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SELF-BOUNDED CONTROLLED INVARIANT SUBSPACES IN MEASURABLE SIGNAL DECOUPLING WITH STABILITY: MINIMAL-ORDER FEEDFORWARD SOLUTION

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The structural properties of self-bounded controlled invariant subspaces are fundamental to the synthesis of a dynamic feedforward compensator achieving insensitivity of the controlled output to a disturbance input accessible for measurement, on the assumption that the system is stable or pre-stabilized by an inner feedback. The control system herein devised has several important features: i) minimum order of the feedforward compensator; ii) minimum number of unassignable dynamics internal to the feedforward compensator; iii) maximum number of dynamics, external to the feedforward compensator, arbitrarily assignable by a possible inner feedback. From the numerical point of view, the design method herein detailed does not involve any computation of eigenspaces, which may be critical for systems of high order. The procedure is first presented for left-invertible systems. Then, it is extended to non-left-invertible systems by means of a simple, original, squaring-down technique.

Keywords: geometric approach, linear systems, self-bounded controlled invariant subspaces, measurable signal decoupling, non-left-invertible systems

AMS Subject Classification: 93B27, 93B50, 93C05, 93C35, 93C55

1. INTRODUCTION

In this paper, the problem of making the system output insensitive to a disturbance input accessible for measurement is treated by means of the geometric approach tools [1, 3, 9, 10]. On the assumption that the system is stable, or pre-stabilized by an inner feedback, an original method is presented for the synthesis of a minimal-order dynamic feedforward compensator, based on the use of self-bounded controlled invariant subspaces. First, the procedure is presented for left-invertible systems. Then, it is extended to non-left-invertible systems by resorting to a simple, strictly geometric, squaring-down technique.

Although self-bounded controlled invariants were introduced in [2] and their basic properties were proved in [8], their potentiality is far from being thoroughly exploited. In fact, their introduction initially led to the statement of an original set of necessary and sufficient conditions for the solution of the disturbance localization problem with stability, based on the use of the minimal (internally stabilizable) controlled invariant self-bounded with respect to the null space of the output and containing the range space of the disturbance input [4]. Afterwards, just few works were devoted to the investigation of the advantages that may derive from adopting self-bounded controlled invariants in the solution of other problems of interest in the field. A relevant contribution is found in [7], where self-bounded controlled invariants and their dual, namely self-hidden conditioned invariants, are employed to synthesize a reduced-order dynamic feedback regulator which guarantees not only that the controlled output is equal to zero for any piecewise-continuous disturbance input if the system is initially in the zero state, but also that the controlled output asymptotically tends to zero if the disturbance input is identically zero, whichever the initial conditions are. In a later work, the features of self-bounded controlled invariants were analyzed in the more general context of singular systems, still with special emphasys on disturbance localization [5].

As aforementioned, self-bounded controlled invariants are herein considered in connection with a relaxed version of disturbance decoupling, where the signal to be localized is accessible for measurement. On the assumption that the system is stable or at least stabilizable by an inner feedback, as is usually required to make a feedforward scheme feasible [6], and that the necessary and sufficient constructive conditions are satisfied, the dynamic feedforward unit designed on the basis of the minimal internally stabilizable controlled invariant self-bounded with respect to the null space of the output, usually denoted by $\mathcal{V}_m$, has several noticeable features. First of all, since the feedforward unit reproduces the sole dynamics corresponding to the internal eigenvalues of $\mathcal{V}_m$, it is of minimal order, if the system is left-invertible. For this same reason, if the system is controllable, the dynamics which are arbitrarily assignable by a possible inner feedback are the maximum number: in fact, they correspond to the external assignable eigenvalues of $\mathcal{V}_m$. In other words, the invariant zeros of the system involved in the design of the feedforward unit, or equivalently, the internal unassignable eigenvalues of $\mathcal{V}_m$, are a subset of the set of the stable invariant zeros of the system, in general. Finally, from the numerical point of view, the detailed procedure avoids any computation of eigenspaces, which may be critical for high-order systems.

Linear discrete time-invariant systems without feedthrough terms are considered. However, the results presented may be extended to systems with feedthrough terms by resorting to the well-known contrivance of inserting a unit delay on the control input, or the controlled output, signal flow [3].

Notation: The symbols $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{C}^{\odot}$ stand for the set of real numbers, the set of complex numbers, and the set of complex numbers inside the unit circle, respectively. Sets, vector spaces, and subspaces are denoted by script capitals like $\mathcal{X}$. The quotient space of a vector space $\mathcal{X}$ over a subspace $\mathcal{V}\subseteq\mathcal{X}$ is denoted by $\mathcal{X}/\mathcal{V}$. The orthogonal complement of a subspace $\mathcal{V}\subseteq\mathcal{X}$ is denoted by $\mathcal{V}^{\perp}$. The dimension of a subspace $\mathcal{V}\subseteq\mathcal{X}$ is denoted by $\dim\mathcal{V}$. Matrices and linear maps are denoted by slanted capitals like A. The restriction of a linear map A to an A-invariant subspace $\mathcal{J}$ is denoted by $A|_{\mathcal{J}}$. The inverse image of a subspace $\mathcal{V}$ through a linear map B

is denoted by $B^{-1}\mathcal{V}$. The range space and the null space of A are denoted by $\operatorname{im} A$ and $\ker A$, respectively. The symbols A^{-1}, A^{+}, and $A^{\top}$ are used for the inverse, the generalized inverse and the transpose of A, respectively. The set of eigenvalues of A is denoted by $\sigma(A)$. The symbol I is used to denote an identity matrix. The symbol O is used to denote a zero matrix of appropriate dimensions. The symbol $\uplus$ is used for aggregation, i.e. union with repetition count. The symbol $\Sigma\equiv(A,B,C,D)$ is used to mean that the system Σ is modeled by the quadruple (A,B,C,D) or any other quadruple equivalent by a state-space similarity transformation.

