KYBERNETIKA — VOLUME 41 (2005), NUMBER 5, PAGES 593 - 600

ONLY A LEVEL SET OF A CONTROL LYAPUNOV FUNCTION FOR HOMOGENEOUS SYSTEMS

HAMADI JERBI AND THOURAYA KHARRAT

In this paper, we generalize Artstein's theorem and we derive sufficient conditions for stabilization of single-input homogeneous systems by means of an homogeneous feedback law and we treat an application for a bilinear system.

Keywords: homogeneous systems, homogeneous feedbacks, stabilizability, sub manifold, vector field

AMS Subject Classification: 93D15

1. INTRODUCTION

For smooth single-input systems that are affine in the control

$$\dot{x} = f(x) + ug(x), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R} \tag{1}$$

where $f(0) = 0$, the basic stabilization Lyapunov condition provided in the previously mentioned works of [1, 9, 10] and [11] can be expressed as follows.

There exists a positive definite real function $V : \mathbb{R}^n \to \mathbb{R}$ (i. e., $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$ near zero) such that for every $x \neq 0$ near zero with $\nabla V \cdot g(x) = 0$ it holds $\nabla V \cdot f(x) < 0$.

In [1] it was shown that if the above condition is fulfilled, then the system (1) is stabilizable at the origin by means of a nonlinear feedback law which is smooth for $x \neq 0$. The same result was proved independently in [9, 10] and [11], where the corresponding stabilizing feedback laws are more explicitly identified. In particular, for single input systems (1) the stabilizing feedback proposed in [11] has the general form

$$\phi(x) = \begin{cases} \left(-\psi(x)\frac{\nabla V \cdot f(x) + \rho(x)}{\nabla V \cdot g(x)} - \nabla V \cdot g(x)\right) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

where ρ is any nonnegative smooth real function and ψ is a suitable real function which is smooth for $x \neq 0$ near zero taking values in the interval $[0, 1]$.

In this paper we will not refer to this general expression because we will restrict our attention to homogeneous systems (i. e. f (respectively g) is differentiable positive homogeneous of degree p (respectively q) (i. e., $f(\lambda x) = \lambda^p f(x)$ (respectively $g(\lambda x) = \lambda^q g(x)$; $\forall \lambda \geq 0$))).

The stability results presented in this paper rely on the properties of homogeneous systems and build on several previous results on stability of homogeneous systems. The basic tools for dilations and homogeneous functions and vector fields are given in the monograph by Goodman [3].

In [2] the authors presents a method to design explicit control Lyapunov functions for affine and homogeneous systems that satisfy the Jurdjevi–Quinn conditions. Hermes has considered the application of homogeneous systems in control theory and has developed approximations which generalize the usual linear approximation theorem [4]. The use of homogeneous structure in stabilization problems has also been considered by Kawski [6], who presented results for low-dimensional control system with drift and defined the notion of exponential stability which we make use of here. Major research effort was directed toward studying the driftless systems (see [8] and [7]). The paper [7] focuses on the problem of exponential stabilization of controllable, driftless systems using time-varying, homogeneous feedback.

In [5] it is proved that if there exists a smooth map θ such that $k \in \mathbb{R}$ is a regular value of θ and θ satisfies

i) θ is proper,

ii) for all $x \in \theta^{-1}\{k\}$, $\langle (\text{grad}_x(\theta))^T | x \rangle \neq 0$,

iii) for $x \in \mathbb{R}^n \setminus \{0\}$ the equation $\theta(\lambda\ x) = k$ has only two solutions λ_1 and λ_2 satisfying $\lambda_1 \lambda_2 < 0$,

iv) if $\langle (\text{grad}_x(\theta))^T | g(x) \rangle = 0$, then $\langle (\text{grad}_x(\theta))^T | f(x) \rangle \langle (\text{grad}_x(\theta))^T | x \rangle < 0$,

then the system (1) is globally asymptotically stabilizable by a positive homogeneous feedback.

The difficulty in this result is the construction of such a function θ that satisfies the conditions (ii) and (iii). This constraint is the essential topic of our paper.

