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EXTENSION TO COPULAS AND QUASI-COPULAS AS SPECIAL 1-LIPSCHITZ AGGREGATION OPERATORS

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Smallest and greatest 1-Lipschitz aggregation operators with given diagonal section, opposite diagonal section, and with graphs passing through a single point of the unit cube, respectively, are determined. These results are used to find smallest and greatest copulas and quasi-copulas with these properties (provided they exist).

Keywords: copula, quasi-copula, 1-Lipschitz aggregation operator, diagonal

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1. INTRODUCTION

Copulas (first mentioned in [17], for an excellent survey see [13]) and quasi-copulas (introduced in [1] and conveniently characterized in [9]) play a key role in the analysis of bivariate distribution functions with given marginals. The basic result in this context is Sklar's Theorem [17, 18] showing that the joint distribution of a random vector and the corresponding marginal distributions are linked by some copula.

A current field of research is the extension of functions defined on a subset of the unit square, e. g., on its diagonal or in a single point, to quasi-copulas or copulas. Several results in this context can be found in [2, 7, 8, 14, 15, 19].

Aggregation operators form a rather new and very general framework to combine different pieces of information (for a recent survey see [3]), and many well-known operations in logic, probability theory, statistics, and decision theory fit into this concept.

As a matter of fact, many results for copulas and quasi-copulas can be derived mainly because they are 1-Lipschitz aggregation operators [11]. Therefore, a careful study of such aggregation operators is helpful for the understanding of the structure of copulas and quasi-copulas, too.

In this paper we look for 1-Lipschitz aggregation operators with given diagonal and opposite diagonal section, as well as those whose graphs pass through a single point of the unit cube. Each of these sets of 1-Lipschitz aggregation operators will be shown to have a smallest and a greatest element.

These results can be carried over to the case of quasi-copulas with the corresponding properties. Again, the sets of quasi-copulas with given diagonal and opposite diagonal section, as well as those whose graphs pass through a single point of the unit cube, have a smallest and a greatest element.

In several cases they also can be used to determine smallest and greatest copulas with the desired properties. However, some sets of copulas, e. g., the set of copulas with given diagonal section, do not always have a greatest element.

2. PRELIMINARIES

Recall that a *(binary) aggregation operator* is a function $A : [0,1]^2 \to [0,1]$ which is non-decreasing (in each component) and satisfies $A(0,0) = 0$ and $A(1,1) = 1$.

An aggregation operator A satisfying the Lipschitz condition with constant 1, i. e., for all $x_1, x_2, y_1, y_2 \in [0,1]$

$$|A(x_1, y_1) - A(x_2, y_2)| \leq |x_1 - x_2| + |y_1 - y_2|,$$

will be called a 1-*Lipschitz aggregation operator.*

Many well-known binary aggregation operators, such as the arithmetic mean, the product, the minimum, the maximum, and weighted means are 1-Lipschitz aggregation operators (for more details see, e. g., [3]). Also copulas and quasi-copulas are special 1-Lipschitz aggregation operators.

A *(two-dimensional) copula* is a function $C : [0,1]^2 \to [0,1]$ such that $C(0,x) = C(x,0) = 0$ and $C(1,x) = C(x,1) = x$ for all $x \in [0,1]$, and C is 2-increasing, i. e., for all $x_1, x_2, y_1, y_2 \in [0,1]$ with $x_1 \leq x_2$ and $y_1 \leq y_2$ for the volume V_C of the rectangle $[x_1, x_2] \times [y_1, y_2]$ we have

$$V_C([x_1, x_2] \times [y_1, y_2]) = C(x_1, y_1) - C(x_1, y_2) + C(x_2, y_2) - C(x_2, y_1) \geq 0. \quad (2.1)$$

A *(two-dimensional) quasi-copula* is a function $Q : [0,1]^2 \to [0,1]$ such that $Q(0,x) = Q(x,0) = 0$ and $Q(1,x) = Q(x,1) = x$ for all $x \in [0,1]$, Q is non-decreasing (in each component), and Q is 1-Lipschitz.

Obviously, each copula is a quasi-copula but not vice versa, and a 1-Lipschitz aggregation operator $A : [0,1]^2 \to [0,1]$ is a quasi-copula if and only if $A(0,1) = A(1,0) = 0$ (see [11]) or, equivalently, if $A \leq M$, where the Fréchet-Hoeffding upper bound M is given by $M(x,y) = \min(x,y)$. To simplify some formulas, we shall also use the infix notations $x \wedge y$ for $\min(x,y)$ and $x \vee y$ for $\max(x,y)$.

Each 1-Lipschitz aggregation operator A satisfies

$$W \leq A \leq W^*, \quad (2.2)$$

where the Fréchet-Hoeffding lower bound W is given by $W(x,y) = (x + y - 1) \vee 0$, and its dual $W^*(x,y) = (x + y) \wedge 1$. Each quasi-copula Q satisfies

$$W \leq Q \leq M, \quad (2.3)$$

and the same holds for copulas.

Note that each of the following sets of functions from $[0,1]^2$ to $\mathbb{R}$ forms a lattice (with respect to the usual pointwise order):

$$\{F : [0,1]^2 \to \mathbb{R} \mid F \text{ is non-decreasing in the first component}\}, \tag{2.4}$$
$$\{F : [0,1]^2 \to \mathbb{R} \mid F \text{ is non-decreasing in the second component}\}, \tag{2.5}$$
$$\{F : [0,1]^2 \to \mathbb{R} \mid F \text{ is 1-Lipschitz}\}, \tag{2.6}$$

i. e., monotonicity and the 1-Lipschitz property are preserved under minimum and maximum (compare [11, 12]).

Starting from a non-decreasing 1-Lipschitz function, it is possible to force the boundary conditions to obtain a 1-Lipschitz aggregation operator and a quasi-copula.

Lemma 2.1.

(i) If $F : [0,1]^2 \to \mathbb{R}$ is non-decreasing and 1-Lipschitz then

$$(W \vee F) \wedge W^* = W \vee (F \wedge W^*)$$

is a 1-Lipschitz aggregation operator.

(ii) If $A : [0,1]^2 \to [0,1]$ is a 1-Lipschitz aggregation operator then $M \wedge A$ is a quasi-copula.

Proof. Observe first that $(W \vee F) \wedge W^* = W \vee (F \wedge W^*)$ follows from $W \leq W^*$. Since the sets in (2.4 – 2.6) are lattices, the functions $(W \vee F) \wedge W^*$ and $M \wedge A$ are both non-decreasing and 1-Lipschitz. The respective boundary conditions are implied by $W \leq (W \vee F) \wedge W^* \leq W^*$ and by the fact that $M \wedge A$ is 1-Lipschitz. □

The following concept is motivated by the Frank functional equation [5], originally studied and solved in the context of associative copulas (compare also [10, 16]):

For each 1-Lipschitz aggregation operator A the function $A^* : [0,1]^2 \to [0,1]$ given by

$$A^*(x,y) = x + y - A(x,y), \tag{2.7}$$

is also a 1-Lipschitz aggregation operator [11]. Clearly, for 1-Lipschitz aggregation operators A, B we have $A^* \leq B^*$ if and only if $A \geq B$.