2. GEOMETRIC APPROACH BACKGROUND

The discrete time-invariant linear system

$$x(t+1) = A\,x(t) + B\,u(t) + H\,h(t), \tag{1}$$

$$y(t) = C\,x(t), \tag{2}$$

is considered, where $x\in\mathcal{X}=\mathbb{R}^n$, $u\in\mathbb{R}^p$, $h\in\mathbb{R}^s$, and $y\in\mathbb{R}^q$ respectively denote the state, the control input, the exogenous input, and the controlled output. The set of all admissible control input functions is defined as the set $\mathcal{U}_f$ of all bounded functions with values in $\mathbb{R}^p$. The set of all admissible exogenous input functions is defined as the set $\mathcal{H}_f$ of all bounded functions with values in $\mathbb{R}^s$. The matrices B, H, and C are assumed to be of full rank. The symbols $\mathcal{B}$, $\mathcal{H}$, and $\mathcal{C}$ are used for $\operatorname{im} B$, $\operatorname{im} H$, and $\ker C$, respectively. The notations $\mathcal{V}^*$ or $\max\mathcal{V}(A,\mathcal{B},\mathcal{C})$ are used for the maximal $(A,\mathcal{B})$-controlled invariant contained in $\mathcal{C}$, $\mathcal{S}^*$ or $\min\mathcal{S}(A,\mathcal{C},\mathcal{B})$ are used for the minimal $(A,\mathcal{C})$-conditioned invariant containing $\mathcal{B}$, and $\mathcal{R}_{\mathcal{V}^*}$ is used for the subspace reachable from the origin on $\mathcal{V}^*$. In the following, some well-known results of the geometric approach are briefly recalled [1, 10, 9, 3]. The subspaces $\mathcal{V}^*$, $\mathcal{S}^*$, and $\mathcal{R}_{\mathcal{V}^*}$ satisfy $\mathcal{R}_{\mathcal{V}^*}=\mathcal{V}^*\cap\mathcal{S}^*$. The triple (A,B,C) is right-invertible if and only if $\mathcal{V}^*+\mathcal{S}^*=\mathbb{R}^n$ or equivalently $\mathcal{S}^*+\mathcal{C}=\mathbb{R}^n$, is left-invertible if and only if $\mathcal{V}^*\cap\mathcal{S}^*=\{0\}$ or equivalently $\mathcal{V}^*\cap\mathcal{B}=\{0\}$, and it is right- and left-invertible if and only if $\mathcal{V}^*\oplus\mathcal{S}^*=\mathbb{R}^n$. A subspace $\mathcal{V}\subseteq\mathcal{X}$ is an $(A,\mathcal{B})$-controlled invariant if and only if there exists at least one real matrix F such that $(A+BF)\mathcal{V}\subseteq\mathcal{V}$. Let $\mathcal{V}\subseteq\mathcal{X}$ be an $(A,\mathcal{B})$-controlled invariant, let F be any real matrix such that $(A+BF)\mathcal{V}\subseteq\mathcal{V}$, and let $\mathcal{R}_{\mathcal{V}}$ be the subspace reachable from the origin on $\mathcal{V}$, i.e. $\mathcal{R}_{\mathcal{V}}=\mathcal{V}\cap\min\mathcal{S}(A,\mathcal{V},\mathcal{B})$. The assignable and the unassignable internal eigenvalues of $\mathcal{V}$ are respectively defined as $\sigma((A+BF)|_{\mathcal{R}_{\mathcal{V}}})$ and $\sigma((A+BF)|_{\mathcal{V}/\mathcal{R}_{\mathcal{V}}})$. Let $\mathcal{R}$ be the reachable set of the pair $(A,\mathcal{B})$. The assignable and the unassignable external eigenvalues of $\mathcal{V}$ are respectively defined as $\sigma((A+BF)|_{(\mathcal{V}+\mathcal{R})/\mathcal{V}})$ and $\sigma((A+BF)|_{\mathcal{X}/(\mathcal{V}+\mathcal{R})})$. Hence, $\mathcal{V}$ is internally stabilizable if and only if there exists at least one real matrix F such that $(A+BF)\mathcal{V}\subseteq\mathcal{V}$ and $\sigma((A+BF)|_{\mathcal{V}})\subset\mathbb{C}^{\odot}$. Likewise, $\mathcal{V}$ is externally stabilizable if and only if there exists at least one real matrix F such that $(A+BF)\mathcal{V}\subseteq\mathcal{V}$ and $\sigma((A+BF)|_{\mathcal{X}/\mathcal{V}})\subset\mathbb{C}^{\odot}$. The unassignable internal eigenvalues of $\mathcal{V}^*$ are also called the invariant zeros of the triple (A,B,C) and denoted by $\mathcal{Z}(A,B,C)$. If (A,B) is stabilizable, any $(A,\mathcal{B})$-controlled invariant is externally stabilizable. Let $\mathcal{V}\subseteq\mathcal{X}$ be an $(A,\mathcal{B})$-controlled invariant contained in $\mathcal{C}$, $\mathcal{V}$ is said to be self-bounded with respect to $\mathcal{C}$ if $\mathcal{V}\supseteq\mathcal{V}^*\cap\mathcal{B}$. The set of all $(A,\mathcal{B})$-controlled invariants self-bounded with respect to $\mathcal{C}$ is a nondistributive

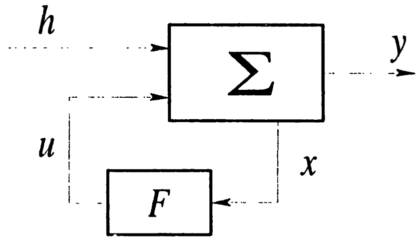

Fig. 1. Block diagram for unaccessible signal decoupling.

lattice with respect to $\subseteq$, $+$, $\cap$, also denoted by $\Phi(\mathcal{B},\mathcal{C})$. The supremum is $\mathcal{V}^*$. The infimum is $\mathcal{R}_{\mathcal{V}^*}$.

