1, \dots, \boldsymbol{W}_r] \in \mathbb{C}^{m\times 2n}, \quad \boldsymbol{W}_i \equiv [\boldsymbol{W}_{i1}, \dots, \boldsymbol{W}_{iq_i}] \in \mathbb{C}^{m\times m_i},$$
$$\boldsymbol{W}_{ij} \equiv [w_{ij,1}, \dots, w_{ij}, p_{ij}] \in \mathbb{C}^{m\times p_{ij}}. \tag{20}$$

This leads to

$$(\lambda_i^2\boldsymbol{M} + \lambda_i\boldsymbol{D} + \boldsymbol{K})v^1_{ij,k} + \boldsymbol{B}w_{ij,k} = -(\lambda_i\boldsymbol{M} + \boldsymbol{D})v^1_{ij,k-1} - \boldsymbol{M}v^2_{ij,k-1}, v_{ij,0} = \mathbf{0}, \quad \forall i,\ j,\ k. \tag{21}$$

The above equation can be equivalently written in the following compact matrix form

$$[\lambda_i^2\boldsymbol{M} + \lambda_i\boldsymbol{D} + \boldsymbol{K}, \boldsymbol{B}] \begin{pmatrix} v^1_{ij,k} \\ w_{ij,k} \end{pmatrix} = -[\lambda_i\boldsymbol{M} + \boldsymbol{D}, \boldsymbol{M}]v_{ij,k-1},\ v_{ij,0} = \mathbf{0}, \forall i,\ j,\ k. \tag{22}$$

Then parameter vectors $v^1_{ij,k} \in \mathbb{C}^n$ and $w_{ij,k} \in \mathbb{C}^m$ are arbitrary chosen under the condition that the columns of matrix $\boldsymbol{V}$ are linearly independent.

The auxiliary vectors in (19) can be rewritten as

$$w_{ij,k} = [\boldsymbol{F}_1, \boldsymbol{F}_2] \begin{pmatrix} v^1_{ij,k} \\ v^2_{ij,k} \end{pmatrix} = [\boldsymbol{F}_1, \boldsymbol{F}_2]v_{ij,k}. \tag{23}$$

A parametric solution to the ESA problem by proportional and derivative feedbacks is derived as

$$[\boldsymbol{F}_1, \boldsymbol{F}_2] = \boldsymbol{W}\boldsymbol{V}^{-1}. \tag{24}$$

There exists a real feedback gain matrix $\boldsymbol{F}$ if the following conditions are satisfied:

1. The desired closed-loop eigenvalues are closed under complex conjugation;
2. The right generalized eigenvectors $\{v_{ij,k} \in \mathbb{C}^n,\ i = 1,\dots,r,\ j = 1,\dots,q_i,\ k = 1,\dots,p_{ij}\}$ are linearly independent and for complex-conjugate poles, $\bar{\lambda}_{i_2} = \lambda_{i_1}$ then $\bar{v}_{i_2 j,k} = v_{i_1 j,k}$; and
3. There exists a set of vectors $\{w_{ij,k} \in \mathbb{C}^m,\ i = 1,\dots,r,\ j = 1,\dots,q_i,\ k = 1,\dots,p_{ij}\}$, satisfying (22) and $\bar{w}_{i_2 j,k} = w_{i_1 j,k}$ for $\bar{\lambda}_{i_2} = \lambda_{i_1}$.

Then the feedback gain matrix is parameterized directly in terms of the eigenstructure of the closed-loop system, which can be selected to ensure robustness by exploiting freedom of these parameters. In the following, we obtain the simple and more general parametric solutions of $v_{ij,k}$ and $w_{ij,k}$ in (22). A complete parametric form is introduced and a new procedure is derived which yields a parametric expression for $\boldsymbol{F}$ involving free parameter vectors.

2.3. Parameterization approach for the eigenstructure assignment

The aim now is to find a parametric solution to the ESA problem via proportional-plus-derivative feedback. We remark that developing parametric solutions to this problem is useful in that one can then think of solving other important variations of the problem, such as the robust ESA problem by exploiting freedom of these parameters. Concerning the controllability of second-order system, the following lemma is introduced [9].