3. 1-LIPSCHITZ AGGREGATION OPERATORS WITH GIVEN DIAGONAL SECTION

Given a 1-Lipschitz aggregation operator A, its diagonal section $\delta_A : [0,1] \to [0,1]$ given by $\delta_A(x) = A(x,x)$ necessarily satisfies the following properties:

(D1) $\delta_A(0) = 0$ and $\delta_A(1) = 1$,

(D2) δ_A is non-decreasing,

(D3) δ_A is 2-Lipschitz.

The question arises whether for each function $\delta : [0,1] \to [0,1]$ satisfying properties (D1) – (D3) (briefly called a *diagonal* in the sequel) there is some 1-Lipschitz aggregation operator whose diagonal section coincides with δ.

Clearly, for each diagonal δ the functions $A_1, A_2 : [0,1]^2 \to [0,1]$ which are given by $A_1(x,y) = \frac{\delta(x)+\delta(y)}{2}$ and $A_2(x,y) = \delta(\frac{x+y}{2})$ are 1-Lipschitz aggregation operators with diagonal section δ.

Moreover, it will turn out that the set of 1-Lipschitz aggregation operators with given diagonal section δ always has a greatest element $\overline{A}^{\delta}$ and a smallest element $\underline{A}^{\delta}$. As a consequence, each 1-Lipschitz aggregation operator A with $\underline{A}^{\delta} \leq A \leq \overline{A}^{\delta}$ also has diagonal section δ.

It is not difficult to see that for each 1-Lipschitz aggregation operator A, for all $(x,y) \in [0,1]^2$ and for each $z \in [x \wedge y, x \vee y]$ we get

$$A(x,y) \leq x \vee y + \delta_A(z) - z. \tag{3.8}$$

The infimum of the right-hand side of this inequality turns out not only to be a 1-Lipschitz aggregation operator, but the greatest 1-Lipschitz aggregation operator with diagonal section δ_A.

Theorem 3.1. For each function $\delta : [0,1] \to [0,1]$ satisfying (D1) – (D3), the function $\overline{A}^{\delta} : [0,1]^2 \to [0,1]$ defined by

$$\overline{A}^{\delta}(x,y) = x \vee y + \bigwedge \{\delta(z) - z \mid z \in [x \wedge y, x \vee y]\}$$

is the greatest 1-Lipschitz aggregation operator with diagonal section δ.

P r o o f. Obviously, the diagonal section of $\overline{A}^{\delta}$ coincides with δ, and the boundary conditions $\overline{A}^{\delta}(0,0) = 0$ and $\overline{A}^{\delta}(1,1) = 1$ hold.

Since the function $\overline{A}^{\delta}$ is commutative it suffices to prove its monotonicity in the first component. Fix arbitrary numbers $x_1, x_2, y \in [0,1]$ with $x_1 < x_2$ and consider the following three cases.

(i) If $y \leq x_1 < x_2$ then we have

$$\begin{aligned}
\overline{A}^{\delta}(x_2,y) &= \bigwedge \{\delta(z) + x_2 - z \mid z \in [y, x_2]\} \\
&= \bigwedge \{\delta(z) + x_2 - z \mid z \in [y, x_1]\} \wedge \bigwedge \{\delta(z) + x_2 - z \mid z \in [x_1, x_2]\} \\
&= (x_2 - x_1 + \overline{A}^{\delta}(x_1,y)) \wedge \bigwedge \{\delta(z) + x_2 - z \mid z \in [x_1, x_2]\} \\
&\geq (x_2 - x_1 + \overline{A}^{\delta}(x_1,y)) \wedge \delta(x_1) \\
&\geq \overline{A}^{\delta}(x_1,y)
\end{aligned}$$

because of $\bigwedge \{\delta(z) + x_2 - z \mid z \in [x_1, x_2]\} \geq \delta(x_1) \geq \overline{A}^{\delta}(x_1,y)$.

(ii) If $x_1 < x_2 \le y$ then we get

$$\begin{aligned}\overline{A}^{\delta}(x_2, y) &= \bigwedge\{\delta(z) + y - z \mid z \in [x_2, y]\} \\ &\ge \bigwedge\{\delta(z) + y - z \mid z \in [x_1, y]\} \\ &= \overline{A}^{\delta}(x_1, y).\end{aligned}$$

(iii) If $x_1 \le y \le x_2$, the first two cases imply

$$\overline{A}^{\delta}(x_2, y) \ge \overline{A}^{\delta}(y, y) \ge \overline{A}^{\delta}(x_1, y).$$

These cases together prove the monotonicity of $\overline{A}^{\delta}$ in its first component.

Because of the commutativity of $\overline{A}^{\delta}$ it suffices to prove the 1-Lipschitz property of $\overline{A}^{\delta}$ in its second component. Fix again arbitrary numbers $x_1, x_2, y \in [0, 1]$ with $x_1 < x_2$ and consider the following three cases.

(i) If $y \le x_1 < x_2$, then similarly as in the corresponding case in the proof of the monotonicity of $\overline{A}^{\delta}$ we get

$$\begin{aligned}\overline{A}^{\delta}(x_2, y) &= \left(x_2 - x_1 + \overline{A}^{\delta}(x_1, y)\right) \wedge \bigwedge\{\delta(z) + x_2 - z \mid z \in [x_1, x_2]\} \\ &\le \overline{A}^{\delta}(x_1, y) + x_2 - x_1.\end{aligned}$$

(ii) If $x_1 < x_2 \le y$ then we get, taking into account that the function δ is 2-Lipschitz,

$$\begin{aligned}\overline{A}^{\delta}(x_1, y) &= \overline{A}^{\delta}(x_2, y) \wedge \bigwedge\{\delta(z) + y - z \mid z \in [x_1, x_2]\} \\ &\ge \overline{A}^{\delta}(x_2, y) \wedge (\overline{A}^{\delta}(x_2, y) - (x_2 - x_1)) \\ &= \overline{A}^{\delta}(x_2, y) - (x_2 - x_1),\end{aligned}$$

i. e., $\overline{A}^{\delta}(x_2, y) \le \overline{A}^{\delta}(x_1, y) + x_2 - x_1$, because of

$$\begin{aligned}\bigwedge\{\delta(z) + y - z \mid z \in [x_1, x_2]\} &\ge \delta(x_2) + y - x_2 - (x_2 - x_1) \\ &\ge \bigwedge\{\delta(z) + y - z \mid z \in [x_2, y]\} - (x_2 - x_1) \\ &= \overline{A}^{\delta}(x_2, y) - (x_2 - x_1).\end{aligned}$$

(iii) If $x_1 \le y \le x_2$, then the first two cases imply

$$\begin{aligned}\overline{A}^{\delta}(x_2, y) &= \overline{A}^{\delta}(x_2, y) - \overline{A}^{\delta}(y, y) + \overline{A}^{\delta}(y, y) \\ &\le x_2 - y + \overline{A}^{\delta}(x_1, y) + y - x_1 \\ &= \overline{A}^{\delta}(x_1, y) + x_2 - x_1.\end{aligned}$$

Cases (i) – (iii) show that $\overline{A}^{\delta}$ is 1-Lipschitz in its second component.