The concepts of self-boundedness and internal stabilizability of an $(A,\mathcal{B})$-controlled invariant play a crucial role in unaccessible and measurable signal decoupling problems with stability when the solutions of minimal complexity are sought. More precisely, the lattice of all $(A,\mathcal{B}+\mathcal{H})$-controlled invariants self-bounded with respect to $\mathcal{C}$, henceforth denoted by $\Phi(\mathcal{B}+\mathcal{H},\mathcal{C})$, and its infimum, $\mathcal{V}_m$, are of primary interest due to the properties briefly recalled below.

Property 1. [2, 8] Let $\mathcal{H} \subseteq \mathcal{V}^*$ or $\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{B}$. Then, $\max \mathcal{V}(A,\mathcal{B}+\mathcal{H},\mathcal{C}) = \mathcal{V}^*$.

Property 2. [2, 8] Let $\mathcal{H} \subseteq \mathcal{V}^*$ $(\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{B})$. Then, for any $\mathcal{V} \in \Phi(\mathcal{B}+\mathcal{H},\mathcal{C})$, $\mathcal{H} \subseteq \mathcal{V}$ $(\mathcal{H} \subseteq \mathcal{V} + \mathcal{B})$.

Lemma 1. [2, 8] Let $\mathcal{H} \subseteq \mathcal{V}^*$ $(\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{B})$. If the minimal $(A,\mathcal{B}+\mathcal{H})$-controlled invariant self-bounded with respect to $\mathcal{C}$, i. e. $\mathcal{V}_m = \mathcal{V}^* \cap \min \mathcal{S}(A,\mathcal{C},\mathcal{B}+\mathcal{H})$, is not internally stabilizable, no internally stabilizable $(A,\mathcal{B})$-controlled invariant $\mathcal{V}$ exists which satisfies both $\mathcal{V} \subseteq \mathcal{C}$ and $\mathcal{H} \subseteq \mathcal{V}$ $(\mathcal{H} \subseteq \mathcal{V} + \mathcal{B})$.

As aforementioned, Property 1, Property 2, and Lemma 1 are fundamental to prove the necessary and sufficient constructive condition for unaccessible signal decoupling with stability.

Problem 1. (Unaccessible signal decoupling with stability) Refer to Figure 1. Let Σ be ruled by (1,2) with $x(0) = 0$. Design a linear algebraic state feedback F such that $\sigma(A+BF) \subset \mathbb{C}^{\odot}$ and, for all admissible $h(t)$ $(t \geq 0)$, $y(t) = 0$ for all $t \geq 0$.

Theorem 1. [3, 4] Consider the system (1,2). Let (A,B) be stabilizable. Problem 1 is solvable if and only if

i) $\mathcal{H} \subseteq \mathcal{V}^*$;

ii) $\mathcal{V}_m$ is internally stabilizable.

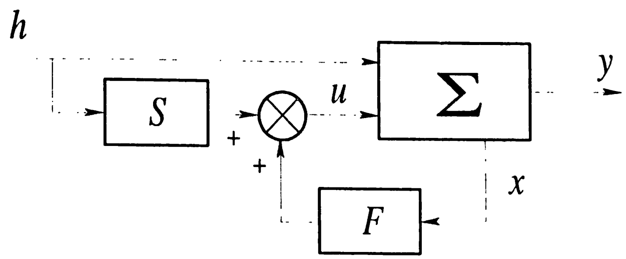

Fig. 2. Block diagram for measurable signal decoupling.

3. MEASURABLE SIGNAL DECOUPLING WITH STABILITY: MINIMAL–ORDER DYNAMIC FEEDFORWARD SOLUTION

The necessary and sufficient constructive condition for measurable signal decoupling with stability can be proved independently of that for unaccessible signal decoupling as suggested in [3]. However, in the following, an original proof is provided, where the measurable signal decoupling problem is reduced to an equivalent problem of unaccessible disturbance localization, thus fostering a unified view of the two issues.

Problem 2. (Measurable signal decoupling with stability) Refer to Figure 2. Let Σ be ruled by (1,2) with $x(0) = 0$. Design a linear algebraic state feedback F and a linear algebraic feedforward S of the measurable exogenous input h on the control input u such that $\sigma(A + BF) \subset \mathbb{C}^{\odot}$ and, for all admissible $h(t)$ $(t \geq 0)$, $y(t) = 0$ for all $t \geq 0$.

Theorem 2. Consider the system (1,2). Let (A, B) be stabilizable. Problem 2 is solvable if and only if

i) $\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{B}$;

ii) $\mathcal{V}_m$ is internally stabilizable.

Proof. If. If $\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{B}$, then $\mathcal{H}$ is decomposable as $\mathcal{H} = \mathcal{H}_{\mathcal{V}^*} + \mathcal{H}_{\mathcal{B}}$ with $\mathcal{H}_{\mathcal{V}^*} \subseteq \mathcal{V}^*$ and $\mathcal{H}_{\mathcal{B}} \subseteq \mathcal{B}$ as is shown below. Denote by V^* a basis matrix of $\mathcal{V}^*$ and consider B and H as basis matrices of $\mathcal{B}$ and $\mathcal{H}$, respectively. Then, the linear algebraic matrix equation $V^* X_1 + B X_2 = H$ admits at least one solution in X_1, X_2 for any given H. In particular, if $\ker [V^* \; B] \neq \{0\}$, the solution is parametrized in $\ker [V^* \; B]$ as

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} V^* & B \end{bmatrix}^+ H + \Omega \Gamma,$$

where Ω denotes a basis matrix of $\ker [V^* \; B]$ and Γ denotes any real matrix of appropriate dimensions. Consequently, $\mathcal{H}_{\mathcal{V}^*}$ and $\mathcal{H}_{\mathcal{B}}$ are respectively defined as

