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A COLLECTOR FOR INFORMATION WITHOUT PROBABILITY IN A FUZZY SETTING

DORETTA VIVONA AND MARIA DIVARI

In the fuzzy setting, we define a collector of fuzzy information without probability, which allows us to consider the reliability of the observers. This problem is transformed in a system of functional equations. We give the general solution of that system for collectors which are compatible with composition law of the kind "inf".

Keywords: information measure, system of functional equations

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1. INTRODUCTION

In the subjective theory of information without probability [9, 10, 11, 12, 15] and in the crisp setting, B. Forte and others [3, 7, 8] have supposed that each group of observers E provides an amount of information $J(A, E)$ from the same event A. Moreover they supposed that, for each E, the information is compositive (in the sense of [13] with the same law with an additive reliability coefficient $\lambda(E)$.

B. Forte has defined a *collector* as a function Φ:

$$J(A, E_1 \cup E_2) = \Phi\left(\lambda(E_1), \lambda(E_2), J(A, E_1), J(A, E_2)\right)$$

for every event A and disjoint groups E_1, E_2.

Putting $x = \lambda(E_1)$, $y = \lambda(E_2)$, $u = J(A, E_1)$, $v = J(A, E_2)$, Aczél, Forte and Ng in [1, 2] gave the solution in the Shannon case:

$$\Phi(x, y, u, v) = -c \log\left(\frac{x\, e^{-u/c} + y\, e^{-v/c}}{x + y}\right),$$

where c is the constant related to the Shannon information; when the information J is of the kind $\wedge$, Benvenuti, Divari and Pandolfi obtained a more general class of solutions (see [4]).

In a previous paper [16] we have defined collectors of $\wedge$-compositive information without probability for fuzzy sets of events, crisp sets of observers with a reliability coefficient defined in a probabilistic space.

In this paper we shall introduce fuzzy collectors for crisp groups of observers with a fuzzy $\vee$-additive measure of reliability.

Evidently, if we restrict our considerations to crisp sets, the collectors studied in [4] are recovered. One of the main aim of this paper is also to enlight interesting ideas from [4] which are not so known in the wider community.

2. PRELIMINARIES

In the setting of *fuzzy sets* [17], we consider the following model:

1) Ω is an abstract space, $\mathcal{F}$ is an algebra of fuzzy sets such that $(\Omega, \mathcal{F})$ is a fuzzy measurable space, the elements of $\mathcal{F}$ are the *observable events*. Recall that for A and $B \in \mathcal{F}$, whose membership functions are f_A and f_B, respectively, it holds: $f_{A\cup B} = f_A \vee f_B, f_{A\cap B} = f_A \wedge f_B, f_{A^c} = 1 - f_A$;

2) $\mathcal{O}$ is another abstract space (space of observers), $\mathcal{E}$ is a σ-algebra contained in $\mathcal{P}(\mathcal{O})$, whose elements are groups of *observers*;

3) a fuzzy $\vee$-additive measure μ is defined on the measurable space $(\mathcal{O}, \mathcal{E})$: $\mu(\emptyset) = 0, \mu(\mathcal{O}) = \overline{\mu} \in]0, +\infty]$, μ is non-decreasing with respect to the inclusion of the elements of $\mathcal{E}$ and $\mu(E_1 \cup E_2) = \mu(E_1) \vee \mu(E_2)\ \forall\, E_1, E_2 \in \mathcal{E}$; if $E \in \mathcal{E}$, $\mu(E)$ is called *fuzzy reliability coefficient*;

4) an information measure J, called *fuzzy information* (see [5, 6]), linked to the group of observers, is a map $J : \mathcal{F} \times \mathcal{E} \to \overline{\mathbb{R}}^+$ such that, fixed $E \in \mathcal{E}, E \neq \emptyset, \neq \mathcal{O}$ for all $A, B \in \mathcal{F}$

4i) $A \subset B \Rightarrow J(A, E) \geq J(B, E)$,

4ii) $J(\emptyset, E) = +\infty,\ J(\Omega, E) = 0$;

5) every information measure $J(\cdot, E)$ is F_E-*compositive* i. e. for every $E \in \mathcal{E}, \mathrm{E} \neq \emptyset$ there exists a map $F_E : \Gamma_E \to \overline{\mathbb{R}}^+$, where $\Gamma_E = \{(x, y) \,/\, \exists\, A,\, B \in \mathcal{F}$ with $x = J(A, E),\ y = J(B, E), f_A \wedge f_B = 0\}$ such that

$$J(A \cup B; E) = F_E\Big(J(A, E), J(B, E)\Big). \tag{1}$$

Evidently F_E is commutative, associative and $F_E(x, +\infty) = x$, for all $x \in \operatorname{Ran} J(\cdot, E)$.