In the present paper, conditions (ii) and (iii) are relaxed by the following one:

$$\exists\, T > 0,\ \exists\, \delta > 0 \text{ such that, for } \|x\| > T \text{ one has } \langle (\text{grad}_x(\theta))^T | x \rangle \geq \delta$$

which is then applied to bilinear systems of the particular structure:

$$\begin{cases} \dot{x}_1 = A_{11}x_1 + A_{12}x_2 + uB_{11}x_1 \\ \dot{x}_2 = A_{22}x_2 + uB_{22}x_2 \end{cases}$$

with $x = (x_1, x_2) \in \mathbb{R}^p \times \mathbb{R}^{n-p}$, A_{11}, A_{12}, A_{22}, B_{11} and B_{22} are matrices which they satisfy some conditions.

In [5] to prove that the equation $\theta(\lambda\ x) = k$ has only two solutions in the application, the author compute these solutions. In the present paper to prove that the equation $\theta(\lambda\ x) = k$ has only two solutions in the application, we can not easily compute these solutions. But using the new condition $(\mathcal{P}_1)$ given in Theorem 2, we prove that the equation $\theta(\lambda\ x) = k$ has only two solutions without doing the resolution of this equation.

2. MAIN RESULT

Definition 1. Let $\mathcal{M}$ be a sub manifold of $\mathbb{R}^n$ of dimension $n-1$. We say that $\mathcal{M}$ is Φ-homotopic to S^{n-1} if the map $\Phi:\ \mathcal{M} \longrightarrow S^{n-1}$, $\quad x \mapsto \frac{x}{\|x\|}$ is a diffeomorphism of C^∞.

Notations.

For $x \in \mathbb{R}^n \setminus \{0\}$, we denote $D_x^+ = \{\lambda x\ ;\ \lambda > 0\}$.

$\langle ..\mid ..\rangle$ denotes the Euclidean inner product, $\|x\| = \sqrt{\langle x \mid x\rangle}$.

For $x \in \mathbb{R}^n$ we define $\langle x\rangle = \{\lambda x\ ;\ \lambda \in \mathbb{R}\}$.

M^T stands for the transpose matrix of M.

Let E_1 and E_2 two subspaces of $\mathbb{R}^n$, we define

$E_1^\perp = \{v \in \mathbb{R}^n$ such that $\langle v \mid w\rangle = 0\ \forall w \in E_1\}$ and the notation

$E = E_1 \oplus E_2$ means that $E_1 \cap E_2 = \{0\}$ and $E = \{v + w\ ;\ v \in E_1$ and $w \in E_2\}$.

Let $\theta:\ \mathbb{R}^n \to \mathbb{R}$ be a smooth map, we denote $\text{grad}_x\theta = (\frac{\partial\theta}{\partial x_1}(x), \dots, \frac{\partial\theta}{\partial x_n}(x))$.

For $T > 0$ we define $B(T) = \{x \in \mathbb{R}^n$ such that $\|x\| \leq T\}$.

For $s \in \mathbb{R}$, we denote $|s| = \sqrt{s^2}$.

We recall the following theorem.

Theorem 1. (For the proof see [5].) We suppose that there exists $\mathcal{M}$ a compact sub manifold of $\mathbb{R}^n$ of dimension $n-1$. $\mathcal{M}$ is Φ-homotopic to S^{n-1} if and only if the following holds

(1) $\forall x \in \mathcal{M}$ one has $\mathbb{R}^n = T_x\mathcal{M} \oplus \langle x\rangle$,

(2) for all $x \in \mathbb{R}^n \setminus \{0\}$ $\mathcal{M} \cap D_x^+$ is a unique point.

Theorem 2. Suppose that $\theta:\ \mathbb{R}^n \longrightarrow \mathbb{R}$ is a map of class C^1 and let $X(x) = (\text{grad}_x(\theta))^T$. If θ is proper and the vector field X satisfies

$(\mathcal{P}_1)$ there exist $T > 0$ and $\delta > 0$ such that for $\|x\| > T$ one has $\langle X(x) \mid x\rangle \geq \delta$,

then there exists $k \in \mathbb{R}$ such that $\theta^{-1}(\{k\}) = \mathcal{M}$ is a sub manifold of $\mathbb{R}^n$ Φ-homotopic to S^{n-1}.

