KYBERNETIKA — VOLUME 41 (2005), NUMBER 2, PAGES 205 – 212

CLASSES OF FUZZY MEASURES AND DISTORTION

ĽUBICA VALÁŠKOVÁ AND PETER STRUK

Distortion of fuzzy measures is discussed. A special attention is paid to the preservation of submodularity and supermodularity, belief and plausibility. Full characterization of distortion functions preserving the mentioned properties of fuzzy measures is given.

Keywords: distorted measure, fuzzy measure, belief measure, plausibility measure

AMS Subject Classification: 28E10

1. INTRODUCTION

For applications, several distinguished classes of fuzzy measures are important. We list some of them in the next definitions for the sake of selfcontainedness, though these well known properties can be found, e. g., in [4, 7, 11].

Definition 1. Let $X = \{1, \dots, n\}, n \in N$, be a fixed universe (set of criteria). A mapping $m : P(X) \to [0, 1]$ is called a fuzzy measure whenever $m(\emptyset) = 0,\ m(X) = 1$ and for all $A \subseteq B \subseteq X$ it holds $m(A) \leq m(B)$.

Distinguished classes of fuzzy measures are determined by their respective properties.

Definition 2. A fuzzy measure m on X is called:

1. Subadditive (submeasure) whenever

$$\forall A, B \in P(X), \quad m(A \cup B) \leq m(A) + m(B),$$

2. Superadditive (supermeasure) whenever

$$\forall A, B \in P(X), \quad A \cap B = \emptyset, \quad m(A \cup B) \geq m(A) + m(B),$$

3. Submodular whenever

$$\forall A, B \in P(X), \quad m(A \cup B) + m(A \cap B) \leq m(A) + m(B),$$

4. Supermodular whenever

$$\forall A, B \in P(X), \quad m(A \cup B) + m(A \cap B) \geq m(A) + m(B),$$

5. Belief whenever
$$m(A_1\cup\cdots\cup A_n) \geq \sum_{i=1}^{n} m(A_i) - \sum_{i<j}^{n} m(A_i\cap A_j) + \cdots + (-1)^{n+1} m(A_1\cap\ldots\cap A_n)$$
for arbitrary $n \in N$ and $A_1, \ldots, A_n \in P(X)$,
6. Plausibility whenever
$$m(A_1\cap\cdots\cap A_n) \leq \sum_{i=1}^{n} m(A_i) - \sum_{i<j}^{n} m(A_i\cup A_j) + \cdots + (-1)^{n+1} m(A_1\cup\ldots\cup A_n)$$
for arbitrary $n \in N$ and $A_1, \ldots, A_n \in P(X)$,
7. Possibility whenever
$$\forall A, B \in P(X), \quad m(A \cup B) = m(A) \vee m(B),$$
8. Necessity whenever
$$\forall A, B \in P(X), \quad m(A \cap B) = m(A) \wedge m(B).$$

Observe that if a fuzzy measure is both submodular and supermodular, it is modular and thus a probability measure on X. For more details about the above properties (classes) of fuzzy measures we recommend [7, 8, 11]. Note also that the relationship of above properties is discussed in [10].

Evidently, each belief is supermodular and each supermodular fuzzy measure is also superadditive. Similarly, each plausibility is submodular, and each submodular fuzzy measure is subadditive. Moreover, possibility measures are also plausibility measures and necessity measures are also belief measures. If we denote by $M_i(X)$, $i = 1, \ldots, 8$ the classes of all fuzzy measures on X possessing the property (i), by $Pr(X)$ the class of all probability measures and by $D(X)$ the class of all Dirac measures on X, then
$$M_8(X) \subset M_5(X) \subset M_4(X) \subset M_2(X),$$
$$M_7(X) \subset M_6(X) \subset M_3(X) \subset M_1(X),$$
$$M_8(X) \cap M_7(X) = M_8(X) \cap M_6(X) = M_7(X) \cap M_5(X) = D(X),$$
$$M_5(X) \cap M_6(X) = M_3(X) \cap M_4(X) = Pr(X).$$
Some of the above properties are dual one to another.

Definition 3. Let m be a fuzzy measure on X. Its dual fuzzy measure m^d is given by $m^d(A) = 1 - m(A^C)$, $A \in P(X)$.

