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SOME REMARKS ON THE PROBLEM OF MODEL MATCHING BY STATE FEEDBACK

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The problem of model matching by state feedback is reconsidered and some of the latest results are discussed.

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1. INTRODUCTION

The problem of model matching represents a succinct abstract formulation of many control problems in which the central role plays the transmission properties of the system, that is to say, the modification of the transfer function is the core problem. As the *regular* static state feedback, which is defined below, forms a basic type of feedback, the discussion concentrates on model matching with this kind of feedback.

Consider a linear time-invariant system described by the equations

$$\dot{x} = Ax + Bu \tag{1}$$

$$y = Cx \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times l}$, $C \in \mathbb{R}^{p\times n}$ with $\operatorname{rank} B = l$ and $\operatorname{rank} C = p$. The system (1) and (2), called also the plant, is supposed to be controllable and observable and its transfer function,

$$T(s) = C(sI - A)^{-1}B \in \mathbb{R}_{sp}^{p\times l}, \tag{3}$$

is supposed to be of rank p (i. e. the system is supposed to be right invertible). Whenever convenient, the system (1) and (2) is also referred to as the triple (C, A, B), or $T(s)$.

As far as notation is concerned, some standard symbols like $:=$, $\mathbb{R}[s]$, and $\mathbb{R}(s)$ denoting the defining equality, the ring of polynomials over the field of real numbers $\mathbb{R}$, and its quotient field, respectively, and $\mathbb{R}_p(s)$ ($\mathbb{R}_{sp}(s)$) standing for the ring of proper (strictly proper) rational functions over $\mathbb{R}$, will frequently be used; some other symbols are defined throughout the text.

Let (C_m, A_m, B_m) be another system, called the model, that is also controllable, observable, right invertible, the dimension of which is $n_m \leq n$ (from now on all the symbols related to the model will have the index m), and gives rise to the transfer function $T_m(s) \in \mathbb{R}_{sp}^{p\times l}(s)$, i.e. $p_m = p$ and $l_m = l$. The problem of model matching then consists of finding a (regular) static state feedback

$$u = Fx + Gv, \tag{4}$$

where $F \in \mathbb{R}^{l\times n}$ and $G \in \mathbb{R}^{l\times l}$ with $\operatorname{rank} G = l$, such that the transfer function of the closed-loop system exactly matches that of the model, i.e.

$$T_m(s) = T_{F,G}(s) \tag{5}$$

where $T_{F,G}(s) := C(sI_n - A - BF)^{-1}BG$.

More generally, the equation (5) leads to studying the equation

$$T_m(s) = T(s)\,C(s) \tag{6}$$

where $C(s) \in \mathbb{R}_p^{l\times l}(s)$ is a compensator transfer function. If a certain type of feedback is used to achieve model matching, the compensator $C(s)$ has to be realizable by this type of feedback. In the case of state feedback (4), it follows that $C(s) = (I_l - F(sI_n - A)^{-1}B)^{-1}G$, which implies that $C(s)$ is a biproper matrix (a unit of the ring $\mathbb{R}_p^{l\times l}(s)$).

The literature concerning the model matching problem by different types of feedback is fairly rich. Most of the contributions however deals with dynamic compensation; see [5, 7, 10, 12, 15, 17] and the references therein. The problem of model matching by state feedback has been defined in [16] for the first time, where also necessary and sufficient conditions of its solvability can be found. In the same year, a solution based on Silvermann's inversion algorithm was established in [11]. Other necessary and sufficient conditions for the existence of a solution to the problem can be found in [5]. These conditions are stated in terms of finite and infinite zeros of the system; however, they are valid just in the case where the system transfer functions are nonsingular. In this paper we build upon the results given in [13, 19], where just necessary conditions of solvability are introduced, and provide necessary and sufficient conditions under which a solution to the model matching problem exists.