$\mathcal{H}_{\mathcal{V}^*} = \text{im}(V^* X_1)$ and $\mathcal{H}_{\mathcal{B}} = \text{im}(B X_2)$. Note that $\mathcal{H}_{\mathcal{V}^*}$ and $\mathcal{H}_{\mathcal{B}}$ depend on the possible parametrization and that their intersection may be different from the sole origin if $\mathcal{V}^* \cap \mathcal{B} \neq \{0\}$. Once $\mathcal{H}_{\mathcal{V}^*}$ and $\mathcal{H}_{\mathcal{B}}$ have been determined, the effect Hh, henceforth denoted by τ, of any $h \in \mathbb{R}^s$ is decomposable as $\tau = \tau_{\mathcal{V}^*} + \tau_{\mathcal{B}}$, with $\tau_{\mathcal{V}^*} \in \mathcal{H}_{\mathcal{V}^*}$ and $\tau_{\mathcal{B}} \in \mathcal{H}_{\mathcal{B}}$. Note that if $\mathcal{H}_{\mathcal{V}^*} \cap \mathcal{H}_{\mathcal{B}} \neq \{0\}$, all possible decompositions of τ differ from a given one, say $\tau = \tau^o_{\mathcal{V}^*} + \tau^o_{\mathcal{B}}$, in a vector belonging to $\mathcal{H}_{\mathcal{V}^*} \cap \mathcal{H}_{\mathcal{B}}$ (hence to $\mathcal{V}^* \cap \mathcal{B}$) to be subtracted from $\tau^o_{\mathcal{V}^*}$ and summed to $\tau^o_{\mathcal{B}}$. Therefore, the particular decomposition of τ does not affect the further discussion. The component $\tau_{\mathcal{B}}$ can be neutralized by a feedforward action Sh such that $BSh = -\tau_{\mathcal{B}}$, since $\tau_{\mathcal{B}}$ belongs to $\mathcal{B}$ and depends on h linearly. The component $\tau_{\mathcal{V}^*}$ can be managed as the effect of an unaccessible disturbance input since the subspace $\mathcal{H}_{\mathcal{V}^*}$, which it belongs to, satisfies both the conditions of Theorem 1. In fact, $\mathcal{H}_{\mathcal{V}^*} \subseteq \mathcal{V}^*$ by construction, and $\mathcal{V}^* \cap \min \mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_{\mathcal{V}^*})$ is internally stabilizable since

$$\begin{aligned}\mathcal{V}^* \cap \min \mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_{\mathcal{V}^*}) &= \mathcal{V}^* \cap \min \mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_{\mathcal{B}} + \mathcal{H}_{\mathcal{V}^*}) \\ &= \mathcal{V}^* \cap \min \mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}) \\ &= \mathcal{V}_m\end{aligned}$$

and $\mathcal{V}_m$ is internally stabilizable by assumption.

Only if. If Problem 2 is solvable, then the effect $\tau = Hh$ of any $h \in \mathbb{R}^s$ is decomposable as $\tau = \tau_{\mathcal{B}} + \tau_{\mathcal{V}}$, where the component $\tau_{\mathcal{B}}$ is neutralized by a feedforward action Sh such that $BSh = -\tau_{\mathcal{B}}$ and the component $\tau_{\mathcal{V}}$ is managed as the effect of an unaccessible disturbance, i. e. it is steered on an $(A + BF)$-invariant contained in $\mathcal{C}$, both internally and externally stable, henceforth denoted by $\mathcal{V}$. Since this is true for all $h \in \mathbb{R}^s$, the subspace $\mathcal{H}$ is decomposable as $\mathcal{H} = \mathcal{H}_{\mathcal{B}} + \mathcal{H}_{\mathcal{V}}$ with $\mathcal{H}_{\mathcal{B}} \subseteq \mathcal{B}$ and $\mathcal{H}_{\mathcal{V}} \subseteq \mathcal{V}$. This latter subspace, in particular, satisfies both the conditions of Theorem 1, i. e. $\mathcal{H}_{\mathcal{V}} \subseteq \mathcal{V}^*$ and $\mathcal{V}^* \cap \min \mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_{\mathcal{V}})$ internally stabilizable. The inclusion $\mathcal{H}_{\mathcal{V}} \subseteq \mathcal{V}^*$ implies $\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{B}$. Internal stabilizability of $\mathcal{V}^* \cap \min \mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_{\mathcal{V}})$ implies internal stabilizability of $\mathcal{V}_m$ since

$$\begin{aligned}\mathcal{V}^* \cap \min \mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_{\mathcal{V}}) &= \mathcal{V}^* \cap \min \mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_{\mathcal{B}} + \mathcal{H}_{\mathcal{V}}) \\ &= \mathcal{V}^* \cap \min \mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}) \\ &= \mathcal{V}_m. \qquad \square\end{aligned}$$

If Σ is stable, the action that, starting from the zero state, is performed by the linear algebraic feedback-feedforward regulator previously considered can also be obtained by means of a linear dynamic feedforward regulator Σ_c, initially assumed in the zero state as well. Let $\Sigma_c \equiv (A_c, B_c, C_c, D_c)$, it is trivial to show that the algebraic feedback-feedforward scheme in Figure 2 can be replaced by the dynamic feedforward one in Figure 3, by setting $A_c = A + BF$, $B_c = H + BS$, $C_c = F$, $D_c = S$. However, the main feature of the dynamic feedforward solution is that of allowing the separation between decoupling and possible stabilization problems, thus allowing the minimization of the regulator order on the basis of the properties of $\mathcal{V}_m$. Note that the hypothesis of stability of Σ, required to make the dynamic feedforward design feasible in the presence of unavoidable model uncertainties, does

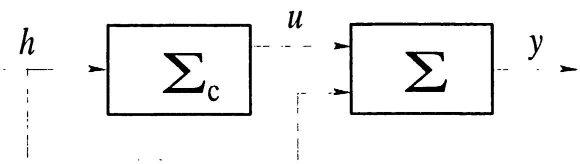

Fig. 3. Block diagram for feedforward measurable signal decoupling.

not cause any loss of generality with respect to the previous discussion. In fact, if (A, B) is stabilizable – as was assumed both in Theorem 1 and in Theorem 2 – Σ can be considered as pre-stabilized by an inner feedback.