Now, it is evident that $(m^d)^d = m$ and that m is a submodular measure if and only if m^d is a supermodular measure. Similarly, m is a belief measure if and only if m^d is a plausibility measure and m is a possibility measure if and only if m^d is a necessity measure. However, the subadditivity and the superadditivity are not more dual properties, i.e., a dual m^d of a subadditive fuzzy measure m need not be a superadditive measure and vice-versa. In this context observe that the participation property $m(A)+m(A^C) = 1$ for all $A \in P(X)$ discussed in [10] is just the selfduality, i.e., m is a participation measure if and only if $m = m^d$.

The aim of this contribution is a discussion of distortions preserving the above mentioned properties. Namely, let $f : [0,1] \to [0,1]$ be a non-decreasing function with $f(0) = 0, f(1) = 1$. The class of all such functions will be denoted by F. We are interested under which conditions on f it holds: If a fuzzy measure m has one of properties from Definition 2 then also $f(m)$ has this property, independently on the universe X.

Note that the problem is well formulated as far as under the above requirements, $f(m)$ is well defined fuzzy measure. Note also that till now, only some partial solutions of our problem were given, compare for example [3]. Especially, convex combinations preserve any of the above properties. Moreover, because of preservation of non-strict inequalities by limits, we have the following result.

Lemma 1. Let F_i, $i = 1, \dots, 8$ be the class of all non-decreasing functions $f : [0,1] \to [0,1]$, $f(0) = 0$, $f(1) = 1$, such that for any X and any $m \in M_i(X)$ also $f(m) \in M_i(X)$. Then each F_i is a closed convex subset of $[0,1]^{[0,1]}$.

Recall that the composition of functions makes F a semigroup and then each F_i is a subsemigroup of F, i.e., each F_i is closed under a composition of functions.

Note that concerning the participation measures (selfduality property), it is evident that for any participation measure m and $f \in F$, also $f(m)$ is a participation measure if and only if $f(x) + f(1-x) = 1$ for all $x \in [0,1]$, i.e., if f commutes with the negation $n(x) = 1 - x$.

2. PRESERVATION OF SUBADDITIVITY, SUPERADDITIVITY, SUBMODULARITY AND SUPERMODULARITY

The following characterization of distortion functions preserving the subadditivity and the superadditivity, respectively, is an immediate consequence of the definition of these properties of fuzzy measures.

Proposition 1. The subadditivity (superadditivity) of fuzzy measures is preserved by subadditive (superadditive) distortion functions, i.e.,

$$F_1 = \{f \in F | f \text{ is subadditive}\} \text{ and } F_2 = \{f \in F | f \text{ is superadditive}\}.$$

Denote by $f_* = 1_{\{1\}}$ the weakest element of F, by $f^* = 1_{]0,1]}$ the strongest element of F and by id the identity on [0,1]. Then $F_1 \cap F_2 = id$, $f_* \in F_2$ and $f^* \in F_1$. All these functions are vertices in the convex sets F_1 and F_2, respectively.

Now, we turn our attention to the distortion functions from F_3 and F_4. First we summarize some equivalent definitions of a convex function from F.

Lemma 2. The following are equivalent for a non-decreasing function $f:[0,1]\to[0,1]$:

(i) f is convex,

(ii) $\forall\, x,\, y, \lambda \in [0,1] : f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$,

(iii) $\forall\, x,\, y \in [0,1] : f(\frac{x+y}{2}) \le \frac{f(x)+f(y)}{2}$ (Jensen's inequality),

(iv) $\forall\, x,\, y, \varepsilon \in [0,1], x \le y, x+\varepsilon, y+\varepsilon \in [0,1] : f(x+\varepsilon) - f(x) \le f(y+\varepsilon) - f(y)$.

From properties of a supermodular fuzzy measure we immediately have the next result.

Lemma 3. A non-decreasing function $f \in F$ preserves the supermodularity of fuzzy measures if and only if

$$\forall\, a,\, b,\, c,\, d \in [0,1], a \le b \le c \le d, a+d \ge b+c \Rightarrow f(a)+f(d) \ge f(b)+f(c).$$

Now we are able to characterize supermodularity preserving functions more precisely.

Theorem 1. A non-decreasing function $f:[0,1]\to[0,1], f(0)=0, f(1)=1$ preserves the supermodularity of fuzzy measures if and only if f is convex, i.e., $F_4 = \{f \in F | f \text{ is convex}\}$.

Proof.