Lemma 1. (See [9].) The second-order system (1) is controllable if and only if

$$\operatorname{rank}[\lambda^2 \boldsymbol{M} + \lambda \boldsymbol{D} + \boldsymbol{K}, \boldsymbol{B}] = n, \quad \forall\, \lambda \in \mathbb{C}. \tag{25}$$

Based on the controllability of the second-order system, the parametric formula is derived. Applying the singular value decomposition (SVD) to the matrix $[\lambda_i^2 \boldsymbol{M} + \lambda_i \boldsymbol{D} + \boldsymbol{K}, \boldsymbol{B}]$ gives

$$[\lambda_i^2 \boldsymbol{M} + \lambda_i \boldsymbol{D} + \boldsymbol{K}, \boldsymbol{B}] = \boldsymbol{X}_i \boldsymbol{\Gamma}_i \boldsymbol{Q}_i^T, \quad \boldsymbol{\Gamma}_i = [\Sigma_i, 0], \quad i = 1,\dots,r \tag{26}$$

where $\boldsymbol{\Gamma}_i \in \mathbb{C}^{n\times(n+m)}$ is a matrix containing all singular values of the matrix $[\lambda_i^2 \boldsymbol{M} + \lambda_i \boldsymbol{D} + \boldsymbol{K}]$, $\Sigma_i \in \mathbb{C}^{n\times n}$ is a nonsingular diagonal matrix, and $\boldsymbol{X}_i \in \mathbb{C}^{n\times n}$ and $\boldsymbol{Q}_i \in \mathbb{C}^{(n+m)\times(n+m)}$ are two orthogonal matrices. The columns of the matrix $\boldsymbol{X}_i$ and columns of the matrix $\boldsymbol{Q}_i$ are the left and right singular vectors of the matrix $[\lambda_i^2 \boldsymbol{M} + \lambda_i \boldsymbol{D} + \boldsymbol{K}, \boldsymbol{B}]$. Then we have

$$\boldsymbol{X}_i^{\mathrm{T}}[\lambda_i^2 \boldsymbol{M} + \lambda_i \boldsymbol{D} + \boldsymbol{K}, \boldsymbol{B}]\, \boldsymbol{Q}_i = [\Sigma_i,\ 0], \quad \Sigma_i = \operatorname{diag}\{\sigma_{i1}, \cdots, \sigma_{in}\}. \tag{27}$$

Pre-multiplying the above equation by Σ_i^{-1}, yields

$$\boldsymbol{P}_i[\lambda_i^2 \boldsymbol{M} + \lambda_i \boldsymbol{D} + \boldsymbol{K}, \boldsymbol{B}]\, \boldsymbol{Q}_i = [\boldsymbol{I}_n, 0] \tag{28}$$

where

$$\boldsymbol{P}_i = \Sigma_i^{-1} \boldsymbol{X}_i^{\mathrm{T}} = \operatorname{diag}\{1/\sigma_{i1}, \cdots, 1/\sigma_{in}\} \boldsymbol{X}_i^{\mathrm{T}} \in \mathbb{C}^{n\times n}.$$

Further, partition matrix Q_i into the following form

$$Q_i = \begin{pmatrix} Q_{i,11} & Q_{i,12} \\ Q_{i,21} & Q_{i,22} \end{pmatrix}, \quad i = 1, \ldots, r$$

where $Q_{i,11} \in \mathbb{C}^{n\times n}$, $Q_{i,12} \in \mathbb{C}^{n\times m}$, $Q_{i,21} \in \mathbb{C}^{m\times n}$ and $Q_{i,22} \in \mathbb{C}^{m\times m}$.

Now, we have the following theorem for solution to the ESA problem for second-order system by proportional and derivative feedback.