Finally, since (3.8) holds for all $z \in [x \wedge y, x \vee y]$, $\overline{A}^{\delta}$ is the greatest 1-Lipschitz aggregation operator whose diagonal section coincides with δ. □

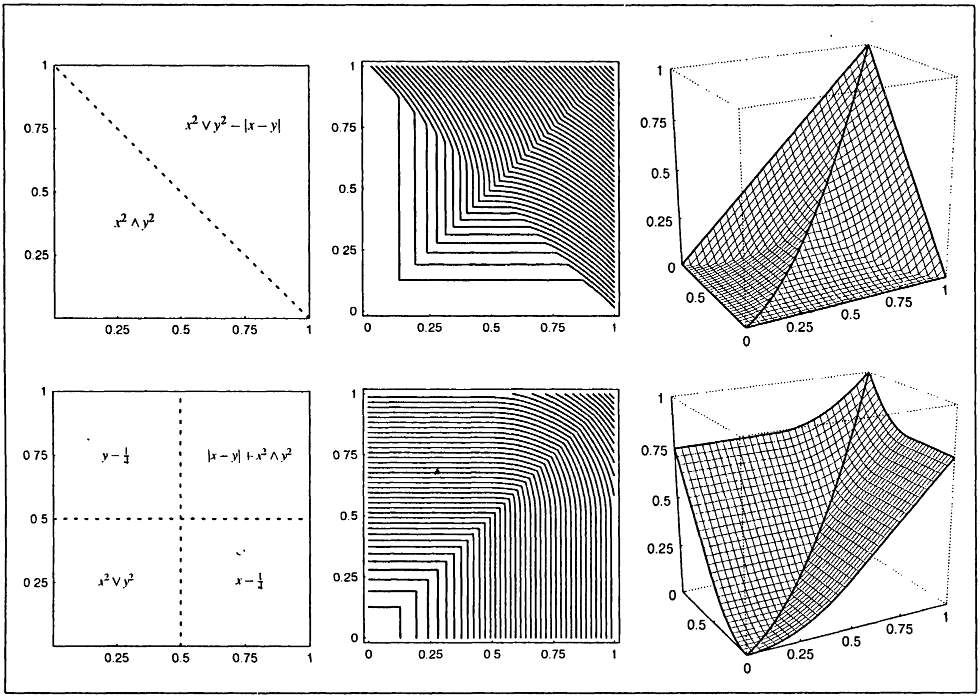

Fig. 1. Domain, contour and 3D plots of the smallest (top) and greatest 1-Lipschitz aggregation operators with diagonal section δ_Π (see Example 3.3 (iii)).

For each diagonal δ, the function $\delta^* : [0,1] \to [0,1]$ defined by $\delta^*(x) = 2x - \delta(x)$ is also a diagonal. Moreover, a 1-Lipschitz aggregation operator A has diagonal section δ if and only if A^*, as defined by (2.7), has diagonal section δ^*.

Since the transition from A to A^* reverses the order between aggregation operators, we get immediately the following result concerning smallest 1-Lipschitz aggregation operator with given diagonal section.

Corollary 3.2. For each function $\delta : [0,1] \to [0,1]$ satisfying (D1) - (D3), the function $\underline{A}^\delta : [0,1]^2 \to [0,1]$ defined by

$$\underline{A}^\delta(x,y) = x \wedge y + \bigvee \{\delta(z) - z \mid z \in [x \wedge y, x \vee y]\} \tag{3.9}$$

is the smallest 1-Lipschitz aggregation operator with diagonal section δ.

Note that this means $\left(\underline{A}^\delta\right)^* = \overline{A}^{\delta^*}$ and $\left(\overline{A}^\delta\right)^* = \underline{A}^{\delta^*}$. Let us illustrate these results for the diagonal sections of the Fréchet-Hoeffding bounds M and W, for the product Π and for some other diagonal.

Example 3.3. Consider the diagonal sections $\delta_M, \delta_W, \delta_\Pi : [0,1] \to [0,1]$ of M, W and Π given by $\delta_M(x) = x$, $\delta_W(x) = (2x-1) \vee 0$ and $\delta_\Pi(x) = x^2$, respectively.

(i) The greatest and smallest 1-Lipschitz aggregation operators $\overline{A}^{\delta_M}$ and $\underline{A}^{\delta_M}$ with diagonal section δ_M are M^* and M, respectively.

(ii) The greatest 1-Lipschitz aggregation operator $\overline{A}^{\delta_W}$ with diagonal section δ_W is given by

$$\overline{A}^{\delta_W}(x,y) = \begin{cases} W(x,y) & \text{if } (x,y) \in \left[0,\tfrac{1}{2}\right]^2 \cup \left]\tfrac{1}{2},1\right]^2, \\ x \vee y - \tfrac{1}{2} & \text{otherwise.} \end{cases}$$

Obviously, W is the smallest 1-Lipschitz aggregation $\underline{A}^{\delta_W}$ operator with diagonal section δ_W.

(iii) The greatest 1-Lipschitz aggregation operator $\overline{A}^{\delta_\Pi}$ and the smallest 1-Lipschitz aggregation operator $\underline{A}^{\delta_\Pi}$ with diagonal section δ_Π are given by

$$\overline{A}^{\delta_\Pi}(x,y) = \begin{cases} x^2 \vee y^2 & \text{if } (x,y) \in \left[0,\tfrac{1}{2}\right]^2, \\ |x-y| + x^2 \wedge y^2 & \text{if } (x,y) \in \left]\tfrac{1}{2},1\right]^2, \\ x \vee y - \tfrac{1}{4} & \text{otherwise.} \end{cases}$$

$$\underline{A}^{\delta_\Pi}(x,y) = \begin{cases} x^2 \wedge y^2 & \text{if } x+y \leq 1, \\ x^2 \vee y^2 - |x-y| & \text{otherwise.} \end{cases}$$

Observe that we have the strict inequalities $\underline{A}^{\delta_\Pi} < \Pi < \overline{A}^{\delta_\Pi}$.

Example 3.4. Consider the function $\delta : [0,1] \to [0,1]$ defined by

$$\delta(x) = (2x-1) \vee (x - \frac{1}{3}) \vee 0. \tag{3.10}$$

Clearly δ satisfies (D1) – (D3), and the greatest 1-Lipschitz aggregation operator $\overline{A}^{\delta}$ and the smallest 1-Lipschitz aggregation operator $\underline{A}^{\delta}$ with diagonal section δ are given by

$$\overline{A}^{\delta}(x,y) = \begin{cases} W(x,y) & \text{if } (x,y) \in \left[0,\tfrac{1}{3}\right]^2 \cup \left[\tfrac{2}{3},1\right]^2, \\ x \vee y - \tfrac{1}{3} & \text{otherwise,} \end{cases} \tag{3.11}$$

$$\underline{A}^{\delta}(x,y) = \begin{cases} x \wedge y - \tfrac{1}{3} & \text{if } (x,y) \in \left[\tfrac{1}{3},\tfrac{2}{3}\right]^2, \\ W(x,y) & \text{otherwise.} \end{cases} \tag{3.12}$$

4. 1-LIPSCHITZ AGGREGATION OPERATORS WITH GIVEN OPPOSITE DIAGONAL SECTION

In this section we show that also the set of 1-Lipschitz aggregation operators with given opposite diagonal section possesses a greatest and a smallest element.

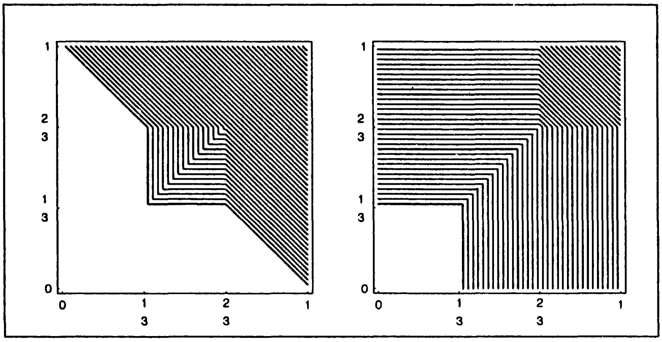

Fig. 2. Smallest (left) and greatest 1-Lipschitz aggregation operators with diagonal section δ (see Example 3.4).