Throughout this paper we deal with universal composition rule $F = \wedge$,

$$J(A \cup B, E) = F[J(A, E), J(B, E)] = J(A, E) \wedge J(B, E). \tag{2}$$

Note that due the idempotency of the operator $\wedge$ we need not to require the disjontness $f_A \wedge f_B = 0$ in the above equality (2).

We call $\wedge$-*compositive fuzzy information* a fuzzy information J which satisfies (2) for every $E \in \mathcal{E}$.

3. COLLECTOR OF $\wedge$-COMPOSITIVE FUZZY INFORMATION

In the previous paper [16] we have defined a collector for crisp sets.

Here, in the fuzzy setting, we give the definition of collector which we shall call *fuzzy collector.*

Definition 3.1. A *fuzzy collector* for a given reliability measure μ is a continuous function Ψ

$$\Psi : \Sigma \rightarrow \overline{\mathbb{R}}^{+}$$

where $\Sigma \subset \left([0,\overline{\mu}] \times \overline{\mathbb{R}}^{+}\right)^2$, $\overline{\mu} = \mu(\mathcal{O})$, such that for every pair of two disjoint groups E_1 and E_2 of observers with reliability coefficients $\mu(E_1)$ and $\mu(E_2)$ it holds

$$J(A, E_1 \cup E_2) = \Psi\left(\mu(E_1), J(A, E_1), \mu(E_2), J(A, E_2)\right). \tag{3}$$

4. PROPERTIES OF A FUZZY COLLECTOR Ψ

In this section we present the properties if a fuzzy collector is expressed by Ψ. They are:

(i) (commutativity):

$$\Psi\left(\mu(E_1), J(A, E_1), \mu(E_2), J(A, E_2)\right) = \Psi\left(\mu(E_2), J(A, E_2), \mu(E_1), J(A, E_1)\right),$$

$\forall\, A \in \mathcal{F}, E_1, E_2 \in \mathcal{E}$, as $J(A, E_1 \cup E_2) = J(A, E_2 \cup E_1)$;

(ii) (associativity):

$$\Psi\left(\mu(E_1) \vee \mu(E_2), \Psi\left(\mu(E_1), J(A, E_1), \mu(E_2), J(A, E_2)\right), \mu(E_3), J(A, E_3)\right)$$

$$= \Psi\left(\mu(E_1), J(A, E_1), \mu(E_2) \ \vee \ \mu(E_3), \Psi\left(\mu(E_2), J(A, E_2), \mu(E_3), J(A, E_3)\right)\right),$$

$\forall\, A \in \mathcal{F},\ E_1, E_2, E_3 \in \mathcal{E}$, as $J(A, (E_1 \cup E_2) \cup E_3) = J(A, E_1 \cup (E_2 \cup E_3)$;

(iii) (universal value $J(\emptyset, E) = +\infty$):

$$\Psi\left(\mu(E_1), +\infty, \mu(E_2), +\infty\right) = +\infty,$$

as $J(\emptyset, E_1 \cup E_2) = +\infty$;

(iv) (universal value $J(\Omega, E) = 0)$):

$$\Psi\left(\mu(E_1), 0, \mu(E_2), 0\right) = 0,$$

as $J(\Omega, E_1 \cup E_2) = 0$.

If the information of the group of observers is $\wedge$-compositive in the sense of (2) we can add another property:

(v) (compatibility condition between the $\wedge$-compositivity of J and the collector Ψ):

$$\Psi\Big(\mu(E_1),\Big[J(A,E_1)\ \wedge J(B,E_1)\Big],\mu(E_2),\Big[J(A,E_2)\ \wedge J(B,E_2)\Big]\Big)$$
$$=\Psi\Big(\mu(E_1),J(A,E_1),\mu(E_2),J(A,E_2)\Big)\ \wedge\Psi\Big(\mu(E_1),J(B,E_1),J(B,E_2),\mu(E_2),\Big)$$
$$\forall A,B\in\mathcal{F},\ \mathrm{E}_1,\mathrm{E}_2\in\mathcal{E}.$$