2. BACKGROUND

First some facts concerning the Morse invariants of (C, A, B) will be introduced. Consider the relationship

$$(C, A, B) \circ \Omega = (C', A', B'),$$

where $C' := HCT^{-1}$, $A' := T(A - BF - LC)T^{-1}$, and $B' := TBG$, describing the action of the Morse group upon the system (C, A, B). The quintuple

$\Omega := (H, T, F, L, G)$ is an element of the Morse group where the matrices T, G, and H are nonsingular and stand for similarity, input space, and output space transformations, respectively, while F represents state feedback and L output injection. Using transformations of this type the system (C, A, B) can be brought into the Morse canonical form [8] that is characterized by certain invariants. These invariants are known as the Morse invariants and correspond to the Kronecker invariants of the system matrix

$$\mathbb{P}(s) := \begin{bmatrix} sI_n - A & -B \\ -C & 0 \end{bmatrix}.$$

Generally, there are four kinds of the Kronecker invariants (invariant polynomials, row and column minimal indices, and infinite zero orders) that are, in the case of the Morse transformations acting on (C, A, B), reduced to infinite zero orders and column minimal indices of $\mathbb{P}(s)$.

There clearly exists a one-to-one correspondence between the aforementioned Morse invariants and some quantities characterizing $T(s)$. This comes from the fact that the matrices C, A, and B are given by a minimal realization of $T(s)$. For example, the infinite zero orders of $\mathbb{P}(s)$ and $T(s)$ are the same and can be obtained from the Smith–McMillan form of $T(s)$ at infinity and the column minimal indices of $\mathbb{P}(s)$ appear in the so-called *extended* interactor, the concept that is defined below.

Lemma 1. ([17]) Let $H(s) \in \mathbb{R}_{sp}^{p \times l}(s)$ be a right invertible matrix. Then there exists a unique matrix $\Phi(s) \in \mathbb{R}^{p \times p}[s]$, called the interactor of $H(s)$, such that

$$\Phi(s)\, H(s) = [I_p \;\, 0]\, B(s) \tag{7}$$

where $B(s)$ is a biproper matrix. The interactor $\Phi(s)$ is of the form

$$\Phi(s) = U_\Phi(s)\, \Lambda_f(s)$$

where $\Lambda_f(s) = \operatorname{diag}\{s^{f_i}\}_{i=1}^{p}$ with f_i being positive integers and

$$U_\Phi(s) = \begin{bmatrix} 1 & & & \\ \varphi_{21}(s) & 1 & & \\ \vdots & \ddots & \ddots & \\ \varphi_{p1}(s) & \dots & \varphi_{p,p-1}(s) & 1 \end{bmatrix}.$$

The polynomials $\varphi_{ij}(s)$ are divisible by s, or are equal to zero.

The relationship (7) shows that $[\Phi^{-1}(s), 0]$ is the Hermite form of $H(s)$ (the $\mathbb{R}_p(s)$ is considered now as a special case of the ring of generalized polynomials [12]). As the biproper matrices play, in the case of the ring $\mathbb{R}_p(s)$, the role of unimodular matrices (or units of the ring $\mathbb{R}_p^{l \times l}(s)$), it easily follows that the interactor is unchanged when $H(s)$ is postmultiplied by a biproper matrix. If the interactor $\Phi(s)$ is row reduced, it can be easily shown that the integers f_i are the infinite zero orders of $H(s)$, and that the row reducedness of $\Phi(s)$ can be achieved just by permuting the rows of $H(s)$; see [6].

The supremal output-nulling controllability subspace $\mathcal{R}^*$ contained in $\operatorname{Ker} C$ plays an important role in the problems like this one. This subspace is characterized by the column minimal (or $\mathcal{R}^*$-controllability) indices of $\mathbb{P}(s)$. To reveal them, we add $m-p$ new rows to the matrix C in such a way that the new matrix, say C_e, will be of rank l and the supremal controllability subspace of the system (C_e, A, B) contained in $\operatorname{Ker} C_e$ will be zero. Such a system (C_e, A, B) is called *the extended system* [3] and has the transfer function

$$T_e(s) := C_e(sI_n - A)^{-1}B.$$

The interactor $\Phi_e(s)$ of $T_e(s)$ is called the *extended interactor* and is of the form

$$\Phi_e(s) = \begin{bmatrix} \Phi_1(s) & 0 \\ \Phi_2(s) & \Phi_3(s) \end{bmatrix}$$

where $\Phi_1(s)$ stands for the interactor of $T(s)$, $\Phi_2(s)$ is a polynomial matrix whose entries $\phi_{ij}(s)$ have the properties stated in Lemma 1, and

$$\Phi_3(s) = \operatorname{diag}\,\{s^{\sigma_i}\}_{i=1}^{m-p}$$

with σ_i being the column minimal indices of $\mathbb{P}(s)$. The indices σ_i are supposed to be non-decreasingly ordered (and the indices $\sigma_{i,m}$ of the model as well).