Proof. Let $\lambda = \sup_{x\in B(T)}(\theta(x))$, we choose $k \in \mathbb{R}_+$ such that $k > \lambda$. To apply Theorem 1, first we prove that $\theta^{-1}(\{k\}) = \mathcal{M}$ is a sub manifold of $\mathbb{R}^n$.

Since $\mathcal{M} \subset \mathbb{R}^n \setminus B(T)$ then $\forall x \in \mathcal{M}$ one has $\langle X(x) \mid x\rangle \geq \delta$. It follows that $\forall x \in \mathcal{M}$ one has $\text{grad}_x(\theta) \neq 0$. So k is a regular value of θ, $\mathcal{M}$ is a sub manifold of $\mathbb{R}^n$ of dimension $n-1$. $\mathcal{M}$ is compact since θ is proper and $T_x\mathcal{M} = \langle\text{grad}_x(\theta)\rangle^\perp$.

$\mathcal{M}$ is a compact sub manifold of $\mathbb{R}^n$ of dimension $n-1$.

Now we prove that for $x \in \mathbb{R}^n$ and $x \neq 0$ one has $\mathcal{M} \cap D_x^+ = \{y\}$. It is clear that $\theta\left(\frac{Tx}{\|x\|}\right) \leq \lambda < k$. Define the map ψ by $\psi(s) = \theta\left(s\frac{Tx}{\|x\|}\right)$. Since $\psi'(s) = \left\langle X\left(s\frac{Tx}{\|x\|}\right) \middle| \left(\frac{Tx}{\|x\|}\right)\right\rangle > 0$ for all $s > 1$, we have ψ is an increasing function for $s > 1$ and $\lim_{s\to+\infty} \psi(s) = +\infty$. So there exists only one $t > 1$ such that $\psi(t) = k$. It follows that $\mathcal{M} \cap D_x^+ = \left\{t\frac{Tx}{\|x\|}\right\}$.

Finally we establish the following, for $x \in \mathcal{M}$ one has $(T_x\mathcal{M}) \oplus \langle x \rangle = \mathbb{R}^n$. Since for $x \in \mathcal{M}$ one has $\langle x \mid X(x)\rangle \neq 0$, then we can easily see that $x \notin \langle \mathrm{grad}_x(\theta)\rangle^{\perp} = (T_x\mathcal{M})$ and so $(T_x\mathcal{M}) \oplus \langle x \rangle = \mathbb{R}^n$. □

Consider the single input homogeneous system (1), we suppose that f is a positive homogeneous function of degree p and g is a positive homogeneous function of degree q.

Definition 2. Let $\mathcal{M}$ be a sub manifold of $\mathbb{R}^n$ of dimension $n - 1$. We suppose that $\mathcal{M}$ is Φ-homotopic to S^{n-1}. We define N_x to be the normal vector of $T_x\mathcal{M}$ pointing towards outside of $\mathcal{M}$ and satisfying $\|N_x\| = 1$.

We recall the following theorem [5].

Theorem 3. If there exists a sub manifold $\mathcal{M}$ of dimension $n-1$ of $\mathbb{R}^n$ which is Φ-homotopic to S^{n-1} and satisfies $\langle f(x) \mid N_x\rangle < 0$ for all $x \in \mathcal{M}$ with $\langle g(x) \mid N_x\rangle = 0$ then there exists an homogeneous feedback of degree $p - q$ stabilizing the system (1).

Theorem 4. Suppose that there exists a map $\theta : \mathbb{R}^n \longrightarrow \mathbb{R}$ which is proper and such that the vector field X defined by $X(x) = (\mathrm{grad}_x(\theta))^T$ satisfies $(\mathcal{P}_1)$ and

$(\mathcal{P}_2)$ $\forall x \in \mathbb{R}^n \setminus B(T)$ such that $\langle X(x) \mid g(x)\rangle = 0$ one has $\langle f(x) \mid X(x)\rangle < 0$,

then for $r > 0$ large enough the positive homogeneous feedback of degree $p - q$

$$u(x) = -r\left\langle X\left(\Phi^{-1}\left(\frac{x}{\|x\|}\right)\right) \middle| g\left(\Phi^{-1}\left(\frac{x}{\|x\|}\right)\right)\right\rangle \left(\frac{\|x\|}{\|\Phi^{-1}\left(\frac{x}{\|x\|}\right)\|}\right)^{(p-q)}$$

$\forall x \in \mathbb{R}^n \setminus \{0\}$ and $u(0) = 0$ stabilizes the system (1).