Given a 1-Lipschitz aggregation operator A, then its opposite diagonal section $\omega_A : [0,1] \to [0,1]$ is defined by $\omega_A(x) = A(x, 1-x)$. For an arbitrary 1-Lipschitz aggregation operator A we can only say that ω_A is a 1-Lipschitz function from $[0,1]$ to $[0,1]$.

It is not difficult to see that, as a consequence of its monotonicity and its 1-Lipschitz property, for each 1-Lipschitz aggregation operator A and for all $(x,y) \in [0,1]^2$ we have

$$A(x,y) \leq W(x,y) + \bigwedge\{\omega_A(z) \mid z \in [x \wedge (1-y), x \vee (1-y)]\}. \tag{4.13}$$

Again, we start with an arbitrary 1-Lipschitz function $\omega : [0,1] \to [0,1]$ and look whether there is some 1-Lipschitz aggregation operator A such that for all $x \in [0,1]$ we have $\omega(x) = A(x, 1-x)$, i. e., whose opposite diagonal section coincides with ω, and try to identify the greatest and smallest 1-Lipschitz aggregation operators with this property, provided they exist.

Motivated by (4.13), we obtain the following result:

Proposition 4.1. For each 1-Lipschitz function $\omega : [0,1] \to [0,1]$, the function $F_\omega : [0,1]^2 \to \mathbb{R}$ defined by

$$F_\omega(x,y) = W(x,y) + \bigwedge\{\omega(z) \mid z \in [x \wedge (1-y), x \vee (1-y)]\} \tag{4.14}$$

is a non-decreasing 1-Lipschitz function with $F_\omega(x, 1-x) = \omega(x)$ for all $x \in [0,1]$.

Proof. The monotonicity and the 1-Lipschitz property of F_ω can be shown in a similar way as in the proof of Theorem 3.1. Evidently, $F_\omega(x, 1-x) = \omega(x)$ for all $x \in [0,1]$. □

For example, for the trivial functions $\omega_0, \omega_1 : [0,1] \to [0,1]$ given by $\omega_0(x) = 0$ and $\omega_1(x) = 1$ we obtain $F_{\omega_0} = W$ and $F_{\omega_1} = W + 1$. Note that F_{ω_1} is not an aggregation operator because of $\mathrm{Ran}(F_{\omega_1}) = [0,2]$. Indeed, in general we only know $F_\omega(0,0) \geq 0$ and $F_\omega(1,1) \geq 1$. Therefore, the function F_ω defined by (4.14) is a 1-Lipschitz aggregation operator if and only if it satisfies the boundary conditions for aggregation operators:

Proposition 4.2. Let $\omega : [0,1] \to [0,1]$ be a 1-Lipschitz function and assume that $F_\omega : [0,1]^2 \to \mathbb{R}$ is as in (4.14). Then the function $\overline{A}_\omega : [0,1]^2 \to [0,1]$ defined by

$$\overline{A}_\omega = F_\omega \wedge W^* \tag{4.15}$$

is the greatest 1-Lipschitz aggregation operator with opposite diagonal section ω.

Proof. That $\overline{A}_\omega$ is a 1-Lipschitz aggregation operator follows from Proposition 4.1 and Lemma 2.1, taking into account $W \leq F_\omega$. Clearly $\overline{A}_\omega(x, 1-x) = \omega(x)$ for each $x \in [0,1]$, and due to (4.13) and (2.2), $\overline{A}_\omega$ is the greatest 1-Lipschitz aggregation operator with this property. □

As an immediate consequence of Proposition 4.2 we get:

Corollary 4.3. Let $\omega : [0,1] \to [0,1]$ be a 1-Lipschitz function. The function $F_\omega : [0,1]^2 \to [0,1]$ defined by (4.14) is the greatest 1-Lipschitz aggregation operator with opposite diagonal section ω if and only if ω satisfies $\bigwedge\{\omega(z) \mid z \in [0,1]\} = 0$.

Note that a 1-Lipschitz aggregation operator A has opposite diagonal section ω_A if and only if the 1-Lipschitz aggregation operator A^* given by (2.7) has opposite diagonal section ω_{A^*}, the latter being given by $\omega_{A^*}(x) = 1 - \omega_A(x)$.

Since the transition from A to A^* reverses the order between aggregation operators, for each 1-Lipschitz function $\omega : [0,1] \to [0,1]$ the smallest 1-Lipschitz aggregation $\underline{A}_\omega$ operator with opposite diagonal section is given by $\underline{A}_\omega = \left(\overline{A}_{\omega^*}\right)^*$, where $\omega^*(x) = 1 - \omega(x)$. To be precise, in analogy to Propositions 4.1 and 4.2 and Corollary 4.3 we get:

Corollary 4.4. Let $\omega : [0,1] \to [0,1]$ be a 1-Lipschitz function.

(i) The function $G_\omega : [0,1]^2 \to \mathbb{R}$ defined by

$$G_\omega(x,y) = W^*(x,y) - 1 + \bigvee\{\omega(z) \mid z \in [x \wedge (1-y), x \vee (1-y)]\} \tag{4.16}$$

is a non-decreasing 1-Lipschitz function with $G_\omega(x, 1-x) = \omega(x)$ for all $x \in [0,1]$.

(ii) The function $\underline{A}_\omega : [0,1]^2 \to [0,1]$ defined by $\underline{A}_\omega = G_\omega \vee W$ is the smallest 1-Lipschitz aggregation operator with opposite diagonal section ω.

(iii) The function G_ω is the smallest 1-Lipschitz aggregation operator with opposite diagonal section ω if and only if $\bigvee\{\omega(z) \mid z \in [0,1]\} = 1$.

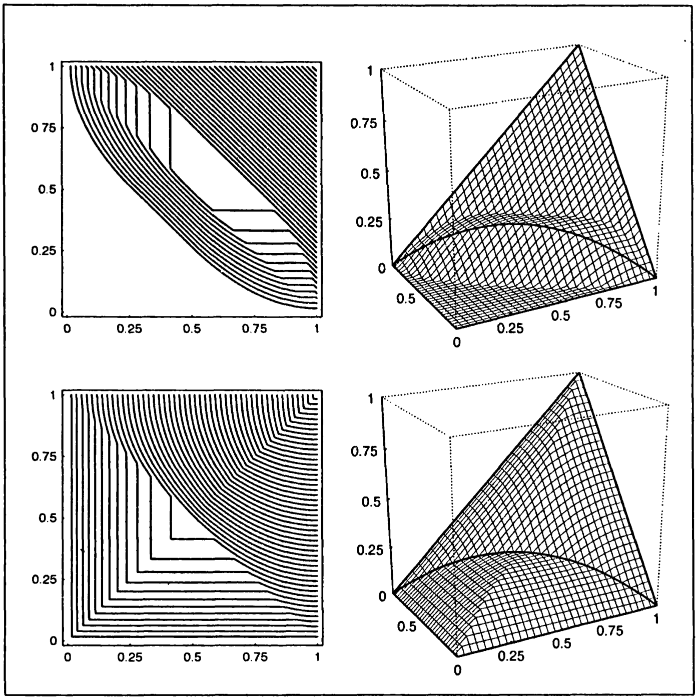

Fig. 3. The smallest (top) and greatest 1-Lipschitz aggregation operators with opposite diagonal section ω_Π (see Example 4.5 (iii)).