In the sequel the following lemma will be useful.

Lemma 2. ([4]) Let $P(s) \in \mathbb{R}^{n\times m}[s]$, $m \le n$, and let $a(s)$ and $b(s)$ be polynomial vectors such that

$$b(s) = P(s)\,a(s).$$

Then $P(s)$ is column reduced if and only if

$$\deg b(s) = \max\{\deg_{ci} P(s) + \deg a_i(s),\ 1 \le i \le m\}.$$

Let now $N(s)$ and $D(s)$ be polynomial matrices that form a normalized matrix fraction description (n.m.f.d.) of $T(s)$, i. e.

$$T(s) = N(s)D^{-1}(s) \tag{8}$$

where $N(s)$, $D(s)$ are right coprime and $D(s)$ is column reduced with column degrees $c_1 \le c_2 \le \ldots \le c_m$. Let further $N_m(s)$ and $D_m(s)$ form a n.m.f.d. of $T_m(s)$ and let $C(s)$ be a state-feedback realizable compensator such that (6) holds. Then using a n.m.f.d. of $T(s)$ and a n.m.f.d. of $T_m(s)$, the relationship (6) can be rewritten in the form

$$\begin{bmatrix} N(s) \\ C^{-1}(s)D(s) \end{bmatrix} = \begin{bmatrix} N_m(s) \\ D_m(s) \end{bmatrix} X(s) \tag{9}$$

where $X(s)$ is a nonsingular polynomial matrix representing a greatest common right divisor of $N(s)$ and $C^{-1}(s)D(s)$. Notice that $C^{-1}(s)D(s) \in \mathbb{R}^{m\times m}[s]$ by assumption. Recall that this relationship describes a necessary and sufficient condition for the compensator $C(s)$ to be realizable with a (regular) static state feedback [2]. In fact the relationship (9) describes the result stated in [16], which is a starting point of our development.

To begin with, a special case of model matching that arises when $T_m(s)$ represents the feedback irreducible system (a closed-loop system $T_{FG}(s)$ having its McMillan degree minimal [1]) will be considered first. To enlighten this concept, consider the relationship (9) again. Applying a state feedback (4) to the system (1) and(2) may result in a zero cancellation between $N(s)$ and $C^{-1}(s)D(s)$. But this not all; another kind of cancellation caused by a non-trivial $\mathcal{R}^*$ of (C, A, B) may take place. To explain that, consider the matrix

$$K(s) := \begin{bmatrix} Q(s) & 0 \\ 0 & I_{m-p} \end{bmatrix} U(s), \tag{10}$$

where $Q(s) \in \mathbb{R}^{p\times p}[s]$ is nonsingular and $U(s)$ is a unimodular matrix given by the equation

$$N(s) = [Q(s)\ 0]\, U(s). \tag{11}$$

Then $K(s)$ and $D(s)$ form a n.m.f.d. of $T_e(s)$ [18].

Further, by Lemma 1,

$$\Phi_e(s)\, T_e(s) = B_e(s) \tag{12}$$

where $B_e(s)$ is a biproper matrix. Next, it follows, from (9), that

$$\begin{bmatrix} N(s) \\ B_e(s)D(s) \end{bmatrix} = \left[\begin{array}{cc} I_p & 0 \\ \hline \Phi_1(s) & 0 \\ \Phi_2(s) & I_{m-p} \end{array}\right] \Gamma(s) \tag{13}$$

with

$$\Gamma(s) := \begin{bmatrix} Q(s) & 0 \\ 0 & \Phi_3(s) \end{bmatrix} U(s).$$

Thus, applying the state feedback (F_Φ, G_Φ) given by $B_e(s)$ to (C, A, B) results in the feedback irreducible system , denoted by (C_Φ, A_Φ, B_Φ), that is a minimal realization of its transfer function $T_\Phi(s) = \Phi_1^{-1}(s)$. Moreover, the relationship (13) reveals all the cancellations that take place in the closed-loop system $(C, A + BF_\Phi, BG_\Phi)$. The matrix $Q(s)$ represents the (finite) pole-zero cancellation while $\Phi_3(s)$ corresponds to the second kind of cancellation. All that is summarized in the following

Proposition 1. Given $T(s)$ and $T_\Phi(s) := \Phi_1^{-1}(s)$, then there exists a state feedback (F_Φ, G_Φ) (given by $B_e(s)$) such that $T_\Phi(s) = T(s)\, B_e(s)$ and the McMillan degree of $T_\Phi(s)$ is the lowest achievable one; its value is given by the sum of the infinite zero orders of $T_\Phi(s)$.