Problem 3. (Measurable signal decoupling with stability by minimal-order linear dynamic feedforward) Refer to Figure 3. Let Σ be ruled by (1,2) with $x(0) = 0$. Let $\sigma(A) \subset \mathbb{C}^{\odot}$. Design a linear dynamic feedforward compensator $\Sigma_c \equiv (A_c, B_c, C_c, D_c)$ of minimal order, such that $\sigma(A_c) \subset \mathbb{C}^{\odot}$ and, for all admissible $h(t)$ $(t \geq 0)$, $y(t) = 0$ for all $t \geq 0$.

Lemma 2. Consider the system (1,2). Let $\mathcal{V}^* \cap \mathcal{B} = \{0\}$ and $\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{B}$. Let F be any real matrix such that $(A + BF)\,\mathcal{V}_m \subseteq \mathcal{V}_m$. Denote by V_m a basis matrix of $\mathcal{V}_m$ and consider B as a basis matrix of $\mathcal{B}$. Perform the similarity transformation $T := [T_1\ T_2\ T_3]$, with $T_1 = V_m$, $T_2 = B$. The matrices A'_F, B', H', C', respectively corresponding to $A + BF$, B, H, C in the new basis, partitioned according to T, have the structures

$$A'_F = \begin{bmatrix} A'_{11} & A'_{12} & A'_{13} \\ O & A'_{22} + B'_{21}F'_{12} & A'_{23} + B'_{21}F'_{13} \\ O & A'_{32} & A'_{33} \end{bmatrix}, \tag{3}$$

$$B' = \begin{bmatrix} O \\ B'_{21} \\ O \end{bmatrix}, \quad H' = \begin{bmatrix} H'_{11} \\ H'_{21} \\ O \end{bmatrix}, \tag{4}$$

$$C' = \begin{bmatrix} O & C'_{12} & C'_{13} \end{bmatrix}, \tag{5}$$

where $A'_{21} + B'_{21}F'_{11}$ has been set to zero by imposing

$$F'_{11} = -\,(B'_{21})^{-1}A'_{21}. \tag{6}$$

P r o o f. The structure of B' is implied by $\mathcal{V}_m \cap \mathcal{B} = \{0\}$, which ensues from $\mathcal{V}^* \cap \mathcal{B} = \{0\}$. The structure of H' depends on $\mathcal{H} \subseteq \mathcal{V}_m + \mathcal{B}$, which is implied by $\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{B}$ by virtue of Property 2. The structure of C' depends on $\mathcal{V}_m \subseteq \mathcal{C}$. The zero submatrices in the first column of A'_F are due to $(A+BF)$-invariance of $\mathcal{V}_m$. $\square$

Theorem 3. Consider the system (1,2). Let $\sigma(A) \subset \mathbb{C}^{\odot}$, $\mathcal{V}^{*} \cap \mathcal{B} = \{0\}$, $\mathcal{H} \subseteq \mathcal{V}^{*} + \mathcal{B}$. Then, $\Sigma_c \equiv (A_c, B_c, C_c, D_c)$ solves Problem 3 if and only if $A_c = A'_{11}$, $B_c = H'_{11}$, $C_c = F'_{11}$, $D_c = -H'_{21}$, where A'_{11}, H'_{11}, F'_{11}, H'_{21} are defined as in (3), (4) and (6), with F being any real matrix such that $(A + BF)\,\mathcal{V}_m \subseteq \mathcal{V}_m$ and $\sigma(A + BF) \subset \mathbb{C}^{\odot}$.

Proof. If. Let the matrices A'_{11}, H'_{11}, F'_{11}, H'_{21} be defined as in (3), (4) and (6), with F such that $(A + BF)\,\mathcal{V}_m \subseteq \mathcal{V}_m$ and $\sigma(A + BF) \subset \mathbb{C}^{\odot}$. Let Σ_c be ruled by

$$\begin{aligned} z(t+1) &= A'_{11}\, z(t) + H'_{11}\, h(t), \\ u(t) &= F'_{11}\, z(t) - H'_{21}\, h(t), \end{aligned}$$

with the initial condition $z(0) = 0$. First, it is shown that, for any admissible $h(t)$ $(t \geq 0)$, the corresponding state trajectory $x(t)$ $(t \geq 0)$, starting from $x(0) = 0$, lies on $\mathcal{V}_m$ since $x(t) = V_m z(t)$ for all $t \geq 0$. In fact, $x(0) = V_m z(0)$, due to the assumptions on the initial conditions. Moreover, for any $t \geq 0$, $x(t) = V_m z(t)$ implies

$$\begin{aligned} x(t+1) &= A\, x(t) + BF'_{11}\, z(t) + V_m H'_{11}\, h(t) \\ &= (A + BF)V_m + V_m H'_{11}\, h(t) \\ &= V_m A'_{11} z(t) + V_m H'_{11}\, h(t) \\ &= V_m z(t+1), \end{aligned}$$

where the relations $H = V_m H'_{11} + B H'_{21}$, $F'_{11} = F V_m$, and $(A+BF)V_m = V_m A'_{11}$ have been taken into account, ordinately. Then, stability of Σ_c is implied by $\sigma(A'_{11}) \subseteq \sigma(A + BF) \subset \mathbb{C}^{\odot}$. Finally, order minimality of Σ_c is implied by $A'_{11} = (A + BF)|_{\mathcal{V}_m}$ and minimality of $\mathcal{V}_m$ as an $(A, \mathcal{B})$-controlled invariant, self-bounded with respect to $\mathcal{C}$, such that $\mathcal{H} \subseteq \mathcal{V}_m + \mathcal{B}$.