Theorem 1. Let the second-order linear system (1) be controllable, and matrix M is nonsingular. Then all the solutions to $v^1_{ij,k}$ and $w_{ij,k}$ in (22) are given by

$$\begin{pmatrix} v^1_{ij,k} \\ w_{ij,k} \end{pmatrix} = Q_i \begin{pmatrix} -P_i[\lambda_i M + D, M]v_{ij,k-1} \\ f_{ij,k} \end{pmatrix}, \quad v_{ij,0} = 0, \quad \forall i, j, k. \tag{29}$$

Then the vectors can be written as

$$\begin{aligned} v^1_{ij,k} &= -Q_{i,11}P_i[\lambda_i M + D, M]v_{ij,k-1} + Q_{i,12}f_{ij,k}, \\ v^2_{ij,k} &= \lambda_i v^1_{ij,k} + v^1_{ij,k-1}, \\ w_{ij,k} &= -Q_{i,21}P_i[\lambda_i M + D, M]v_{ij,k-1} + Q_{i,22}f_{ij,k}, v_{ij,0} = 0, \quad \forall i, j, k \end{aligned} \tag{30}$$

where $f_{ij,k} \in \mathbb{C}^m$, $i = 1, \ldots, r$, $j = 1, \ldots, q_i$, $k = 1, \ldots, p_{ij}$, are a set of arbitrarily free parameter vectors satisfying the following constraints:

$$\det(V) \neq 0 \text{ and } \bar{f}_{i_2 j,k} = f_{i_1 j,k} \text{ if } \bar{\lambda}_{i_2} = \lambda_{i_1}, \quad \text{(for real gain)}$$

and P_i and Q_i are matrices satisfying (28).

Proof. We need to prove that the set of vectors satisfying (22) and the set of vectors given by (29) are equal. Then using (22) and (29), yields

$$\begin{aligned} &[\lambda_i^2 M + \lambda_i D + K, B] \begin{pmatrix} v^1_{ij,k} \\ w_{ij,k} \end{pmatrix} \\ &= [\lambda_i^2 M + \lambda_i D + K, B]\, Q_i \begin{pmatrix} -P_i[\lambda_i M + D, M]v_{ij,k-1} \\ f_{ij,k} \end{pmatrix} \\ &= P_i^{-1}[I_n, 0] \begin{pmatrix} -P_i[\lambda_i M + D, M]v_{ij,k-1} \\ f_{ij,k} \end{pmatrix} \\ &= -[\lambda_i M + D, M]v_{ij,k-1}, \quad v_{ij,0} = 0, \quad \forall i, j, k. \end{aligned} \tag{31}$$

Therefore, the vectors given by (29) satisfy (22). Now, we show that vectors $v^1_{ij,k}$ and $w_{ij,k}$ in (22) ($i = 1, \ldots, r$, $j = 1, \ldots, q_i$, $k = 1, \ldots, p_{ij}$) can be expressed in the form of (29). From (28) one can obtain

$$P_i[\lambda_i^2 M + \lambda_i D + K, B] = [I_n, 0]\, Q_i^{-1}, \quad i = 1, \ldots, r. \tag{32}$$

The above equation can be expressed as

$$P_i[\lambda_i^2 M + \lambda_i D + K, B] \begin{pmatrix} v^1_{ij,k} \\ w_{ij,k} \end{pmatrix} = [I_n, 0]\, Q_i^{-1} \begin{pmatrix} v^1_{ij,k} \\ w_{ij,k} \end{pmatrix}, \quad \forall i, j, k. \tag{33}$$

Utilizing (22), then

$$-P_i[\lambda_i M + D, M]v_{ij,k-1} = [I_n, 0]\begin{pmatrix} e_{ij,k} \\ f_{ij,k} \end{pmatrix}, \quad v_{ij,k} = 0, \quad \forall i, j, k \tag{34}$$

where

$$\begin{pmatrix} e_{ij,k} \\ f_{ij,k} \end{pmatrix} = Q_i^{-1}\begin{pmatrix} v_{ij,k}^1 \\ w_{ij,k} \end{pmatrix}, \quad \forall i, j, k. \tag{35}$$

Then from (34) we obtain

$$e_{ij,k} = -P_i[\lambda_i M + D, M]v_{ij,k-1}, \quad v_{ij,0} = 0, \quad \forall i, j, k. \tag{36}$$

Substituting (36) into (35) we obtain (29). □

Theorem 1 gives complete and explicit parametric solutions with the complete and explicit freedom of the ESA using proportional and derivative feedbacks. These solutions are expressed by the eigenvalues and a group of free parameter vectors, $f_{ij,k}$. By especially choosing the free parameter vectors, solutions with desired properties can be obtained. The vectors $f_{ij,k}$ represent the degrees of the freedom of ESA using proportional and derivative feedback.