Example 4.5. Consider the opposite diagonal sections $\omega_W, \omega_M, \omega_\Pi : [0,1] \to [0,1]$ of W, M and Π given by $\omega_W(x) = 0$, $\omega_M(x) = x \wedge (1-x)$ and $\omega_\Pi(x) = x \cdot (1-x)$, respectively.

(i) W is the only 1-Lipschitz aggregation operator with opposite diagonal section ω_W.

(ii) The smallest 1-Lipschitz aggregation operator with opposite diagonal section ω_M is $(\langle 0, \frac{1}{2}, W\rangle, \langle \frac{1}{2}, 1, W\rangle)$, i. e., an ordinal sum of two copies of the Fréchet-Hoeffding lower bound W. It can be shown that M is the greatest 1-Lipschitz aggregation operator with opposite diagonal section ω_M.

(iii) The greatest 1-Lipschitz operator $\overline{A}_{\omega_\Pi}$ and the smallest 1-Lipschitz operator $\underline{A}_{\omega_\Pi}$ with opposite diagonal section ω_Π are given by

$$\overline{A}_{\omega_\Pi}(x,y) = \begin{cases} (x \wedge y) \cdot (1 - x \wedge y) & \text{if } x + y \leq 1, \\ W(x,y) + (x \vee y) \cdot (1 - x \vee y) & \text{otherwise,} \end{cases}$$

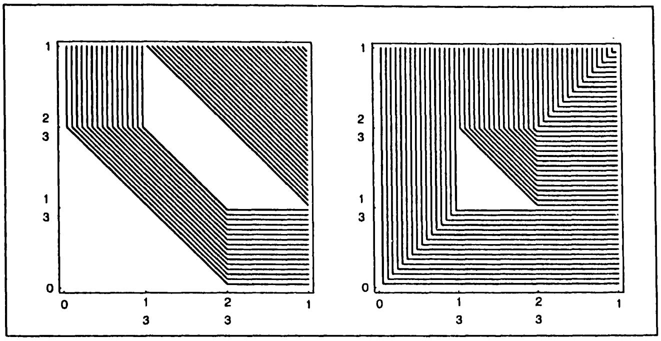

Fig. 4. The smallest (left) and greatest 1-Lipschitz aggregation operators with opposite diagonal section ω (see Example 4.6).

$$\underline{A}_{\omega_\Pi}(x,y) = \begin{cases} (x+y-\frac{3}{4}) \vee 0 & \text{if } (x,y) \in \left[0,\frac{1}{2}\right]^2, \\ (x+y-1) \vee \frac{1}{4} & \text{if } (x,y) \in \left]\frac{1}{2},1\right]^2, \\ x(1-x) & \text{if } x \in \left[0,\frac{1}{2}\right] \text{ and } y \in \left]1-x,1-x^2\right], \\ y(1-y) & \text{if } y \in \left[0,\frac{1}{2}\right] \text{ and } x \in \left]1-y,1-y^2\right], \\ y-(1-x)^2 & \text{if } x \in \left]\frac{1}{2},1\right] \text{ and } y \in \left[(1-x)^2,1-x\right], \\ x-(1-y)^2 & \text{if } y \in \left]\frac{1}{2},1\right] \text{ and } x \in \left[(1-y)^2,1-y\right], \\ W(x,y) & \text{otherwise.} \end{cases}$$

Example 4.6. Consider the 1-Lipschitz function $\omega : [0,1] \longrightarrow [0,1]$ defined by $\omega(x) = x \wedge (1-x) \wedge \frac{1}{3}$. The greatest 1-Lipschitz operator $\overline{A}_\omega$ and the smallest 1-Lipschitz operator $\underline{A}_\omega$ with opposite diagonal section ω are given by

$$\begin{aligned} \overline{A}_\omega &= \left(\langle \frac{1}{3}, \frac{2}{3}, W \rangle\right), \\ \underline{A}_\omega(x,y) &= x \wedge y \wedge \left((x+y-\frac{2}{3}) \vee 0\right) \wedge \left((x+y-1) \vee \frac{1}{3}\right). \end{aligned}$$

5. 1-LIPSCHITZ AGGREGATION OPERATORS DETERMINED IN A SINGLE POINT

Now we look for smallest and greatest 1-Lipschitz aggregation operators whose graphs pass through a point $(x_0, y_0, z_0) \in [0,1]^3$, and we shall show that the set of all 1-Lipschitz aggregation operators with this property has a greatest and a smallest element.

Because of (2.2) it is clear that

$$x_0 + y_0 - 1 \leq z_0 \leq x_0 + y_0 \tag{5.17}$$

is a necessary condition for the existence of such 1-Lipschitz aggregation operators. If $(x_0, y_0, z_0) \in [0,1]^3$ then the functions $\overline{L}^{x_0,y_0,z_0}, \underline{L}^{x_0,y_0,z_0} : [0,1]^2 \to \mathbb{R}$ given by

$$\begin{aligned} \overline{L}^{x_0,y_0,z_0}(x,y) &= z_0 + (x - x_0) \vee 0 + (y - y_0) \vee 0, \\ \underline{L}^{x_0,y_0,z_0}(x,y) &= z_0 + (x - x_0) \wedge 0 + (y - y_0) \wedge 0, \end{aligned}$$

obviously are the greatest and the smallest non-decreasing 1-Lipschitz functions, respectively, whose graphs pass through the point (x_0, y_0, z_0). By definition we have $\underline{L}^{x_0,y_0,z_0} \leq W^*$ and $W \leq \overline{L}^{x_0,y_0,z_0}$.

Proposition 5.1. Let $(x_0, y_0, z_0) \in [0,1]^3$ such that (5.17) holds. Then the functions $\overline{A}^{x_0,y_0,z_0}$, $\underline{A}^{x_0,y_0,z_0} : [0,1]^2 \to [0,1]$ defined by

$$\begin{aligned} \overline{A}^{x_0,y_0,z_0} &= W^* \wedge \overline{L}^{x_0,y_0,z_0}, \\ \underline{A}^{x_0,y_0,z_0} &= W \vee \underline{L}^{x_0,y_0,z_0}, \end{aligned}$$

are the greatest and smallest 1-Lipschitz aggregation operators, respectively, whose graphs pass through the point (x_0, y_0, z_0).

Proof. This is an immediate consequence of Lemma 2.1 (i). □

Proposition 5.2. Assume that $(x_0, y_0, z_0) \in [0,1]^3$ satisfies (5.17). Then we have:

(i) The 1-Lipschitz aggregation operator $\underline{A}^{x_0,y_0,z_0}$ has neutral element 1 if and only if $z_0 \leq x_0 \wedge y_0$.

(ii) The 1-Lipschitz aggregation operator $\overline{A}^{x_0,y_0,z_0}$ has neutral element 0 if and only if $z_0 \geq x_0 \vee y_0$.