Only if. Let $\Sigma_c \equiv (A_c, B_c, C_c, D_c)$ solve Problem 3. Then, for any admissible $h(t)$ $(t \geq 0)$, the corresponding state trajectory $x(t)$ $(t \geq 0)$, starting from $x(0) = 0$, is steered on an $(A + BF)$-invariant, say $\mathcal{V}$, both internally and externally stable, contained in $\mathcal{C}$, such that $\mathcal{H} \subseteq \mathcal{V} + \mathcal{B}$. Denote by V a basis matrix of $\mathcal{V}$ and consider B as a basis matrix of $\mathcal{B}$. Perform the similarity transformation $T := [\,T_1\ T_2\ T_3\,]$, with $T_1 = V$, $T_2 = B$. The matrices A'_F, B', H', C', respectively corresponding to $A + BF$, B, H, C in the new basis, partitioned according to T, have the same structures as those in (3), (4) and (5) and the respective submatrices will henceforth be denoted by the same symbols. Thus, for any admissible $h(t)$ $(t \geq 0)$, $z(t)$ $(t \geq 0)$ exists, such that the state equation (1) can also be written as

$$V\, z(t+1) = (A + BF)\, V\, z(t) + V\, H'_{11}\, h(t), \tag{7}$$

with the initial condition $z(0) = 0$. The control law $u(t) = F'_{11}\, z(t) - H'_{21}\, h(t)$ follows from the comparison of (7) with (1), by taking $F'_{11} = FV$ and $H = VH'_{11} + BH'_{21}$ into account. The regulator state equation $z(t+1) = A'_{11}\, z(t) + H'_{11}\, h(t)$ directly follows from (7) by considering that $(A + BF)V = VA'_{11}$. Note that the dynamic order of Σ_c is equal to the dimension of $\mathcal{V}$. Hence, the condition that Σ_c is of minimal order, implies $\mathcal{V} = \mathcal{V}_m$. Therefore, $\Sigma_c \equiv (A'_{11}, H'_{11}, F'_{11}, -H'_{21})$, where A'_{11}, H'_{11}, F'_{11}, H'_{21} are the same as those in (3), (4) and (6), since the similarity transformation T which must be considered is the same. □

In the light of the previous results, it is worth pointing out the advantages, made even more clear by the dynamic feedforward scheme, of the solution based on the use of $\mathcal{V}_m$, i.e. the minimal internally stabilizable $(A,\mathcal{B})$-controlled invariant self-bounded with respect to $\mathcal{C}$ such that $\mathcal{H} \subseteq \mathcal{V}_m + \mathcal{B}$, in comparison with the solution, often considered in the literature, based on the use of the maximal internally stabilizable $(A,\mathcal{B})$-controlled invariant contained in $\mathcal{C}$, usually denoted by $\mathcal{V}_g^*$. First, let us observe that, in general, since $\mathcal{R}_{\mathcal{V}^*} \subseteq \mathcal{V}_m \subseteq \mathcal{V}_g^*$, the subspaces $\mathcal{V}_m$ and $\mathcal{V}_g^*$ have the same number of internal assignable eigenvalues, while the number of the internal unassignable eigenvalues of $\mathcal{V}_g^*$ is greater than or equal to the number of the internal unassignable eigenvalues of $\mathcal{V}_m$, the difference being equal to $\dim \mathcal{V}_g^* - \dim \mathcal{V}_m$. Moreover, if $\mathcal{R} = \mathcal{X}$, the number of the external assignable eigenvalues of $\mathcal{V}_m$ is greater than or equal to the number of the external assignable eigenvalues of $\mathcal{V}_g^*$, the difference still being $\dim \mathcal{V}_g^* - \dim \mathcal{V}_m$. More precisely, the internal unassignable eigenvalues of $\mathcal{V}_g^*$ which are not internal unassignable eigenvalues of $\mathcal{V}_m$ are, indeed, external assignable eigenvalues of $\mathcal{V}_m$. Hence, the use of $\mathcal{V}_m$ in place of $\mathcal{V}_g^*$, not only guarantees that the dynamic feedforward unit which is synthesized is of minimal order if the system is left-invertible, but it also implies that the number of dynamics arbitrarily assignable by a possible inner feedback is maximal. Furthermore, from the numerical point of view, the evaluation of a basis matrix of $\mathcal{V}_g^*$ involves computation of eigenspaces, which is rather critical for systems of high order.

4. EXTENSION TO NON-LEFT-INVERTIBLE SYSTEMS

In Lemma 2 and Theorem 3, the system (1,2) was assumed to be left-invertible with respect to the control input. However, the design procedure introduced in the previous section is suitable for extension to non-left-invertible systems, provided that they are treated in the light of the following results.

Lemma 3. Consider the system (1,2). Let $\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{B}$. Let F be any real matrix such that $(A+BF)\mathcal{V}^* \subseteq \mathcal{V}^*$. Perform the similarity transformations $T := [\,T_1\ T_2\ T_3\ T_4\,]$, with $\operatorname{im} T_1 = \mathcal{R}_{\mathcal{V}^*}$, $\operatorname{im}\,[T_1\ T_2] = \mathcal{V}^*$, $\operatorname{im}\,[T_1\ T_3] = \mathcal{S}^*$, and $U := [\,U_1\ U_2\,]$, with $\operatorname{im} U_1 = B^{-1}\mathcal{V}^*$, $\operatorname{im} U_2 = \left(B^{-1}\mathcal{V}^*\right)^{\perp}$. The matrices A'_F, B', H', C', respectively corresponding to $A+BF$, B, H, C in the new bases, partitioned according to T and U, have the structures

$$A'_F = \begin{bmatrix} A'_{F11} & A'_{F12} & A'_{F13} & A'_{F14} \\ O & A'_{22} & A'_{23} & A'_{24} \\ O & O & A'_{F33} & A'_{F34} \\ O & O & A'_{43} & A'_{44} \end{bmatrix}, \tag{8}$$

$$B' = \begin{bmatrix} B'_{11} & B'_{12} \\ O & O \\ O & B'_{32} \\ O & O \end{bmatrix}, \qquad H' = \begin{bmatrix} H'_{11} \\ H'_{21} \\ H'_{31} \\ O \end{bmatrix}, \tag{9}$$

$$C' = [\ O \quad O \quad C'_{13} \quad C'_{14}\ ], \tag{10}$$

where $A'_{F1j} = A'_{1j} + B'_{11}F'_{1j} + B'_{12}F'_{2j}$, with $j = 1,2,3,4$, $A'_{F3j} = A'_{3j} + B'_{32}F'_{2j}$, with $j = 3,4$, and where $A'_{F3j} = A'_{3j} + B'_{32}F'_{2j}$, with $j = 1,2$, have been set to zero by imposing $F'_{2j} = -(B'_{32})^{+}A'_{3j}$ for $j = 1,2$, respectively.