Proof. Suppose that there exists a map $\theta : \mathbb{R}^n \longrightarrow \mathbb{R}$ which is proper and such that the vector field X defined by $X(x) = (\mathrm{grad}_x(\theta))^T$ satisfies $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$. According to Theorem 2 there exists $k \in \mathbb{R}_+$ such that $\mathcal{M} = \theta^{-1}\{k\}$ is a sub manifold of $\mathbb{R}^n$ Φ-homotopic to S^{n-1}. The proof follows from the previous theorem. □

3. APPLICATION

As an application for Theorem 4, we consider the following bilinear system:

$$\begin{cases} \dot{x} = A_{11}x + A_{12}y + uB_{11}x \\ \dot{y} = A_{22}y + uB_{22} \\ x \in \mathbb{R}^p, \quad y \in \mathbb{R}^q \quad \text{and} \quad u \in \mathbb{R}. \end{cases} \tag{2}$$

Notations. Let M be a matrix, M^T denotes the transpose of M.
For n, p and q, the three integers, we denote by $\mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q$ with $p+q=n$, $y = (y_1, \cdots, y_q)^T$, $z = \begin{pmatrix} x \\ y \end{pmatrix}$ for x (resp. y) is in $\mathbb{R}^p$ (resp. $\mathbb{R}^q$) and $\rho = \|z\| = \sqrt{\|x\|^2 + \|y\|^2}$.

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}; \quad B = \begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix}.$$

We suppose that there exist orthogonal basis $\{e_1, e_2, \ldots, e_p\}$ of $\mathbb{R}^p$ and $\{v_1, v_2, \ldots, v_q\}$ of $\mathbb{R}^q$ such that we have the following conditions:

There exists a positive real a such that $\langle A_{11}x \mid x\rangle \le -a\|x\|^2 \; \forall x \in \mathbb{R}^p$ and $a > \|A_{12}\|$.

$\langle B_{22}y \mid y\rangle = 0, \quad \forall y \in \mathbb{R}^q$,

$$A_{22} + A_{22}^T = 2 \begin{pmatrix} \alpha_1 & 0 & . & . & . & 0 \\ 0 & \alpha_2 & 0 & . & . & 0 \\ . & & . & & & . \\ . & & & . & & . \\ . & & & & . & . \\ 0 & . & . & . & . & \alpha_q \end{pmatrix}$$

with $\alpha_1 \ge 0$ and $\alpha_i < 0$ for all $i \in \{2, \ldots, q\}$.

$v_1 \in \operatorname{Im} B_{22}$, i.e. there exists a vector $b \in \mathbb{R}^q$ such that $B_{22}b = v_1$.

Denote $\alpha = -\sup_{i=2,\cdots,q} \alpha_i$.

Under these assumptions, we have the following result.

Proposition 1. We consider the map defined on $\mathbb{R}^p \times \mathbb{R}^{n-p}$ by
$\theta(z) = \gamma \ln \rho + \frac{1}{2}\|y - b\|^2$, where γ is a positive constant satisfying $\gamma > \|\frac{b}{2}\|^2$.

The vector field X defined by $X(z) = (\operatorname{grad}_z(\theta))^T$ satisfies

i) $\exists\, \delta > 0$ such that $\langle X(z) \mid z\rangle \ge \delta \;\; \forall z \ne 0$,

ii) $\langle X(z) \mid Bz\rangle = 0$ gives $y_1 = -\frac{\gamma}{\rho^2}\langle x \mid B_{11}x\rangle$.

Proof. The vector field X defined by $X(z) = (\text{grad}_z(\theta))^T$ is given by

$$X: \quad \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$z = \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \frac{\gamma z}{\rho^2} + \begin{pmatrix} 0 \\ y - b \end{pmatrix}.$$

Let $z = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q$, we have

i) $\langle X(z) \mid z\rangle = \gamma + ||y||^2 - \langle y \mid b\rangle = \gamma + ||y - \frac{b}{2}||^2 - ||\frac{b}{2}||^2$

If we choose $0 < \delta < \gamma - ||\frac{b}{2}||^2$, then for all $z \neq 0$ one has $\langle X(z) \mid z\rangle \geq \delta$.

ii) A simple computation gives

$\langle X(z) \mid Bz\rangle = \frac{\gamma}{\rho^2}\langle x \mid B_{11}x\rangle - \langle B_{22}^T b \mid y\rangle = \frac{\gamma}{\rho^2}\langle x \mid B_{11}x\rangle + y_1$.