1. Let f be supermodular preserving function. In Lemma 3, put $b=c=\frac{a+d}{2}$, $0 \le a \le d$. Then $f(a)+f(d) \ge f(\frac{a+d}{2}) + f(\frac{a+d}{2})$, i.e., $f(\frac{a+d}{2}) \le \frac{f(a)+f(d)}{2}$ and due to Lemma 2 (Jensen's inequality), f is convex.
2. Let for some $a,\, b,\, c,\, d \in [0,1]$, $a \le b \le c \le d$ and $a+d \ge b+c$ we have $f(a)+f(d) \le f(b)+f(c)$. Put $d^* = b+c-a \le d$. Then for $\varepsilon = b-a$ it holds that $d^*-c=\varepsilon$, $a+d^*=b+c$, $f(a)+f(d^*) \le f(a)+f(d) < f(b)+f(c)$ and hence $f(c+\varepsilon)-f(c) < f(a+\varepsilon)-f(a)$ violating (iv) of Lemma 2, and then f cannot be convex. □

Consequently we have the characterization of submodularity preserving functions.

Corollary 1. A function $f:[0,1]\to[0,1]$, which is non-decreasing, $f(0)=0$, $f(1)=1$, preserves the submodularity if and only if f is concave, i.e., $F_3 = \{f \in F | f \text{ is concave}\}$.

Proof. The result follows from the fact that f preserves submodularity of fuzzy measures if and only if its dual function$f^d:[0,1]\to[0,1]$ given by $f^d = 1-f(1-x)$ preserves the supermodularity. However, the convexity of f^d is equivalent to the concavity of f.

Note that the convex class F_4 of distortion functions can be characterized by its vertices which are f_* and for $a \in [0,1[$, $f_a \in F$, $f_a(x) = \max(0, \frac{x-a}{1-a})$. By duality the convex class F_3 is characterized by its vertices f^* and $g_a \in F$, $a \in [0,1[$, $g_a(x) = \min(1, \frac{x}{1-a})$. □

3. PRESERVATION OF BELIEF, PLAUSIBILITY, POSSIBILITY AND NECESSITY MEASURES

It is well-known that the product of belief measures on $P(X)$ is again a belief measure, see e.g. [5]. Therefore, $p_n \in F$ given by $p_n(x) = x^n, n \in N$ preserve belief fuzzy measures, i.e., $p_n \in F_5$. Moreover, as far as F_5 is a convex class, also each polynomial $p = \sum_{i=1}^{n} a_i p_n \in F$, $a_i \in [0,1]$, $\sum_{i=1}^{n} a_i = 1$, is a member of F_5. Further, due to the closedness of F_5, also each infinite Taylor series $p = \sum_{n=1}^{\infty} a_n p_n \in F$, $a_n \in [0,1]$, $\sum_{n=1}^{\infty} a_n = 1$, is contained in F_5. We conjecture that continuous members of F_5 form the convex hull of $\{p_n | n \in N\}$, i.e., that each continuous belief preserving distortion function is a Taylor series (finite or infinite) with non-negative coefficients.

Theorem 2. Let $f : [0,1] \to [0,1]$, $f(0) = 0$, $f(1) = 1$ be a non-decreasing function having all derivatives on $]0,1[$. If function f preserves belief, then for all $k \in N$ it holds $f^{(k)}(x) \geq 0$ for all $x \in]0,1[$.

Proof. Suppose that there exist $k \in N$ and $x_0 \in [0,1]$ such that $f^{(k)}(x_0) < 0$. We will prove the existence of fuzzy measure m, for which $f(m)$ is not belief.

Since kth derivative of f is continuous, there exist an interval $]a,b[$ such that $x_0 \in]a,b[$ and $f^{(k)}(x) < 0$ for all $x \in]a,b[$. We can choose rational numbers x_1, x_2 such that $x_0 \leq x_1 \leq x_2 \leq b$. Due to [6] it holds

$$^{h}\Delta_f^k(x_1) < 0 \text{ for arbitrary positive step } h \text{ with } kh + x_1 \leq b. \tag{1}$$

We notice that

$$^{h}\Delta_f^k(x) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(x_1 + ih)$$

is kth difference of f in x_1 with the step h.

Let $h = \frac{x_2 - x_1}{k}$, than x_1, x_2, h are rational numbers and we can express them by quotients with joint denominator $x_1 = \frac{p}{q}, x_2 = \frac{r}{q}, h = \frac{s}{q}$.

Let $X = \{1, 2, \ldots, q\}$ and $m : P(X) \to [0,1]$, $m(A) = \frac{|A|}{q}$ for all $A \in P(X)$, m is obviously belief fuzzy measure. In order to composed set function $\mu = f(m)$ be belief, it must fulfill the condition (5) in Definition 2.