Proof. The structure of B' is due to $\mathcal{B} \subseteq \mathcal{S}^*$ and $\mathcal{V}^* \cap \mathcal{B} \subseteq \mathcal{R}_{\mathcal{V}^*}$. The structure of H' is implied by $\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{B}$. The structure of C' is implied by $\mathcal{V}^* \subseteq \mathcal{C}$. The zero submatrices in the third and fourth row of A'_F are due to $(A + BF)$-invariance of $\mathcal{V}^*$. The zero submatrix in the second row of A'_F is due to $(A + BF)$-invariance of $\mathcal{R}_{\mathcal{V}^*}$ for any F such that $(A + BF)\,\mathcal{V}^* \subseteq \mathcal{V}^*$. □

Theorem 4. Consider the system (1,2). Let the triple (A, B, C) be non-left-invertible. Let F be any real matrix such that $(A + BF)\,\mathcal{V}^* \subseteq \mathcal{V}^*$. Let U_2 be a basis matrix of $\left(B^{-1}\mathcal{V}^*\right)^{\perp} \neq \{0\}$. Set $\tilde{A}{:=}A{+}BF$ and $\tilde{B}{:=}B\,U_2$. Then, $\mathcal{V}^* = \max \mathcal{V}(\tilde{A}, \tilde{B}, C)$ and the triple $(\tilde{A}, \tilde{B}, C)$ is left-invertible.

Proof. Consider the triple $(A + BF, B, C)$ and perform the similarity tranformations T and U defined as in Lemma 3. First, note that the matrix $\tilde{B}'{:=}B'U'_2$, corresponding to $\tilde{B}$ in the new bases, matches the second column of B', since $U'_2{:=}U^{-1}U_2 = [\,O\ I\,]^{\mathsf{T}}$. Also note that, in the new basis, $\mathcal{V}^* = \operatorname{im}[\,T'_1\ T'_2]$, with $T'_1 = [\,I\ O\ O\ O\,]^{\mathsf{T}}$ and $T'_2 = [\,O\ I\ O\ O\,]^{\mathsf{T}}$. Consequently, $\mathcal{V}^* \cap \tilde{\mathcal{B}} = \{0\}$ is simply derived by comparing the basis matrices of $\mathcal{V}^*$ and $\tilde{\mathcal{B}}$ in the new coordinates, being B'_{32} a full-rank matrix. On the other hand, $\mathcal{V}^*$, which is the maximal $(A, \mathcal{B})$-controlled invariant contained in $\mathcal{C}$, is also the maximal $(A + BF)$-invariant contained in $\mathcal{C}$. Hence, $\mathcal{V}^*$ is also the maximal $(A + BF, \tilde{\mathcal{B}})$-controlled invariant contained in $\mathcal{C}$, i. e. $\mathcal{V}^* = \max \mathcal{V}(\tilde{A}, \tilde{\mathcal{B}}, \mathcal{C})$, which completes the proof. □

Corollary 1. Consider the system (1,2). Let the triple (A, B, C) be non-left-invertible. Let F be any real matrix such that $(A + BF)\,\mathcal{V}^* \subseteq \mathcal{V}^*$. Let U_2 be a basis matrix of $\left(B^{-1}\mathcal{V}^*\right)^{\perp} \neq \{0\}$. Consider the triple $(\tilde{A}, \tilde{B}, C)$, with $\tilde{A}{:=}A{+}BF$ and $\tilde{B}{:=}B\,U_2$. Then,

$$\mathcal{Z}(\tilde{A}, \tilde{B}, C) = \sigma((A + BF)|_{\mathcal{R}_{\mathcal{V}^*}}) \uplus \mathcal{Z}(A, B, C).$$

Proof. The statement follows from Lemma 3 and Theorem 4, by considering that all the internal eigenvalues of $\max \mathcal{V}(\tilde{A}, \tilde{\mathcal{B}}, \mathcal{C})$ are unassignable. □

In view of Corollary 1, it is worth pointing out that the derivation of the left-invertible triple $(\tilde{A}, \tilde{B}, C)$ from the original triple (A, B, C) implies the arbitrary assignment of the eigenvalues internal to $\mathcal{R}_{\mathcal{V}^*}$.

Theorem 5. Consider the system (1,2). Let the triple (A, B, C) be non-left-invertible. Let F be any real matrix such that $(A + BF)\,\mathcal{V}^* \subseteq \mathcal{V}^*$. Let U_2 be a basis matrix of $\left(B^{-1}\mathcal{V}^*\right)^{\perp} \neq \{0\}$. Consider the system $\tilde{\Sigma}$, ruled by

$$\tilde{x}(t+1) = \tilde{A}\,\tilde{x}(t) + \tilde{B}\,\tilde{u}(t) + H\,h(t), \tag{11}$$

$$\tilde{y}(t) = C\,\tilde{x}(t), \tag{12}$$

with $\tilde{A}{:=}A{+}BF$ and $\tilde{B}{:=}B\,U_2$. If Problem 2 stated for system (1,2) is solvable, then Problem 2 stated for system (11, 12) is solvable.