Proof. In order to show (i) assume first that 1 is the neutral element of $\underline{A}^{x_0,y_0,z_0}$. Then for all $x \in [0,1]$ we have $x = \underline{A}^{x_0,y_0,z_0}(x,1) = (z_0 + (x - x_0) \wedge 0) \vee x$, which implies $z_0 + (x - x_0) \wedge 0 \leq x$ for all $x \in [0,1]$. Putting $x = x_0$ we obtain $z_0 \leq x_0$. Similarly, from the equality $\underline{A}^{x_0,y_0,z_0}(1,y) = y$ for all $y \in [0,1]$ we derive $z_0 \leq y_0$, so $z_0 \leq x_0 \wedge y_0$.

Conversely, if $z_0 \leq x_0 \wedge y_0$ holds then for each $x \in [0,1]$

$$\underline{A}^{x_0,y_0,z_0}(x,1) = (z_0 + (x - x_0) \wedge 0) \vee x = \left\{ \begin{array}{ll} (z_0 + x - x_0) \vee x & \text{if } x \leq x_0 \\ z_0 \vee x & \text{if } x > x_0 \end{array} \right\} = x.$$

Similarly, we obtain $\underline{A}^{x_0,y_0,z_0}(1,y) = y$ for all $y \in [0,1]$, i. e., 1 is the neutral element of $\underline{A}^{x_0,y_0,z_0}$.

The proof of (ii) is analogous. □

Example 5.3. If we are looking for 1-Lipschitz aggregation operators A whose graphs pass through a certain point (x_0, x_0, z_0) on the diagonal section, i.e., satisfying $A(x_0, x_0) = z_0$, we necessarily must have $(2x_0 - 1) \vee 0 \leq z_0 \leq 2x_0 \wedge 1$ because of (5.17).

Then the greatest diagonal $\overline{\delta_{x_0,z_0}}$ and the smallest diagonal $\underline{\delta_{x_0,z_0}}$ of a 1-Lipschitz aggregation operator containing (x_0, x_0, z_0) are given by

$$\begin{aligned}\overline{\delta_{x_0,z_0}}(x) &= (z_0 \vee (z_0 + 2(x - x_0))) \wedge 2x \wedge 1, \\ \underline{\delta_{x_0,z_0}}(x) &= (z_0 \wedge (z_0 + 2(x - x_0))) \vee (2x - 1) \vee 0,\end{aligned}$$

and $\overline{A}^{\overline{\delta_{x_0,z_0}}}$ and $\underline{A}^{\underline{\delta_{x_0,z_0}}}$ are the greatest and smallest 1-Lipschitz aggregation operators whose graphs pass through the point (x_0, x_0, z_0), respectively.

6. CONSEQUENCES FOR QUASI-COPULAS

Most results of Sections 3 – 5 can be carried over to the case of quasi-copulas. In particular, each of the sets of quasi-copulas with given diagonal section, with given opposite diagonal section, and whose graphs pass through a single point of the unit cube, respectively, always has a greatest and a smallest element.

Each quasi-copula Q is a 1-Lipschitz aggregation operator bounded from above by M. Therefore its diagonal section $\delta_Q : [0,1] \to [0,1]$ satisfies the conditions (D1) – (D3) and, additionally,

(D4) $\quad \delta_Q \leq \mathrm{id}_{[0,1]}$.

For each diagonal δ in this context, i.e., a function $\delta : [0,1] \to [0,1]$ satisfying (D1) – (D4), the functions $Q_1, Q_2 : [0,1] \to [0,1]$ given by

$$\begin{aligned}Q_1(x,y) &= M(x,y) \wedge \frac{\delta(x) + \delta(y)}{2}, \\ Q_2(x,y) &= M(x,y) \wedge \delta\left(\frac{x+y}{2}\right)\end{aligned}$$

are quasi-copulas with diagonal section δ.

Now we can use our results for 1-Lipschitz aggregation operators to obtain the greatest and the smallest quasi-copula with a given diagonal section (introduced in [15]).

Proposition 6.1. For each function $\delta : [0,1] \to [0,1]$ satisfying (D1) – (D4), the function $\overline{Q}^{\delta} : [0,1] \to [0,1]$ defined by $\overline{Q}^{\delta} = M \wedge \overline{A}^{\delta}$ is the greatest quasi-copula with diagonal section δ.

Proof. This is an immediate consequence of Theorem 3.1 and Lemma 2.1 (ii).□

Since the smallest 1-Lipschitz aggregation operator $\underline{A}^{\delta}$, as defined in (3.9), with diagonal section δ (satisfying (D1) – (D4)) is always a quasi-copula, we obtain the following result.

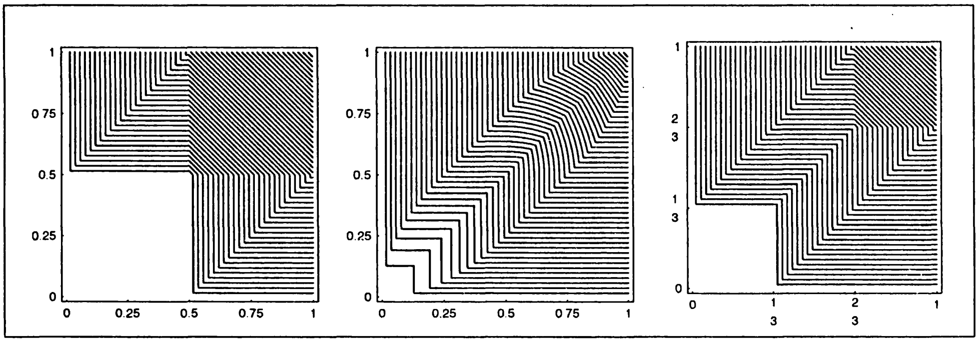

Fig. 5. Greatest quasi-copulas with diagonal sections δ_W (left), δ_Π (center), and δ (see Examples 6.3 (ii) – (iii) and 6.4).

Proposition 6.2. For each function $\delta : [0,1] \to [0,1]$ satisfying (D1) – (D4), the function $\underline{A}^\delta$ defined by (3.9) is the smallest quasi-copula with diagonal section δ.

Example 6.3.

(i) M is the only quasi-copula with diagonal section δ_M.

(ii) The greatest quasi-copula $\overline{Q}^{\delta_W}$ with diagonal section δ_W is given by

$$\overline{Q}^{\delta_W}(x,y) = \begin{cases} W(x,y) & \text{if } (x,y) \in \left[0,\frac{1}{2}\right]^2 \cup \left]\frac{1}{2},1\right]^2, \\ M(x,y) & \text{if } |x-y| > \frac{1}{2}, \\ x \vee y - \frac{1}{2} & \text{otherwise.} \end{cases}$$

(iii) The greatest quasi-copula $\overline{Q}^{\delta_\Pi}$ with diagonal section δ_Π is given by

$$\overline{Q}^{\delta_\Pi}(x,y) = \begin{cases} x^2 \vee y^2 & \text{if } x^2 \vee y^2 \leq x \wedge y \leq x \vee y \leq \frac{1}{2}, \\ x \vee y - \frac{1}{4} & \text{if } x \wedge y \leq \frac{1}{2} \leq x \vee y \leq x \wedge y + \frac{1}{4}, \\ |x-y| + x^2 \wedge y^2 & \text{if } \frac{1}{2} \leq x \wedge y \leq x \vee y \leq 2(x \wedge y) - x^2 \wedge y^2, \\ M(x,y) & \text{otherwise.} \end{cases}$$

Example 6.4. Let $\delta : [0,1] \to [0,1]$ be again the diagonal defined by (3.10) which obviously satisfies also (D4). The greatest quasi-copula $\overline{Q}^\delta$ with diagonal section δ is given by

$$\overline{Q}^{\delta}(x,y) = \begin{cases} W(x,y) & \text{if } (x,y) \in \left[0,\frac{1}{3}\right]^2 \cup \left[\frac{2}{3},1\right]^2, \\ M(x,y) & \text{if } |x-y| > \frac{1}{3}, \\ x \vee y - \frac{1}{3} & \text{otherwise.} \end{cases} \tag{6.18}$$

Turning our attention to quasi-copulas with given opposite diagonal section, note first that the opposite diagonal section ω_Q of each quasi-copula Q must be a 1-Lipschitz function satisfying $\omega_W \leq \omega_Q \leq \omega_M$ because of (2.3). Note also that an arbitrary 1-Lipschitz function $\omega : [0,1] \to [0,1]$ satisfies $0 \leq \omega(x) \leq x \wedge (1-x)$ for each $x \in [0,1]$ if and only if $\omega(0) = \omega(1) = 0$.