Let

$$\begin{aligned}
A &= \{1, 2, \ldots, p + ks\}, \\
A_1 &= A - \{1, 2, \ldots, s\}, \\
A_2 &= A - \{s + 1, \ldots, 2s\}, \\
&\vdots \\
A_k &= A - \{(k-1)s + 1, \ldots, ks\}.
\end{aligned}$$

Obviously $\bigcup_{i=1}^{k} A_i = A$, $\mu(A) = f(m(A)) = f\left(\frac{|A|}{q}\right) = f\left(\frac{p+ks}{q}\right) = f(x_1 + kh)$.

$$\mu(A_i) = f(m(A_i)) = f\left(\frac{|A_i|}{q}\right) = f\left(\frac{p+(k-1)s}{q}\right) = f(x_1 + (k-1)h),\ i \in \{1, 2, \dots, k\},$$

$$\mu(A_i \cap A_j) = f\left(\frac{|A_i \cap A_j|}{q}\right) = f\left(\frac{p+(k-2)s}{q}\right) = f(x_1 + (k-2)h),\ i < j,$$

$i, j \in \{1, 2, \dots, k\}$, and finally

$$\mu\left(\bigcap_{i=1}^{k} A_i\right) = f\left(\frac{\left|\bigcap_{i=1}^{k} A_i\right|}{q}\right) = f\left(\frac{p}{q}\right) = f(x_1).$$

Condition (5) in Definition 2 for our sets $A_1, \dots, A_k \in P(X)$ has the form

$$f(x_1+kh) - k.f(x_1+(k-1)h) + \binom{k}{2} f(x_1+(k-2)h) - \dots + (-1)^k \binom{k}{k} f(x_1) \geq 0$$

that is ${}^h\Delta_f^k(x_1) \geq 0$ contradicting (1).

Based on generalized differences and the generalized Euler's formula, also the sufficiency of non-negativity of all derivatives on $]0,1[$ of a function $f \in F_4$ to preserve belief fuzzy measures was recently shown in [2]. Moreover, the non-existence of some derivative $f^{(k)}$ in some point $x \in]0,1[$ excludes f from F_5. Summarizing all above results, we have the next characterization of F_5.

Theorem 3. A function $f \in F$ preserves belief if and only if it has all derivatives on $]0,1[$ which are non-negative and $f \leq id$, i.e.,

$$F_5 = \{f \in F | f \leq id \text{ and } f^{(k)}(x) \geq 0 \text{ for all } k \in N, x \in]0,1[\}.$$

Denote the convex hull of $\{p_n | n \in N\}$ by CF_5 (conjecturing the coincidence of continuous elements of F_5 with elements of CF_5). Observe also that $f_* \in F_5$.

Lemma 4. f_* and $p_n, n \in N$ are the vertices of F_5.

For f_*, the claim is evident. For a given $n \in N$, suppose that $p_n = \lambda f + (1-\lambda)g$ for some $\lambda \in]0,1[$, $f, g \in F_5$. Then $p_n^{(k)} = \lambda f^{(k)} + (1-\lambda)g^{(k)} = 0$ for all $k > n$ (on $]0,1[$), and thus due to Theorem 2 also $f^{(k)} = g^{(k)} = 0$ on $]0,1[$ for all $k > n$. Hence $f = \sum_{i=1}^{n} a_i x^i$ and $g = \sum_{i=1}^{n} b_i x^i$ on $]0,1[$ with $a_i \geq 0, b_i \geq 0$, $\sum_{i=1}^{n} a_i \leq 1$, $\sum_{i=1}^{n} b_i \leq 1$. However, then

$$\begin{aligned} 0 &= \lambda a_1 + (1-\lambda)b_1 \\ &\vdots \\ 0 &= \lambda a_{n-1} + (1-\lambda)b_{n-1} \\ 1 &= \lambda a_n + (1-\lambda)b_n, \end{aligned}$$

i.e., $f = g = p_n$. Therefore p_n is a vertex of F_5.

Lemma 5. Each non-continuous element $f \in F_5$, $f \neq f_*$, is a convex combination of f_* and a continuous element $g \in F_5$.

Proof. Due to Theorem 3, the continuity of all distortion function $f \in F_5$ on $[0,1[$ is granted. The non-continuity of f means that $f(1^-) = \lambda < 1$ and from $f \neq f_*$ we conclude $\lambda > 0$. Put $g = \frac{1}{\lambda}(f + (1-\lambda)f^*)$. Then g is continuous on $[0,1]$, $g^{(k)} = \frac{1}{\lambda}f^{(k)} \geq 0$ for all $k \in N$ on $]0,1[$, and moreover $g \leq id$. Thus g is a continuous member of F_5 and $f = \lambda g + (1-\lambda)f^*$. □

It is evident that the convex hull of $\{f_*, p_n | n \in N\}$ is a subset of F_5. Due to the above lemmas, we conjecture the equality of these two convex sets.