Proof. Since $\mathcal{V}^* = \max\mathcal{V}(\tilde{A},\tilde{\mathcal{B}},\mathcal{C})$ by virtue of Theorem 4 and $\mathcal{V}^*+\mathcal{B} = \mathcal{V}^*+\tilde{\mathcal{B}}$ by definition of $\tilde{\mathcal{B}}$, the following equivalence holds

$$\mathcal{H} \subseteq \mathcal{V}^*+\mathcal{B} \Longleftrightarrow \mathcal{H} \subseteq \max\mathcal{V}(\tilde{A},\tilde{\mathcal{B}},\mathcal{C})+\tilde{\mathcal{B}}. \tag{13}$$

Let the inclusions in (13) hold. The subspace $\mathcal{V}_m$, which is the minimal $(A,\mathcal{B})$-controlled invariant self-bounded with respect to $\mathcal{C}$ such that $\mathcal{H} \subseteq \mathcal{V}_m+\mathcal{B}$, is also the minimal $(A+BF)$-invariant contained in $\mathcal{C}$ and containing $\mathcal{V}^*\cap\mathcal{B}$ such that $\mathcal{H} \subseteq \mathcal{V}_m+\mathcal{B}$. Hence, $\mathcal{V}_m$ is the minimal $(A+BF,\tilde{\mathcal{B}})$-controlled invariant contained in $\mathcal{C}$ and containing $\mathcal{V}^*\cap\mathcal{B}$ such that $\mathcal{H} \subseteq \mathcal{V}_m+\mathcal{B}$. By definition of $\tilde{\mathcal{B}}$ and by virtue of the inclusions $\mathcal{V}^*\cap\mathcal{B} \subseteq \mathcal{R}_{\mathcal{V}^*} \subseteq \mathcal{V}_m$, it follows that $\mathcal{V}_m+\mathcal{B} = \mathcal{V}_m+\tilde{\mathcal{B}}$, which, in turn, implies that $\mathcal{V}_m$ is the minimal $(\tilde{A},\tilde{\mathcal{B}})$-controlled invariant contained in $\mathcal{C}$ and containing $\mathcal{V}^*\cap\mathcal{B}$ such that $\mathcal{H} \subseteq \mathcal{V}_m+\tilde{\mathcal{B}}$. On the other hand, the subspace

$$\tilde{\mathcal{V}}_m = \max\mathcal{V}(\tilde{A},\tilde{\mathcal{B}},\mathcal{C}) \cap \min\mathcal{S}(\tilde{A},\mathcal{C},\tilde{\mathcal{B}}+\mathcal{H}) \tag{14}$$

is the minimal $(\tilde{A},\tilde{\mathcal{B}})$-controlled invariant contained in $\mathcal{C}$ such that $\mathcal{H} \subseteq \tilde{\mathcal{V}}_m+\tilde{\mathcal{B}}$. Consequently, the inclusion $\tilde{\mathcal{V}}_m \subseteq \mathcal{V}_m$ holds. Due to this latter inclusion, internal stabilizability of $\mathcal{V}_m$ implies internal stabilizability of $\tilde{\mathcal{V}}_m$.

Finally, the inclusion on the right-hand side of (13) and the internal stabilizability of $\tilde{\mathcal{V}}_m$ imply solvability of Problem 2 stated for system (11,12), by virtue of Theorem 2. □

Theorem 6. Let $\tilde{\Sigma}_c \equiv (\tilde{A}_c,\tilde{B}_c,\tilde{C}_c,\tilde{D}_c)$ solve Problem 3 stated for system (11,12). Then, the compensator Σ_c solving Problem 3 stated for the original system (1,2) is defined by the quadruple (A_c,B_c,C_c,D_c), where $A_c=\tilde{A}_c$, $B_c=\tilde{B}_c$, $C_c = FV_m + U_2\tilde{C}_c$, $D_c = U_2\tilde{D}_c$, with V_m denoting a basis matrix of $\mathcal{V}_m$, F any real matrix such that $(A+BF)\mathcal{V}^* \subseteq \mathcal{V}^*$, and U_2 a basis matrix of $\left(B^{-1}\mathcal{V}^*\right)^{\perp} \neq \{0\}$.

Proof. The state equations of the feedforward connection of Σ_c and Σ, shown in Figure 3, are

$$\left\{\begin{array}{l} x(t+1) = A\,x(t) + BC_c\,z(t) + BD_c\,h(t) + H\,h(t), \\ z(t+1) = A_c\,z(t) + B_c\,h(t), \end{array}\right. \tag{15}$$

with the initial conditions $x(0)=0$, $z(0)=0$. The state equations of the corresponding feedforward connection of $\tilde{\Sigma}_c$ and $\tilde{\Sigma}$ are

$$\left\{\begin{array}{l} \tilde{x}(t+1) = (A+BF)\,\tilde{x}(t) + BU_2\tilde{C}_c\,\tilde{z}(t) + BU_2\tilde{D}_c\,h(t) + H\,h(t), \\ \tilde{z}(t+1) = \tilde{A}_c\,\tilde{z}(t) + \tilde{B}_c\,h(t), \end{array}\right. \tag{16}$$

with the initial conditions $\tilde{x}(0)=0$, $\tilde{z}(0)=0$. Hence, the thesis follows by imposing $x(t)=\tilde{x}(t)$ and $z(t)=\tilde{z}(t)$ for all $t \geq 0$. □

5. CONCLUDING REMARKS

An original method to design a dynamic feedforward compensator achieving localization of measurable signals with stability has been presented. The structural properties of self-bounded controlled invariant subspaces have been exploited to obtain a feedforward controller with the minimum dynamic order and the minimum number of internal unassignable dynamics. Meanwhile, the number of dynamics arbitrarily assignable by a possible inner feedback is maximized. Left-invertible systems are considered first. Then an original, strictly geometric, squaring-down technique enables extension to non-left-invertible systems. All the algorithms involved in the design method are supported by appropriate software, based on the fundamental routines for the geometric approach.

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