3. MODEL MATCHING BY STATE FEEDBACK

It has been shown in [1] that the transfer functions $T_{F,G}(s)$ can be ordered with respect to their McMillan degrees, i. e.

$$\partial(T_\Phi(s)) \leq \partial(T_m) = \partial(T_{F,G}(s)) \leq \partial(T(s)).$$

The matter in question now is a characterization of all the transfer functions $T_{F,G}(s)$. To that end, write the relationship (12) in the form

$$D(s) = B_T^{-1}(s)\Phi_e(s)K(s) \tag{14}$$

and similarly, for the model,

$$D_m(s) = B_{Tm}^{-1}(s)\Phi_{e,m}(s)K_m(s) \tag{15}$$

and consider the relationship (9) where $C(s)$ represents a state-feedback realizable compensator. Substituting (14) and (15) into (9) gives

$$\begin{bmatrix} N(s) \\ B(s)\Phi_e(s)K(s) \end{bmatrix} = \begin{bmatrix} N_m(s) \\ \Phi_{e,m}(s)K_m(s) \end{bmatrix} X(s) \tag{16}$$

where $B(s) := B_{Tm}(s)\, C^{-1}(s)\, B_T^{-1}(s)$ is a biproper matrix that is state-feedback realizable. This can further be simplified using (10), (11), and (12) such that

$$[Q(s)\ 0] = [Q_m\ 0]Z(s) \tag{17}$$

and

$$B(s) \begin{bmatrix} \Phi_1(s)\, Q(s) & 0 \\ \Phi_2(s)\, Q(s) & \Phi_3(s) \end{bmatrix} = \begin{bmatrix} \Phi_{1,m}(s)\, Q(s) & 0 \\ \Phi_{2,m}(s)\, Q(s) & \Phi_{3,m}(s) \end{bmatrix} Z(s) \tag{18}$$

where $B(s)$ and $Z(s) := U_m(s)X(s)U^{-1}(s)$ are of the form

$$B(s) = \begin{bmatrix} B_{11}(s) & 0 \\ B_{21}(s) & B_{22}(s) \end{bmatrix}, \qquad Z(s) = \begin{bmatrix} Z_{11}(s) & 0 \\ Z_{21}(s) & Z_{22}(s) \end{bmatrix}.$$

Based on the relationships (17) and (18), necessary and sufficient conditions for the existence of a state feedback compensator $C(s)$ satisfying (6) can now be established.

Theorem 1. Let $T(s)$ and $T_m(s)$ be given transfer functions. Then there exists a state-feedback realizable compensator $C(s)$ such that $T_m(s) = T(s)\, C(s)$ if and only if

(a) the interactors of $T(s)$ and $T_m(s)$ are the same;

(b) the matrices $T_m(s)$ and $[T(s)\ T_m(s)]$ have the same finite zero structures;

(c) $\sigma_i \geq \sigma_{i,m}$ for $i = 1, 2, \dots, m - p$;

(d) There exist polynomial matrices $Z_{21}(s)$ and $Z_{22}(s)$ nonsingular such that

$$\deg_{ci} \Gamma(s)V(s) \leq \deg_{ci} \Phi_1(s)\, Q(s)V(s), \quad i = 1, 2, \dots, p, \tag{19}$$

where $\Gamma(s) := \Phi_{2m}(s)\, Q(s) - \Phi_{3m}(s)\, Z_{22}(s)\Phi_3^{-1}(s)\Phi_2(s)\, Q(s) + \Phi_{3m}(s)\, Z_{21}(s)$ and $V(s)$ is a unimodular matrix making the product $\Phi_1(s)\, Q(s)$ column reduced.

Proof. (Necessity). The claim (a) follows from the properties of the interactor; see Lemma 1. To prove (b), write $[T(s)\ T_m(s)]$ in the form

$$[T(s)\ T_m(s)] = [N(s)\ N_m(s)] \begin{bmatrix} D(s) & 0 \\ 0 & D_m(s) \end{bmatrix}^{-1},$$

which is a n.m.f.d. for $[T(s)\ T_m(s)]$. The finite zero structure of $[T(s)\ T_m(s)]$ is given by the greatest common left divisor of $N(s)$ and $N_m(s)$, which is the matrix $Q_m(s)$ in view of (17). To show that (c) holds, consider the equality

$$B_{22}(s)\Phi_3(s) = \Phi_{3,m}(s)\, Z_{22}(s) \tag{20}$$

where $B_{22}(s)$ is a biproper matrix and $Z_{22}(s)$ a nonsingular polynomial matrix. The following lemma gives an answer.