In fact, from (2) it is $J(A\cup B,E_1\cup E_2)=J(A,E_1\cup E_2)\wedge J(B,E_1\cup E_2)$, and, on the other hand, from (3), we get $J(A\cup B,E_1\cup E_2)=$

$$\Psi\Big(\mu(E_1),J(A\cup B,E_1),\mu(E_2),J(A\cup B,E_2)\Big)$$
$$=\Psi\Big(\mu(E_1),\Big[J(A,E_1)\ \wedge J(B,E_1)\Big],\mu(E_2),\Big[J(A,E_2)\wedge J(B,E_2)\Big]\Big).$$

5. SYSTEM OF FUNCTIONAL EQUATIONS

Put $\mu(E_1)=x,\mu(E_2)=y,\mu(E_3)=z$, with $x,y,z\in[0,1]$. The function Ψ given in (3) is defined in the domain $\Sigma^2=([0,\overline{\mu}]\times\overline{\mathbb{R}}^+)^2$. Moreover we set $J(A,E_1)=u,\ J(A,E_2)=v,\ J(B,E_1)=u',\ J(B,E_2)=v',\ J(A,E_3)=w$.

Now we rewrite the conditions $[(i)-(v)]$ in order to obtain a system of functional equations. The equations are:

$$\begin{cases} (i')\quad \Psi\Big(x,u,y,v\Big)=\Psi\Big(y,v,x,u\Big)\\ (ii')\quad \Psi\Big(x,u,y\vee z,\Psi(y,v,z,w)\Big)=\Psi\Big(x\vee y,\Psi(x,y,u,v),z,w\Big)\\ (iii')\quad \Psi\Big(x,+\infty,y,+\infty\Big)=+\infty\\ (iv')\quad \Psi\Big(x,0,y,0\Big)=0\\ (v')\quad \Psi\Big(x,u\wedge u',y,v\wedge v'\Big)=\Psi\Big(x,u,y,v\Big)\wedge\Psi\Big(x,u',y,v'\Big).\end{cases}$$

In the setting of crisp sets, an analogous system was studied and solved by Benvenuti–Divari–Pandolfi in [4]. We study the system $[(i')-(v')]$ and we give the general solution step by step.

Theorem 5.1. Main Theorem. The function $\Psi\Big(x,u,y,v\Big)$ is solution of the system $[(i')-(v')]$ if and only if

$$\Psi\Big(x,u,y,v\Big)=g(x,y,u)\wedge g(y,x,v) \tag{4}$$

where the function $g:[0,\overline{\mu}]^2\times\overline{\mathbb{R}}\to\overline{\mathbb{R}}$ fulfills the following properties:

(α) g is non decreasing with respect to u and continuous,

(β) $g(x,y,+\infty)=+\infty$,

(γ) $g(x,y,0)\wedge g(y,x.0)=0$,

(δ) $g[x\vee z,y,g(x,z,u)]=g(x,y\vee z,u)$.

P r o o f. Putting $g(x, y, u) = \Psi(x, u, y, +\infty)$, from (v') for $u' = v$, we have the (4). It is easy to verify that every function Ψ with the form (4) and the properties $[(\alpha) - (\delta)]$ is a solution of the system $[(i') - (v')]$. □

For every function $g(x, y, u)$ which satisfies the properties $[(\alpha) - (\delta)]$, we can prove the following Lemmas.

Lemma 5.2. For every function $g(x, y, u)$ which satisfies the properties $[(\alpha) - (\delta)]$, we have $g(x, y, 0) = 0$.

P r o o f. From (δ), for $u = 0$ it is

$$g(x \vee z, y, g(x, z, 0)) = g(x, y \vee z, 0), \tag{5}$$

and then, changing x with z

$$g(z \vee x, y, g(z, x, 0)) = g(z, y \vee x, 0). \tag{6}$$

Because of (γ), either $g(x, z, 0) = 0$ or $g(z, x, 0) = 0$, from (α), (4) and (5) we get