So $\langle X(z) \mid Bz\rangle = 0$ gives $y_1 = -\frac{\gamma}{\rho^2}\langle x \mid B_{11}x\rangle$. □

Denote by $\tilde{b} = A_{22}^T b$, we can write $\mathbb{R}^q = \langle\tilde{b}\rangle \oplus \langle\tilde{b}\rangle^\perp$. So for all $y \in \mathbb{R}^q$, there exist $v \in \langle\tilde{b}\rangle^\perp$ and $r \in \mathbb{R}$ such that $y = v + r\tilde{b}$. We obtain $\rho^2 = ||x||^2 + ||v||^2 + r^2||\tilde{b}||^2$.

There exists $l > 0$ such that $|\langle x \mid B_{11}x\rangle| \leq l||x||^2$.

Theorem 5. If γ is chosen such that

$$\sup\left(||\tfrac{b}{2}||^2, \tfrac{4||\tilde{b}||^2}{a\alpha}\right) < \gamma < \tfrac{a - ||A_{12}||}{4(\alpha_1 + \alpha)l^2} \text{ and } d^2 = \sup\left(\gamma, \tfrac{4\gamma a||\tilde{b}||^2}{\alpha^2\gamma a - 4\alpha||\tilde{b}||^2}, \tfrac{\gamma a}{\alpha}, \gamma\tfrac{||A_{12}||}{\alpha}\right),$$

then for all $z \in \mathbb{R}^p \times \mathbb{R}^q$ such that $||z|| > d$,
$\langle X(z), Bz\rangle = 0$ gives $\langle X(z) \mid Az\rangle < 0$.

Proof. Let $z \in \mathbb{R}^n \setminus B(d)$ such that $\langle X(z) \mid Bz\rangle = 0$. Under Proposition 1 one has $\langle X(z) \mid Bz\rangle = 0$ gives $y_1 = -\frac{\gamma}{\rho^2}\langle x \mid B_{11}x\rangle$.
So

$$\begin{aligned}\langle X(z) | Az\rangle &= \frac{\gamma}{\rho^2}\langle x, A_{11}x\rangle + \frac{\gamma}{\rho^2}\langle x, A_{12}y\rangle + (\frac{\gamma}{\rho^2} + 1)\langle y | A_{22}y\rangle - \langle b | A_{22}y\rangle \\ &= \frac{\gamma}{\rho^2}\langle x | A_{11}x\rangle + \frac{\gamma}{\rho^2}\langle x | A_{12}y\rangle + (\frac{\gamma}{\rho^2} + 1)\langle y | A_{22}y\rangle - \langle A_{22}^T b | y\rangle.\end{aligned}$$

$$\langle X(z) | Az\rangle \leq -\frac{\gamma a}{\rho^2}||x||^2 + \frac{\gamma}{\rho^2}||A_{12}||||x||||y|| + (\frac{\gamma}{\rho^2} + 1)(\alpha_1 y_1^2 + \alpha_2 y_2^2 + \ldots + \alpha_q y_q^2) - r||\tilde{b}||^2$$

$$\leq -\tfrac{\gamma a}{\rho^2}||x||^2 + \tfrac{\gamma}{\rho^2}||A_{12}||(\tfrac{||x||^2 + ||y||^2}{2}) + (\tfrac{\gamma}{\rho^2} + 1)((\alpha_1 + \alpha)l^2\gamma^2\tfrac{||x||^4}{\rho^4} - \alpha||y||^2) - r||\tilde{b}||^2$$

$$\leq \gamma(-a + \tfrac{||A_{12}||}{2})\tfrac{||x||^2}{\rho^2} + (-\alpha + \gamma\tfrac{||A_{12}||}{2\rho^2})||y||^2 + (\tfrac{\gamma}{\rho^2} + 1)(\alpha_1 + \alpha)l^2\gamma^2\tfrac{||x||^4}{\rho^4} - \alpha\gamma\tfrac{||y||^2}{\rho^2} - r||\tilde{b}||^2$$