Proposition 6.5. Let $\omega : [0,1] \to [0,1]$ be a 1-Lipschitz function such that $\omega(0) = \omega(1) = 0$. Then we have:

(i) The function $F_\omega : [0,1]^2 \to [0,1]$ defined by (4.14) is the greatest quasi-copula with opposite diagonal section ω.

(ii) The function $\underline{A}_\omega : [0,1]^2 \to [0,1]$ defined by $\underline{A}_\omega = G_\omega \vee W$, with $G_\omega : [0,1]^2 \to [0,1]$ as in (4.16), is the smallest quasi-copula with opposite diagonal section ω.

Proof. Because of Corollary 4.3, the function F_ω is the greatest 1-Lipschitz aggregation operator with opposite diagonal section ω, and because of $F_\omega(0,1) = F_\omega(1,0) = 0$ it is the greatest quasi-copula with this property. The proof of (ii) is analogous, using Corollary 4.4 (ii). □

Example 6.6. As a consequence of Proposition 6.5, all the greatest and smallest 1-Lipschitz aggregation operators with opposite diagonal sections ω_W, ω_M, ω_Π (considered in Example 4.5) and ω (considered in Example 4.6), respectively, are also the greatest and smallest quasi-copulas with the respective opposite diagonal section.

As an immediate consequence of Propositions 5.1 and 5.2, we have the following results for quasi-copulas determined in a single point (compare [15]).

Corollary 6.7. Let $(x_0, y_0, z_0) \in [0,1]^3$. If $x_0 + y_0 - 1 \leq z_0 \leq x_0 \wedge y_0$ then $\underline{A}^{x_0,y_0,z_0}$ and $M \wedge \overline{A}^{x_0,y_0,z_0}$ are the smallest and greatest quasi-copulas, respectively, whose graphs pass through the point (x_0, y_0, z_0).

Example 6.8. Any quasi-copula Q whose graph is passing through some point (x_0, x_0, z_0) on the diagonal, i. e., satisfying $Q(x_0, x_0) = z_0$ with $(2x_0 - 1) \vee 0 \leq z_0 \leq x_0$ because of $W \leq Q \leq M$, has a diagonal section δ_Q such that $\underline{\delta_{x_0,z_0}} \leq \delta_Q \leq \overline{\delta_{x_0,z_0}}$, where $\overline{\delta_{x_0,z_0}}$ and $\underline{\delta_{x_0,z_0}}$ are given by

$$\begin{aligned} \overline{\delta_{x_0,z_0}}(x) &= (z_0 \vee (z_0 + 2(x - x_0))) \wedge x, \\ \underline{\delta_{x_0,z_0}}(x) &= (z_0 \wedge (z_0 + 2(x - x_0))) \vee (2x - 1) \vee 0. \end{aligned}$$

Consequently, $\overline{Q}^{\overline{\delta_{x_0,z_0}}}$ and $\underline{Q}^{\underline{\delta_{x_0,z_0}}}$ are the greatest and smallest quasi-copulas whose graphs pass through the point (x_0, x_0, z_0), respectively.

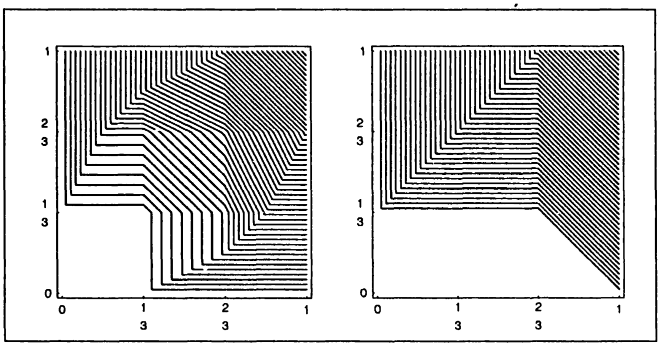

Fig. 6. Two incomparable copulas with diagonal section δ: the maximal copula $\overline{C}_c^\delta$ (left) and the non-commutative copula C given in Example 7.1.

A closer look shows that each quasi-copula with diagonal section $\overline{\delta_{x_0,z_0}}$ has an ordinal sum structure $(\langle z_0, 2x_0 - z_0, Q\rangle)$ (see [10, 13, 16]), where Q is some quasi-copula with diagonal section δ_W. In particular, we have

$$\overline{Q}^{\overline{\delta_{x_0,z_0}}} = (\langle z_0, 2x_0 - z_0, \overline{Q}^{\delta_W}\rangle). \tag{6.19}$$

7. CONSEQUENCES FOR COPULAS

There are several methods to construct copulas with given diagonal section. If $\delta : [0,1] \to [0,1]$ is a diagonal satisfying (D1) - (D4), then from [7, 13, 14] we know that the function $\overline{C}_c^\delta : [0,1]^2 \to [0,1]$ given by

$$\overline{C}_c^\delta(x,y) = M(x,y) \wedge \frac{\delta(x)+\delta(y)}{2}$$

is a commutative copula with diagonal section δ. It is called a *diagonal copula*, and it is the greatest commutative copula with diagonal section δ.

Moreover, $\overline{C}_c^\delta$ is also a maximal copula with diagonal section δ. To see this, assume that C is a (necessarily non-commutative) copula with diagonal section δ such that $C > \overline{C}_c^\delta$. But then C_c defined by $C_c(x,y) = \frac{1}{2}(C(x,y) + C(y,x))$ is a commutative copula with diagonal section δ and $C_c > \overline{C}_c^\delta$, which is a contradiction.

We also mention that in [6] it was shown that an Archimedean copula is uniquely determined by its diagonal section δ whenever $\delta'(1^-) = 2$.

However, in general there is no greatest element in the set of copulas with diagonal section δ. In [15, Theorem 3.4] it was shown that there is a greatest copula with diagonal section δ if and only if $\overline{Q}^\delta = \overline{C}_c^\delta$.

The following is a copula with diagonal section δ which is incomparable with the maximal copula $\overline{C}_c^\delta$:

Example 7.1. Let $\delta : [0,1] \to [0,1]$ be again the diagonal defined by (3.10), and consider the function $C : [0,1]^2 \to [0,1]$ given by

$$C(x,y) = \begin{cases} \overline{Q}^{\delta}(x,y) & \text{if } x \leq y, \\ \underline{A}^{\delta}(x,y) & \text{otherwise}, \end{cases} \tag{7.20}$$

where $\overline{Q}^{\delta}$ and $\underline{A}^{\delta}$ are defined by (6.18) and (3.12), respectively. Then C is a copula (in fact, it is a shuffle of M [13]) with diagonal section δ which is non-commutative and incomparable with $\overline{C}_c^{\delta}$ since we have, on the one hand, $\overline{C}_c^{\delta}(\frac{4}{5}, \frac{1}{10}) > C(\frac{4}{5}, \frac{1}{10})$ and $\overline{C}_c^{\delta}(\frac{3}{10}, \frac{13}{30}) < C(\frac{3}{10}, \frac{13}{30})$, on the other hand.