$$g(x \vee z, y, 0) = g(x, y \vee z, 0) \wedge g(z, y \vee x, 0), \tag{7}$$

and from (7) for $y = 0$

$$g(x \vee z, 0, 0) = g(x, z, 0) \wedge g(x, z, 0) \tag{8}$$

i.e., due to (γ),

$$g(x \vee z, 0, 0) = 0 \ \forall x,\ z. \tag{9}$$

Finally, from (8) and (9), for $x = z$, we get

$$g(z, z, 0) = 0. \tag{10}$$

For $x \leq z$

$$g(z, x, 0) = g(x, z, 0) \wedge g(z, x, 0),$$

so we obtain, due to (γ),

$$g(z, x, 0) = 0 \ \forall x \leq z. \tag{11}$$

Putting in (7) $y = x$ and for $x > z$

$$g(x, x, 0) = g(x, x, 0) \wedge g(z, x, 0),$$

$$g(x \vee z, x, 0) = g(x, x \vee z, 0) \wedge g(z, x \vee x, 0).$$

From (δ), for $u = 0$

$$g(x \vee z, y, g(x, z, 0)) = g(x, y \vee z, 0).$$

By contradiction we suppose $g(z, x, 0) = \lambda > 0$, i.e. $g(x, y, \lambda) = g(x, y \vee z, 0)$. For $y > z$, we get $g(x, y, \lambda) = g(x, y, 0)$: this is impossible as g is non-decreasing with respect to u, then

$$g(z, x, 0) = 0 \ \forall x,\ z. \tag{12}$$

□

Lemma 5.3. For every function g which enjoys $[(\alpha)-(\delta)]$, we have

$$g(x,0,u)=u \text{ in } [0,\overline{\mu}]\times\overline{\mathbb{R}}^{+}. \tag{13}$$

Proof. As, from (γ) and (δ), $g(x,0,0)=0$ and $g(x,0,+\infty)=+\infty$, for every $v\in\overline{\mathbb{R}}^{+}$ there exists u such that $g(x,0,u)=v$.

From (γ), for $y=z=0$, we have $g(x,0,g(x,0,v))=g(x,0,u)$. □

Lemma 5.4. Every function g which satisfies $[(\alpha)-(\delta)]$ has the following representation:

$$g(x,y,u)=h[x\vee y,h^{-1}(x,u)]\ (x,y,u)\in\ [0,1]^2\times\overline{\mathbb{R}}^{+} \tag{14}$$

with $h:[0,1]\times\overline{\mathbb{R}}^{+}\to\overline{\mathbb{R}}^{+}$, continuous, non decreasing with respect to u and h^{-1} its pseudo-inverse [14], defined by $h^{-1}(x,v)=\mathrm{Inf}\{\xi\ /\ h(x,\xi)=v\}$.

Proof. Putting $h(x,u)=g(0,x,u)$, for (α) and (β) the function h is continuous, monotone and $h(x,0)=0$, $h(x,+\infty)=+\infty$, therefore its pseudo-inverse h^{-1} is defined on $[0,1]\times\overline{\mathbb{R}}^{+}$. From (δ), for $x=0$ and $u=h^{-1}(z,v)$, we have $g(z,y,g(0,z,h^{-1}(z,v)))=g(0,y\wedge z,h^{-1}(z,v))$, i.e. $g(z,y,g(0,z,h^{-1}(z,v)))=h(y\wedge z,h^{-1}(z,v))$. The thesis follows from $h(z,h^{-1}(z,v))=v$. □

Remark. We observe that continuity of g and condition (β) imply that $h(x,u)=g(0,x,u)$ is not (definitely) null or constant (unless $=+\infty$). Indeed, if we hold the following situation: $g(x,y,u)=\dfrac{x\ u}{x\vee y}$ (with $0\cdot+\infty=0$), then we couldn't find h^{-1}, but clearly $g(0,x,u)=0$, contrary to (β).

This situation corresponds to the following example:

Let $\mathcal{O}=\{1,2,\ldots,\mathrm{n}\}$ be the set of observers, $\mu(E)=\max E$ and $J(A,E)=\frac{-\log\inf f_A}{\mu(E)}$. So, we have: $g(x,y,u)=\frac{x\ u}{x\vee y}$, $h(x,u)=0$ and the collector is: $\Psi(x,u,y,v)=\frac{x\ u\wedge y\ v}{x\vee y}$.

Lemma 5.5. For every function g which satisfies $[(\alpha)-(\delta)]$, the corresponding function h given by (14) enjoys the following properties:

$$h(0,v)=v\in\overline{\mathbb{R}}^{+} \tag{15}$$

and

$$h(x,u)=h(x,v)\Rightarrow h(y,u)=h(y,v)\ \forall y>x. \tag{16}$$

Proof. The condition (15) follows from the definition of the function h and from Lemma 5.4. Now, we shall prove the (16): in (δ) setting $x=0$ it is $g(z,y,g(0,z,u))=g(0,y\wedge z,u)$ and for (14) we get $h(z\vee y,h^{-1}(z,h(z,h^{-1}(0,u))))=h(y\vee z,h^{-1}(0,u))$, i.e. $h(z\vee y,h^{-1}(z,h(z,u))=h(y\vee z,u)$.