$$\leq \gamma(-\tfrac{a}{2} + \tfrac{||A_{12}||}{2})\tfrac{||x||^2}{\rho^2} + (-\tfrac{\alpha}{2} + \gamma\tfrac{||A_{12}||}{2\rho^2})||y||^2 + (\tfrac{\gamma}{\rho^2} + 1)(\alpha_1 + \alpha)l^2\gamma^2\tfrac{||x||^4}{\rho^4} - \alpha\gamma\tfrac{||y||^2}{\rho^2}$$
$$- r||\tilde{b}||^2 - \tfrac{\gamma a}{2}\tfrac{||x||^2}{\rho^2} - \tfrac{\alpha}{2}||y||^2$$

Denote by $D_1 = \gamma(-\frac{a}{2} + \frac{||A_{12}||}{2})\frac{||x||^2}{\rho^2} + (\frac{\gamma}{\rho^2} + 1)(\alpha_1 + \alpha)l^2\gamma^2\frac{||x||^4}{\rho^4}$

and

$$\begin{aligned}D_2 &= (-\tfrac{\alpha}{2} + \gamma\tfrac{||A_{12}||}{2\rho^2})||y||^2 - \alpha\gamma\tfrac{||y||^2}{\rho^2} - r||\tilde{b}||^2 - \tfrac{\alpha}{2}||y||^2 - \tfrac{\gamma a}{2}\tfrac{||x||^2}{\rho^2} \\ &= (-\tfrac{\alpha}{2} + \gamma\tfrac{||A_{12}||}{2\rho^2})||y||^2 - \alpha\gamma\tfrac{||y||^2}{\rho^2} - r(1 + \tfrac{\alpha}{2}r)||\tilde{b}||^2 - \tfrac{\alpha}{2}||v||^2 - \tfrac{\gamma a}{2}\tfrac{||x||^2}{\rho^2}\end{aligned}$$

If $\gamma < \frac{a-\|A_{12}\|}{4(\alpha_1+\alpha)l^2}$, then $D_1 < 0$.

If $\rho^2 > \gamma\frac{\|A_{12}\|}{\alpha}$ and $r(1+\frac{\alpha}{2}r) \geq 0$ (i.e. $r \in]-\infty, -\frac{2}{\alpha}] \cup [0, +\infty[$), then $D_2 < 0$.

If $r(1+\frac{\alpha}{2}r) < 0$ (i.e. $r \in]-\frac{2}{\alpha}, 0[$), we can write

$$-r\|\tilde{b}\|^2 - \frac{\gamma a}{2}\frac{\|x\|^2}{\rho^2} - \frac{\alpha}{2}\|v\|^2 = -r\|\tilde{b}\|^2 - \frac{\gamma a}{2}\frac{\rho^2 - \|v\|^2 - r^2\|\tilde{b}\|^2}{\rho^2} - \frac{\alpha}{2}\|v\|^2$$
$$= -r\|\tilde{b}\|^2 - \frac{\gamma a}{2} + \frac{\gamma a}{2}\frac{\|\tilde{b}\|^2}{\rho^2}r^2 + (\frac{\gamma a}{2\rho^2} - \frac{\alpha}{2})\|v\|^2$$

Let $\mathcal{P}(r) = \frac{\gamma a}{2}\frac{\|\tilde{b}\|^2}{\rho^2}r^2 - r\|\tilde{b}\|^2 - \frac{\gamma a}{2}$.

$\mathcal{P}(0) = -\frac{\gamma a}{2} < 0$, $\mathcal{P}(\frac{-2}{\alpha}) = \gamma a\frac{\|\tilde{b}\|^2}{\rho^2}\frac{2}{\alpha^2} + \frac{2}{\alpha}\|\tilde{b}\|^2 - \frac{\gamma a}{2}$.

If $\gamma > \frac{4\|\tilde{b}\|^2}{a\alpha}$ and $\rho^2 > \frac{4\gamma a\|\tilde{b}\|^2}{\alpha^2\gamma a - 4\alpha\|\tilde{b}\|^2}$, then $\mathcal{P}(\frac{-2}{\alpha}) < 0$ and if in addition $\rho^2 > \frac{\gamma a}{\alpha}$, then $D_2 < 0$.