Remark 1. It should be noted that for the case of distinct eigenvalues ($m_i = q_i = 1$, $r = 2n$) the computations of v_i and w_i, take the simple form, and are given by

$$v_i^1 = Q_{i,12}f_i, v_i^2 = \lambda_i Q_{i,12}f_i, w_i = Q_{i,22}f_i, \quad i = 1, \ldots, 2n. \tag{37}$$

Then the feedback gain is

$$[F_1, F_2] = (Q_{1,22}f_1, \cdots Q_{2n,22}f_{2n})\begin{pmatrix} Q_{1,12}f_1 & \cdots & Q_{2n,12}f_{2n} \\ \lambda_1 Q_{1,12}f_1 & \cdots & \lambda_{2n}Q_{2n,12}f_{2n} \end{pmatrix}^{-1}. \tag{38}$$

Remark 2. For single-input system ($m = 1$), the parameter vectors $f_{ij,k}$ reduce to scalars and accordingly, the feedback gain is unique regardless of the choice of $f_{ij,k}$.

Remark 3. In the case that the closed-loop eigenvalues λ_i, $i = 1, \ldots, 2n$ are not known *a priori*, one may seek, instead of the matrices P_i and Q_i satisfying (28), the unimodular polynomial matrices $P(\lambda)$ and $Q(\lambda)$ satisfying the following equation

$$P(\lambda)[\lambda^2 M + \lambda D + K, B]\, Q(\lambda) = [I_n, 0]. \tag{39}$$

These reductions can be completed by a series of simple elementary matrix transformations. The Smith canonical form is used that exploits the fact that for a controllable second-order system (1) the matrix $[\lambda^2 M + \lambda D + K, B]$ maintains full rank for all values of λ. The Smith canonical form constructs two unimodular matrices

$P(\lambda)$ and $Q(\lambda)$ that diagonalize a given polynomial matrix as (39). Consequently, the augmented matrix

$$G = \begin{pmatrix} I_n & \lambda^2 M + \lambda D + K & B \\ 0 & I_{n+m} & \end{pmatrix} \tag{40}$$

can be changed into the form of

$$H = \begin{pmatrix} P(\lambda) & I_n & 0 \\ 0 & Q(\lambda) & \end{pmatrix}. \tag{41}$$

By applying a series of row elementary transformations within the upper n rows and a series of column elementary transformations within the last $n+m$ columns the matrices $P(\lambda)$ and $Q(\lambda)$ in matrix H are unimodular and automatically satisfying (39).

Based on the discussion and analysis above, an algorithm for solving the ESA problem for second-order system can be given as follows:

ESA Algorithm.

Input. Real matrices M, D, K, B, where the system is controllable and M is nonsingular, and a set of $2n$ self-conjugate complex numbers.

Step 1. Using Singular value decomposition (SVD) to obtain the matrices P_i and Q_i, $i = 1, \dots, r$, as in (28), or a series of simple elementary matrix transformations if the desired eigenvalues are not known *a priori.*

Step 2. Choose arbitrary parameter vectors $f_{ij,k} \in \mathbb{C}^m$, $i = 1, \dots, r$, $j = 1, \dots, q_i$, $k = 1, \dots, p_{ij}$, in such a way that $\bar{f}_{i_2 j,k} = f_{i_1 j,k}$ if $\bar{\lambda}_{i_2} = \lambda_{i_1}$.

Step 3. Calculate the eigenvectors $v_{ij,k} \in \mathbb{C}^{2n}$, $i = 1, \dots, r$, $j = 1, \dots, q_i$, $k = 1, \dots, p_{ij}$, using (30). If the matrix V is singular, then return to Step 2 and select different parameters $f_{ij,k}$, until V is nonsingular.

Step 4. Compute the vectors $w_{ij,k} \in \mathbb{C}^m$, $i = 1, \dots, r$, $j = 1, \dots, q_i$, $k = 1, \dots, p_{ij}$, using (30) and construct matrix W.