If $h(z,v)=h(z,u)$ with $v<u$, from definition of h^{-1}, we have $h^{-1}(z,h(z,u))=\mathrm{Inf}\{\xi\,/\,h(z,\xi)=h(z,u)\}=v'\le v$ and therefore $h(y\wedge z,v')=h(y\wedge z,u)$.

If $v>u$, for the monotonicity of the function h and the arbitrary of y, we obtain the (16). □

Lemma 5.6. The expression (14) with the function $h(x,u)$ satisfying the conditions of the Lemmas 5.4 and 5.5 gives the general form of the continuous solutions of the system $[(\alpha)-(\delta)]$.

Proof. We shall, now, verify that every function $g(x,y,u)$ defined by (14)

$$g(x,y,u)=h(x\vee y,h^{-1}(x,u))$$

with $h(x,u)$ satisfying the conditions of the Lemmas 5.4 and 5.5 is solution of the system $[(\alpha)-(\delta)]$. In fact, for the properties of h in Lemma 5.5, the properties (α) and (β) are verified. The (δ) becomes $g(x\vee z,y,g(x,z,u)=g(x,y\vee z,u)$ and then

$$h\left(x\vee z\vee y,h^{-1}(x\vee z,h(x\vee z,h^{-1}(x,u)))\right)=h\left(x\vee z\vee y,h^{-1}(x,u)\right). \tag{17}$$

Putting $h^{-1}(x,u)=v$, the (17) becomes $h(x\vee z\vee y,h^{-1}(x\vee z,h(x\vee z,v)))=h(x\vee z\vee y,v)$. Moreover $h^{-1}(x\vee z,h(x\vee z,v))=\mathrm{Inf}\{\xi\,/\,h(x\vee z,\xi)=h(x\vee z,v)\}=v'\le v$, with $h(x\vee z,v')=h(x\vee z,v)$. For the (16), as $x\vee z\vee y\ge x\vee z$ and $h(x\vee y\vee z,v')=h(x\vee y\vee z,v)$, we have the (δ). □

Summarizing the previous Lemmas, we obtain the following main result:

Theorem 5.7. The general solution of the system $[(i')-(v')]$ is the function

$$\Psi(x,y,u,v)=h\left(x\vee y,h^{-1}(x,u)\wedge h^{-1}(y,v)\right)$$

where $h:[0,1]\times\overline{\mathbb{R}}^+\to\overline{\mathbb{R}}^+$ satisfies the following conditions:

— $h(x,\cdot)$ is non-decreasing, continuous, $h(x,0)=0$, $h(x,+\infty)=+\infty$, $\forall x\in(0,\overline{\mu}]$,

— $h(x,u)=h(x,v)\Rightarrow h(y,u)=h(y,v)$ for every $y>x$.

Example: Let $h(x,u)=e^x\,u$, this function satisfies the hypotheses of the Theorem above; its pseudo-inverse is $h^{-1}(x,v)=\dfrac{v}{e^x}$. Then the function g is

$$g(x,y,u)=h(x\vee y,h^{-1}(x,u))=e^{x\vee y}\,h^{-1}(x,u)=e^{x\vee y}\,u\,e^{-x}=u\,e^{(x\vee y)-x}.$$

Then the collector Ψ has the following expression:

$$\begin{aligned}\Psi\left(x,y,u,v\right)&=g(x,y,u)\wedge g(y,x,v)\\&=u\,e^{(x\vee y)-x}\wedge v\,e^{(y\vee x)-y}=e^{x\vee y}\left(\frac{u}{e^x}\wedge\frac{v}{e^y}\right).\end{aligned} \tag{18}$$

Let J be an information measure on crisp sets such that $J(E)=e^{-\lambda(E)}$ with λ a fuzzy measure $\vee$-additive and $J(A,E)$ an information depending on the set of observers.

From (3) and (18), we get

$$J(A,E_1\cup E_2)=\frac{J(A,E_1)J(E_1)\wedge J(A,E_2)J(E_2)}{J(E_1\cup E_2)}.$$

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Doretta Vivona and Maria Divari, Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate, Università "La Sapienza", Via A. Scarpa n. 16 – 00161 Roma. Italy. e-mail: vivona@dmmm.uniroma1.it