Lemma 3. Let $P(s), Q(s) \in \mathbb{R}^{n \times n}[s]$ be column reduced with column degrees $\alpha_1 \leq \alpha_2 \leq \dots \alpha_n$, $\beta_1 \leq \beta_2 \leq \dots \beta_n$, respectively. Then there exist a biproper matrix $V(s)$ and a polynomial matrix $Z(s)$ such that

$$V(s)\, P(s) = Q(s)\, Z(s) \tag{21}$$

if and only if $\alpha_i \geq \beta_i,\ i = 1, 2, \dots, n$.

Proof. As $V(s)$ is biproper, the product $V(s)\, P(s)$ is clearly column reduced with $\deg_{ci} V(s)\, P(s) = \alpha_i$, $i = 1, 2, \dots, n$. This means that the product $Q(s)\, Z(s)$ is column reduced, too, and has the column degrees α_i. Then, by Lemma 3,

$$\alpha_j = \max\{\beta_i + \deg\ z_{ij}(s), 1 \leq i \leq n\}$$

for $j = 1, 2, \dots, n$, which implies that $\alpha_j \geq \beta_j$, $j = 1, 2, \dots, n$.

To prove the sufficiency part, define

$$Z(s) = \operatorname{diag}\{s^{\alpha_i - \beta_i}\}_{i=1}^{n} \text{ and } V(s) := L(s)\, P^{-1}(s)$$

where $L(s)$ is a column reduced matrix with $\deg_{ci} = \alpha_i$, $i = 1, 2, \dots, n$. The matrix $V(s)$ is clearly biproper while the product $Q(s)\, Z(s)$ is column reduced with column degrees α_i. It follows that (21) holds. □

By definition, $\Phi_3(s)$ and $\Phi_{3,m}(s)$ are clearly column reduced with the column degrees σ_i and $\sigma_{i,m}$, respectively, which means that the inequalities (c) hold.

To prove (d), consider the equation

$$B_{21}(s)\Phi_1(s)\,Q(s) + B_{22}(s)\Phi_2(s)\,Q(s) = \Phi_{2,m}(s)\,Q(s) + \Phi_{3,m}(s)\,Z_{21}(s), \tag{22}$$

where $B_{21}(s)$ is proper rational, $B_{22}(s)$ biproper, and $Z_{21}(s)$ polynomial. Substituting now $\Phi_{3m}(s)\,Z_{22}(s)\Phi_3(s)$ for $B_{22}(s)$ and $F^{-1}(s)\,G(s)$ for $B_{21}(s)$, where the matrices $F(s), G(s)$ form a n.m.f.d. of $B_{21}(s)$, the relationship (22) can be written in the form

$$B_{21}(s) := F^{-1}(s)\,G(s) = \Gamma(s)\big[\Phi_1(s)\,Q(s)\big]^{-1}, \tag{23}$$

where $\Gamma(s)$ is defined in (d). As the matrix $B_{21}(s)$ is proper, it implies that

$$\deg_{ci}\Gamma(s) \le \deg_{ci}\Phi_1(s)\,Q(s), \quad i = 1, 2, \dots, p. \tag{24}$$

Postmultiplying the matrix $\begin{bmatrix} \Gamma(s) \\ \Phi_1(s)\,Q(s) \end{bmatrix}$ by the unimodular matrix $V(s)$ then gives (19).

(Sufficiency). To prove the sufficiency part, a biproper matrix B(s) and polynomial matrix Z(s) will be constructed such that the relationship (18) will hold. Notice first that the relationship (17) implies that $Z_{11}(s) = Q_m^{-1}(s)\,Q(s)$. Further, the equality $\Phi_1(s) = \Phi_{1m}(s)$ gives $B_{11} = I_m$. The rest of the proof follows from the assumption that there exist matrices $Z_{21}(s)$ and $Z_{22}(s)$ such that (20) and (19) hold. Then $B_{21}(s)$ is given by (23) and $B_{22}(s)$ can be computed from (20). □

The following corollary concerns a special case in which both extended interactors $\Phi_e(s)$ and $\Phi_{e,m}(s)$ are diagonal.