Step 5. Compute the proportional and derivative feedback gain matrix using

$$[F_1, F_2] = WV^{-1}.$$

From the above results we can observe that the system poles can always be assigned by proportional-plus-derivative feedback controller for any controllable system if and only if the mass matrix M is nonsingular. Based on the controllability of the second order system, this work proposes a solution to the ESA problem. Complete parametric expressions for both the closed-loop eigenvector matrices and the feedback gains are established in the terms of the closed-loop eigenvalues and a group of free parameter vectors. Both the closed-loop eigenvalues and these parameters can be properly chosen to produce a closed-loop system with desired system specifications. The proposed approach is simple because the main computations involved are only the singular value decomposition and it utilizes directly the original system data. In the case of single-input, $m = 1$, there is only at most one solution. In the case of multi-input, $m > 1$, the solution is generally non-unique, and extra conditions must be imposed to specify the solution.

In the following, two numerical examples are included to demonstrate the effectiveness of this procedure.

3. ILLUSTRATIVE EXAMPLES

In this section, we present numerical examples to illustrate feasibility and effectiveness of the proposed technique using a MATLAB version 6.5.

Example 1. Consider the mechanical system shown in Figure 1. The system consisting of five material points linked by elastic springs [7], the points can slide without friction along their respective axes. Two external forces acting at masses 1 and 5 control the system. Mass, distance to the origin at the equilibrium, and spring stiffness are given for each point in Table 1.

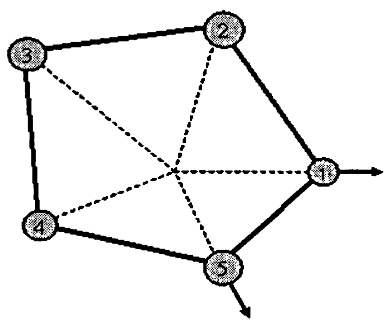

Fig. 1. Five masses linked by an elastic spring.

Table 1. System data.

Point	Mass	Distance	Spring	Stiffness
1	0.5093	0.8034	1–2	1.461
2	0.9107	0.7430	2–3	1.369
3	0.7224	0.9456	3–4	1.088
4	0.8077	0.8810	4–5	1.203
5	0.8960	0.7282	5–1	1.468

The dynamical system equations are given by equations (1) where

$$\boldsymbol{M} = \boldsymbol{I}_5, \quad \boldsymbol{D} = \boldsymbol{0}_5,$$

$$K = \begin{pmatrix} 2.565 & 1.080 & 0 & 0 & 1.089 \\ 0.6038 & 0.8206 & 0.4766 & 0 & 0 \\ 0 & 0.6009 & 1.504 & 0.4808 & 0 \\ 0 & 0 & 0.4300 & 1.114 & 0.5131 \\ 0.6190 & 0 & 0 & 0.4626 & 0.8352 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1.964 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1.116 & 0 \end{pmatrix}.$$

The open-loop eigenvalues are all purely imaginary and located at

$$\{\pm \mathrm{j}1.7828,\quad \pm\mathrm{j}1.3800,\quad \pm\mathrm{j}1.1451,\quad \pm\mathrm{j}0.5674 \text{ and } \pm\mathrm{j}0.3506\}.$$

They correspond to the eigenfrequences of free vibrations of the masses.

In the following, we consider the assignment of three different cases:

Case 1: The desired closed-loop eigenvalues are

$$\{-1,\ -1.5,\ -2,\ -2.5,\ -3,\ -3.5,\ -4,\ -4.5,\ -5 \text{ and } -5.5\}.$$

Specially choosing

$f_1 = [1, 6]^T$, $f_2 = [1, 3]^T$, $f_3 = [3, 2]^T$, $f_4 = [5, 1]^T$, $f_5 = [4, 5]^T$,
$f_6 = [3, 1]^T$, $f_7 = [1, 2]^T$, $f_8 = [5, 1]^T$, $f_9 = [6, 0]^T$ and $f_{10} = [2, 1]^T$

and using the SVD as in (28) the matrices P_i and Q_i, $i = 1, \ldots, 5$, can be obtained.