Example. The following continuous functions belong to CF_5:

$$f(x) = \frac{a^x - 1}{a - 1}, a \in]1, \infty[; \; f(x) = \frac{\ln(1 - ax)}{\ln(1 - a)}, a \in]0, 1[.$$

By duality, we obtain the characterization of F_6.

Corollary 2. A function $f \in F$ preserves plausibility if and only if it has all derivatives on $]0,1[$ which are alternating, and $f \geq id$, i. e.,

$$F_6 = \{f \in F | f \geq id, (-1)^k f^{(k)}(x) \leq 0 \text{ for all } k \in N, x \in]0, 1[\}.$$

Characterization of classes F_7 and F_8 preserving the possibility measures and the necessity measures, respectively, is straightforward from the definition.

Proposition 2. Each distortion function preserves both possibility measures and necessity measures, i. e., $F_7 = F_8 = F$.

Observe that each possibility measure m on $P(X)$ is induced by a fuzzy subset $\varphi : X \to [0,1]$ (possibility distribution) with non-empty kernel $\ker(\varphi) = \{x \in X | \varphi(x) = 1\}$. Then $f_*(m)$ is induced by the possibility distribution $\varphi_* = 1_{\ker(\varphi)}$, while $f^*(m)$ is induced by $\varphi^* = 1_{\mathrm{supp}(\varphi)}$, where the support $\mathrm{supp}(\varphi) = \{x \in X | \varphi(x) > 0\}$. Similar observations are valid also for the necessity measures.

4. CONCLUSION

We have discussed and characterized distortion function preserving distinguished classes of fuzzy measures. Especially important seems to be the characterization of function preserving the belief measures. Moreover, here we conjecture that these functions are exactly the elements of the convex hull of $\{f_*, p_n | n \in N\}$. We expect similar results for k-ary aggregation operators preserving the above discussed classes of fuzzy measures. For example in the case of belief measures, the crucial role of polynomials $x_1^{n_1} \dots x_k^{n_k}$, $n_i \in N \cup \{0\}$, $\sum_{i=1}^k n_i > 0$, is expected.

ACKNOWLEDGEMENT

This work is supported by Science and Technology Assistance Agency under the contract No. APVT-20-023402. A partial support of the grant VEGA 1/1145/04 is also kindly announced.

(Received June 12, 2004.)

REFERENCES

[1] R. J. Aumann and L. S. Shapley: Values of Non-Atomic Games. Princeton University Press, Princeton 1974.

[2] A. G. Bronevich: Aggregation operators of fuzzy measures. Properties of inheritance, submitted.

[3] A. G. Bronevich and A. E. Lepskiy: Operators for Convolution of Fuzzy Measures. In: Soft Methods in Probability, Statistics and Data Analysis, Advances in Soft Computing, Physica–Verlag, Heidelberg 2002, pp. 84–91.

[4] D. Denneberg: Non-Additive Measure and Integral. Kluwer Academic Publishers, Dordrecht 1994.

[5] D. Dubois and H. Prade: Possibility Theory. Plenum Press, New York 1998.

[6] V. K. Dzjadyk: Vvedenie v teoriju ravnomernogo približenia funkcij polinomami. Nauka, Moskva 1977.

[7] E. Pap: Null-Additive Set Functions. Kluwer Academic Publishers, Dordrecht – Boston – London and Ister Science, Bratislava 1995.

[8] E. Pap (ed.): Handbook on Measure Theory. Elsevier, Amsterdam 2002.

[9] P. Struk and Ľ. Valášková: Preservation of distinguished fuzzy measure classes by distortion. In: Uncertainty Modelling 2003, Publishing House of STU, Bratislava 2003, pp. 48–51.

[10] A. Stupňanová and P. Struk: Pessimistic and optimistic fuzzy measures on finite sets. In: MaGiA 2003, Publishing House of STU, Bratislava 2003, pp. 94–100.

[11] Z. Wang and G. Klir: Fuzzy Measure Theory. Plenum Press, New York – London 1992.

Ľubica Valášková and Peter Struk, Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak University of Technology, Radlinského 11, 813 68 Bratislava. Slovak Republic.
e-mails: luba@math.sk, struk@math.sk