Finally, if γ is chosen such that $\sup(\|\frac{b}{2}\|^2, \frac{4\|\tilde{b}\|^2}{a\alpha}) < \gamma < \frac{a-\|A_{12}\|}{4(\alpha_1+\alpha)l^2}$ and $d^2 = \sup(\gamma, \frac{4\gamma a\|\tilde{b}\|^2}{\alpha^2\gamma a - 4\alpha\|\tilde{b}\|^2}, \frac{\gamma a}{\alpha}, \gamma\frac{\|A_{12}\|}{\alpha})$, then for all $z \in \mathbb{R}^p \times \mathbb{R}^q$ such that $\|z\| > d$, $\langle X(z), Bz\rangle = 0$ gives $\langle X(z)\,|\,Az\rangle < 0$.

The conditions of Theorem 4 are satisfied and the bilinear system (2) is globally asymptotically stabilizable by means of a homogeneous feedback of degree zero. □

Example. As an application of the previous result, we consider system (2) in which matrices A_{11}, A_{12}, A_{22}, B_{11} and B_{22} are the following

$$A_{11} = \begin{pmatrix} -3 & 0 \\ 0 & -4 \end{pmatrix}, \quad A_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B_{11} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix},$$
$$A_{22} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \quad \text{and} \quad B_{22} = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}.$$

We can easily verify that the vector $b = (0, 0, \frac{1}{2})^T$ satisfies $B_{22}b = v_1$ and $\tilde{b} = A_{22}b = (0, 0, -\frac{1}{4})^T$. A simple computation gives $\|A_{12}\| = 1$ and the positive real $a = 3$ satisfies $\langle A_{11}x\,|\,x\rangle \leq -a\|x\|^2$.

So, if we choose $\gamma = \frac{11}{60}$ then all the conditions mentioned previously are satisfied and the bilinear system (2) with matrices defined below is globally asymptotically stabilizable by means of an homogeneous feedback of degree zero.

(Received October 25, 2004.)

REFERENCES

[1] Z. Artstein: Stabilization with relaxed controls. Nonlinear. Anal. *7* (1983), 1163–1173.

[2] L. Faubourg and J. P. Pomet: Control Lyapunov functions for homogeneous "Jurdjevic-Quinn" systems. ESAIM: Control, Optimization and Calculus of Variations *5* (2000), 293–311.

[3] R. W. Goodman: Nilpotent Lie Groups: Structure and Applications to Analysis. (Lecture Notes in Mathematics 562.) Springer–Verlag, Berlin 1976.

[4] H. Hermes: Nilpotent and high-order approximations of vector field systems. SIAM Rev. *33* (1991), 238–264.

[5] H. Jerbi: A manifold-like characterization of asymptotic stabilizability of homogeneous systems. Systems Control Lett. *45* (2002), 173–178.

[6] M. Kawski: Homogeneous stabilizing feedback laws. Control Theory and Advanced Technology *6* (1990), 497–516.

[7] R. Mac Closkey and M. Murray: Exponential stabilization of driftless nonlinear control systems using homogeneous feedback. IEEE Trans. Automat. Control *42* (1997), 614–628.

[8] R. Mac Closkey and P. Morin: Time-varying homogeneous feedback: design tools for exponential stabilization of systems with drift. Internat. J. Control *71* (1998), 837–869.

[9] E. D. Sontag: A "universal" construction of Artstein's Theorem on nonlinear stabilization. Systems Control Lett. *13* (1989).

[10] J. Tsinias: Stabilization of affine in control nonlinear systems. Nonlinear Anal. *12* (1988), 1238–1296.

[11] J. Tsinias: Sufficient Lyapunov like conditions for stabilization. Math. Control Signals Systems *2* (1989), 343–357.

Hamadi Jerbi, Faculté des sciences de Sfax, Département de Mathématique, Route Soukra Km 4 3018 BP 802, Sfax. Tunisie.
e-mail: hjerbi@voila.fr

Thouraya Kharrat, Faculté des sciences de Sfax, Département de Mathématique, Route Soukra Km 4 3018 BP 802, Sfax. Tunisie.
e-mail: thouraya.kharrat@fss.rnu.tn