Then the proportional and derivative feedback gain matrices are

$$F_1 = 10^3 \begin{pmatrix} -0.0411 & 1.2666 & 1.6800 & 1.0221 & 0.0150 \\ 0.0993 & -1.4734 & -2.3437 & -1.1630 & 0.0534 \end{pmatrix}$$

and

$$F_2 = 10^3 \begin{pmatrix} -0.0023 & 0.4460 & -2.2141 & 0.2378 & 0.0092 \\ 0.0113 & -0.7263 & 1.8127 & -0.5468 & 0.0038 \end{pmatrix}.$$

The computed closed-loop eigenvalues are

$$\begin{array}{ll} -1.00000000000004, & -1.50000000000472, \\ -1.99999999998233, & -2.50000000000172, \\ -3.00000000004541, & -3.49999999991008, \\ -3.99999999996453, & -4.49999999999473, \\ -5.00000000018671, & -5.49999999986991. \end{array}$$

Case 2: The desired closed-loop poles are

$$\{1\pm j,\ -2\pm j,\ -3\pm j,\ -4\pm j \text{ and } -5\pm j\}.$$

Choosing

$$f_1 = f_2 = [1,\ 2]^T,\ f_3 = f_4 = [3,\ 1]^T,\ f_5 = f_6 = [2,\ 1]^T,\ f_7 = f_8 = [1,\ 3]^T$$

and

$$f_9 = f_{10} = [2,\ 3]^T.$$

A stabilizing controller is obtained as

$$F_1 = \begin{pmatrix} 28.6151 & -399.3427 & -532.2271 & -289.2215 & 55.6172 \\ 64.9285 & -735.7233 & -923.2887 & -498.2693 & 32.5990 \end{pmatrix}$$

and

$$F_2 = \begin{pmatrix} 2.3729 & -218.9939 & 403.3329 & -270.3257 & 11.2184 \\ 8.9003 & -416.6206 & 815.5803 & -275.1871 & 2.9371 \end{pmatrix}.$$

Case 3: The desired eigenvalues are

$$\{-1,\ -1,\ -2,\ -2,\ -3,\ -3,\ -4,\ -4,\ -5 \text{ and } -5\}.$$

Choosing

$$f_{11,1} = f_{21,1} = f_{31,1} = f_{41,1} = f_{51,1} = [1,2]^T$$

and

$$f_{12,1} = f_{22,1} = f_{32,1} = f_{42,1} = f_{52,1} = [2,1]^T.$$

Therefore

$$F_1 = \begin{pmatrix} 4.2169 & 14.9640 & -28.8808 & 12.3616 & 48.6339 \\ 58.7223 & -439.7042 & -790.0286 & -279.8865 & 8.9084 \end{pmatrix}$$

and

$$F_2 = \begin{pmatrix} 0.4773 & -20.8923 & -151.3115 & -157.2235 & 11.2471 \\ 8.8840 & -320.3067 & 281.9555 & -1.3527 & 0.4311 \end{pmatrix}.$$

Example 2. Consider a linear system with $n = 3$ and $m = 2$ (cf. [3]). The equations of motion can be written in the form of (1) with

$$\boldsymbol{M} = \text{diag}\{10, 10, 10\}, \boldsymbol{D} = \boldsymbol{0}, \boldsymbol{K} = \begin{pmatrix} 40 & -40 & 0 \\ -40 & 80 & -40 \\ 0 & -40 & 80 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \\ 3 & 4 \end{pmatrix}.$$

The system has zero damping and the open-loop eigenvalues are

$$\{\pm j3.6039,\ \pm j2.4940 \text{ and } \pm j0.8901\}$$

again expressing the eigenfrequences of free vibrations of the considered system.

For this system, a pair of unimodular matrices $\boldsymbol{P}(\lambda)$ and $\boldsymbol{Q}(\lambda)$ satisfying (39) can be obtained as

$$\boldsymbol{P}(\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1.5 & -0.5 & 1 \end{pmatrix}$$

and

$$\boldsymbol{Q}(\lambda) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1/140 \\ 0 & 0 & \dfrac{-1}{5\lambda^2+20} & \dfrac{1}{30}+\dfrac{1}{5\lambda^2+20} & \dfrac{1}{140}+\dfrac{1}{5\lambda^2+20} \\ 0 & 0 & 0 & 1/60 & 1/70 \\ -0.5 & 0.5 & 1+\dfrac{8}{\lambda^2+4} & -\dfrac{\lambda^2}{6}-\dfrac{8}{3}-\dfrac{8}{\lambda^2+4} & -\dfrac{6}{7}-\dfrac{8}{\lambda^2+4} \\ 0.75 & -0.25 & \dfrac{-\lambda^2-20}{2\lambda^2+8} & 1.5+\dfrac{\lambda^2}{12}+\dfrac{\lambda^2+20}{2\lambda^2+8} & \dfrac{1}{14}-\dfrac{\lambda^2}{28}+\dfrac{\lambda^2+20}{2\lambda^2+8} \end{pmatrix}.$$