Note that, for general diagonal sections δ, functions C as constructed in (7.20) need not be copulas.

It was shown in [2, 8] (compare also [13, 15]) that $\underline{A}^{\delta}$ (which is called a *Bertino* copula) is the smallest (commutative) copula with diagonal section δ. This result can easily be derived from Corollary 3.2 and Proposition 6.2:

Corollary 7.2. For each function $\delta : [0,1] \to [0,1]$ satisfying (D1) - (D4), the function $\underline{A}^{\delta}$ given by (3.9) is the smallest copula with diagonal section δ.

If, for a diagonal section $\delta : [0,1] \to [0,1]$ there is some $x_0 \in [0, \frac{1}{2}]$ such that $\delta(x) = 0$ for all $x \in [0, x_0]$ and $(\delta - \mathrm{id}_{[0,1]})|_{[x_0,1]}$ is non-decreasing, then it was shown in [4] that $\underline{A}^{\delta}$ has the following simple form:

$$\underline{A}^{\delta}(x,y) = (\delta(x \vee y) - |x-y|) \vee 0.$$

The greatest quasi-copula with given opposite diagonal section (given in Proposition 6.5) even turns out to be a copula:

Proposition 7.3. Let $\omega : [0,1] \to [0,1]$ be a 1-Lipschitz function such that $\omega(0) = \omega(1) = 0$. Then the function F_ω defined by (4.14) is the greatest copula with opposite diagonal section ω.

Proof. As a consequence of Proposition 6.5 it suffices to prove that F_ω is 2-increasing.

Consider first a square $R_1 = [x_1, x_2] \times [1-x_2, 1-x_1]$. Then from the continuity of ω it follows that $\bigwedge\{\omega(z) \mid z \in [x_1,x_2]\} = \omega(z_0)$ for some $z_0 \in [x_1,x_2]$, and by the 1-Lipschitz property of ω we get $\omega(x_2) - \omega(z_0) \leq x_2 - z_0$ and $\omega(x_1) - \omega(z_0) \leq z_0 - x_1$, leading to

$$V_{F_\omega}(R_1) = x_2 - x_1 - \omega(x_1) - \omega(x_2) + 2\bigwedge\{\omega(z) \mid z \in [x_1,x_2]\} \geq 0.$$

If $R_2 = [x_1,x_2] \times [y_1,y_2]$ is a rectangle with $1 - y_2 \leq 1 - y_1 \leq x_1 \leq x_2$, then

$$\begin{aligned} V_{F_\omega}(R_2) &= \bigwedge\{\omega(z) \mid z \in [1-y_1, x_1]\} - \bigwedge\{\omega(z) \mid z \in [1-y_1, x_2]\} \\ &\quad + \bigwedge\{\omega(z) \mid z \in [1-y_2, x_2]\} - \bigwedge\{\omega(z) \mid z \in [1-y_2, x_1]\}. \end{aligned}$$

Choose $z_0 \in [1-y_2, x_2]$ such that $\bigwedge\{\omega(z) \mid z \in [1-y_2, x_2]\} = \omega(z_0)$. As a consequence of $[1-y_2, x_2] = [1-y_2, 1-y_1] \cup [1-y_1, x_1] \cup [x_1, x_2]$, we distinguish the following three cases:

(i) If $z_0 \in [1-y_2, 1-y_1]$, then $[1-y_1, x_1] \subseteq [1-y_1, x_2]$ implies

$$V_{F_\omega}(R_2) = \bigwedge\{\omega(z) \mid z \in [1-y_1, x_1]\} - \bigwedge\{\omega(z) \mid z \in [1-y_1, x_2]\} \geq 0.$$

(ii) If $z_0 \in [1-y_1, x_1]$, then $V_{F_\omega}(R_2) = 0$.

(iii) If $z_0 \in [x_1, x_2]$, then because of $[1-y_1, x_1] \subseteq [1-y_2, x_1]$ we obtain

$$V_{F_\omega}(R_2) = \bigwedge\{\omega(z) \mid z \in [1-y_1, x_1]\} - \bigwedge\{\omega(z) \mid z \in [1-y_2, x_1]\} \geq 0.$$

If $R_3 = [x_1, x_2] \times [y_1, y_2]$ is a rectangle such that $x_1 \leq x_2 \leq 1-y_2 \leq 1-y_1$, then $V_{F_\omega}(R_3) \geq 0$ can be shown in complete analogy.

Any other rectangle $R \subseteq [0,1]^2$ is a union of finitely many rectangles of types R_1, R_2 and R_3, and the inequality $V_{F_\omega}(R) \geq 0$ follows from the additivity of the measure V_{F_ω}. $\square$

Example 7.4.

(i) As a consequence of Propositions 6.5 and 7.3, each greatest 1-Lipschitz aggregation operator with opposite diagonal section ω_W, ω_M, ω_Π (considered in Example 4.5) and ω (considered in Example 4.6), respectively, is also the greatest copula with the respective opposite diagonal section.

(ii) The smallest 1-Lipschitz aggregation operators with opposite diagonal sections ω_W and ω_M (considered in Example 4.5), respectively, are also the smallest copulas with the respective opposite diagonal section.

(iii) The smallest 1-Lipschitz aggregation operator $\underline{A}_{\omega_\Pi}$ with opposite diagonal section ω_Π (considered in Example 4.5) is the smallest quasi-copula with this property because of Proposition 6.5, but not a copula because of

$$\underline{A}_{\omega_\Pi}\left(\frac{3}{8}, \frac{7}{16}\right) - \underline{A}_{\omega_\Pi}\left(\frac{3}{8}, \frac{9}{16}\right) + \underline{A}_{\omega_\Pi}\left(\frac{5}{8}, \frac{9}{16}\right) - \underline{A}_{\omega_\Pi}\left(\frac{5}{8}, \frac{7}{16}\right) = -0.1171875 < 0.$$

(iv) Similarly, the smallest 1-Lipschitz aggregation operator $\underline{A}_\omega$ with opposite diagonal section ω (considered in Example 4.6) is the smallest quasi-copula with this property, but not a copula because of

$$\underline{A}_\omega\left(\frac{1}{3}, \frac{1}{3}\right) - \underline{A}_\omega\left(\frac{1}{3}, \frac{2}{3}\right) + \underline{A}_\omega\left(\frac{2}{3}, \frac{2}{3}\right) - \underline{A}_\omega\left(\frac{2}{3}, \frac{1}{3}\right) = -\frac{1}{3} < 0.$$

Remark 7.5. The greatest quasi-copula $\overline{Q}^{\delta_W}$ with diagonal section δ_W (see Example 6.3 (ii)) is a shuffle of M [13] and, therefore, also the greatest copula $\overline{Q}^{\delta_W}$ with diagonal section δ_W. As a consequence, the function $\overline{Q}^{\delta_{x_0,z_0}}$ given in (6.19) is the greatest copula whose graph passes through the point (x_0, x_0, z_0).

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