In the following, we consider the assignment of three different cases:

Case 1: The desired closed-loop eigenvalues are

$$\{-1,\ -2,\ -3,\ -4,\ -5\ \text{and}\ -6\}.$$

Choosing

$$f_1 = [1,\ 3]^{\mathrm{T}},\ f_2 = [1,\ 2]^{\mathrm{T}},\ f_3 = [3,\ 1]^{\mathrm{T}},\ f_4 = [1,\ 1]^{\mathrm{T}},\ f_5 = [4,\ 1]^{\mathrm{T}}$$

and

$$f_6 = [3,\ 2]^{\mathrm{T}}.$$

The proportional and derivative feedback gain matrices are

$$\boldsymbol{F}_1 = \begin{pmatrix} -251.3475 & -114.3680 & 451.0617 \\ 150.5340 & 61.5276 & -164.3802 \end{pmatrix}$$

and

$$\boldsymbol{F}_2 = \begin{pmatrix} -28.9236 & 30.9118 & 45.1396 \\ 60.8117 & -14.3807 & -20.5231 \end{pmatrix}.$$

Case 2: The desired closed-loop poles are

$$\{-1\pm j2,\ -2\pm j2\ \text{and}\ -3\pm j2\}.$$

Choosing

$$f_1 = f_2 = [1,\ 2]^{\mathrm{T}},\ f_3 = f_4 = [3,\ 1]^{\mathrm{T}}\quad \text{and}\quad f_5 = f_6 = [2,\ 1]^{\mathrm{T}}.$$

Then the feedback gains are

$$\boldsymbol{F}_1 = \begin{pmatrix} 155.2533 & -66.0528 & 176.7034 \\ -47.0345 & 49.3438 & -63.4364 \end{pmatrix}$$

and

$$\boldsymbol{F}_2 = \begin{pmatrix} 87.3612 & 22.2568 & 14.3689 \\ -12.3185 & -8.5007 & -8.9000 \end{pmatrix}.$$

Case 3: The desired eigenvalues are

$$\{-1, \; -1, \; -2, \; -2, \; -3 \;\text{ and }\; -3\}.$$

Taking

$$f_{11,1} = f_{21,1} = f_{31,1} = [1,\, 2]^{\mathrm{T}} \quad \text{and} \quad f_{12,1} = f_{22,1} = f_{32,1} = [2,\, 1]^{\mathrm{T}}.$$

Therefore

$$\boldsymbol{F}_1 = \begin{pmatrix} -8.8357 & -43.8065 & -39.7076 \\ -8.6067 & 32.2746 & 66.6708 \end{pmatrix}$$

and

$$\boldsymbol{F}_2 = \begin{pmatrix} -99.7076 & -6.5444 & 57.5545 \\ 76.6708 & 3.5775 & -23.4548 \end{pmatrix}.$$

4. CONCLUSIONS

In this paper, a complete parametric approach for solving the eigenstructure assignment problem for the second-order linear systems using linear proportional-plus-derivative feedback is presented. The necessary conditions to ensure solvability are that the system is completely controllable and the mass matrix is nonsingular. A complete parametric form for both the closed-loop eigenvector matrices and the feedback gains are established. This parametric solution describes the available degrees of freedom offered by the proportional-plus-derivative feedback in selecting the associated eigenvectors from an admissible class. The extra degrees of freedom of the choice of feedback gains are exploited to further improve the closed-loop robustness against perturbation. The main computation involves only the singular value decomposition and manipulates only the original system matrices. The principle benefits of the explicit characterization of parametric class of feedback controllers lie in the ability to directly accommodate various additional design criteria.

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