Corollary 1. Given a plant $T(s)$ and model $T_m(s)$ with the interactors $\Phi_1(s) = \operatorname{diag}\{s^{n_i}\}_{i=1}^{p}$ and $\Phi_{1,m}(s) = \operatorname{diag}\{s^{n_{i,m}}\}_{i=1}^{p}$ where both the integers n_i and $n_{i,m}$ are non-decreasingly ordered, and with the extended interactors $\Phi_e(s)$ and $\Phi_{em}(s)$ in which $\Phi_2(s) = 0$, $\Phi_{2,m}(s) = 0$, $\Phi_3(s) = \operatorname{diag}\{s^{\sigma_i}\}_{i=1}^{l-p}$, and $\Phi_{3,m}(s) = \operatorname{diag}\{s^{\sigma_{i,m}}\}_{i=1}^{l-p}$. Then there exists a state feedback (4) such that (6) holds if and only if

(α) $n_i = n_{i,m}$ for $i = 1, 2, \dots, p$,

(β) the matrices $T_m(s)$ and $[T(s)\ T_m(s)]$ have the same finite zero structures,

(γ) $\sigma_i \ge \sigma_{i,m}$ for $i = 1, 2, \dots, p$,

(δ) There exist a polynomial matrix $Z_{21}(s)$ and a proper rational matrix $B_{21}(s)$ such that

$$B_{21}(s)\Phi_1(s)\,Q(s) = \Phi_{3,m}(s)\,Z_{21}(s). \tag{25}$$

Another special case, in which necessary and sufficient conditions of its solvability are known, arises when both $T(s)$ and $T_m(s)$ are square and nonsingular

Corollary 2. Given nonsingular $T(s)$, $T_m(s) \in \mathbb{R}_{sp}^{l\times l}(s)$, there exists a state-feedback realizable compensator $C(s)$ such that (6) holds if and only if

(i) $\Phi(s) = \Phi_m(s)$,

(ii) $N(s) = N_m(s)X(s)$ for some nonsingular $X(s) \in \mathbb{R}^{l\times l}[s]$.

It is readily seen that the condition (ii) is just the condition (b) of Theorem 1. In other words, the conditions (a) and (b) of Theorem 1 are necessary and sufficient if $T(s)$ and $T_m(s)$ are nonsingular.

It should also be noted that the condition (i) and (ii) of Corollary 2 are equivalent to the conditions established in [5] that are stated as equality between finite and infinite zero structures of the matrices $T(s)$ and $[T(s), T_m(s)]$. It can be shown that this result is an easy consequence of Corollary 2 and subsequent

Lemma 4. Given nonsingular $T(s)$, $T_m(s) \in \mathbb{R}_{sp}^{l\times l}(s)$, then $\Phi(s) = \Phi_m(s)$ if and only if the infinite zero orders of the matrices $T(s)$ and $[T(s), T_m(s)]$ are the same.

4. DYNAMIC COMPENSATION

The general problem of model matching is described by the equation (6), that is,

$$T_m(s) = T(s)\,C(s) \tag{26}$$

where $T_m(s) \in \mathbb{R}_{sp}^{p\times q}$, $T(s) \in \mathbb{R}_{sp}^{p\times l}$, and $C(s) \in \mathbb{R}_{p}^{l\times q}$. More precisely, given a plant $T(s)$ of rank p and a full rank model $T_m(s)$, the problem is to find a compensator $C(s)$ of rank q such that (26) holds. Such a compensator is called admissible.

The equation (26) is a system of equations over the ring $\mathbb{R}_p(s)$, which implies that an admissible compensator exists if $q \le p$.

Theorem 2. ([17]) Given $T(s)$ and $T_m(s)$ (having the above stated propeties) with $q \le p$, then there exists an admissible compensator $C(s)$ satisfying (26) if and only if $\Phi(s)\Phi_m^{-1}(s) \in \mathbb{R}_p^{p\times p}(s)$ where $\Phi(s)$ and $\Phi_m(s)$ stand for the interactors of $T(s)$ and $T_m(s)$, respectively.

A special case arises when the compensator is a feedback compensator, like state feedback (4). In practise, the dynamic output feedback

$$u(s) = K(s)y(s) + v(s), \quad K(s) \in \mathbb{R}_p^{l\times p}(s) \tag{27}$$

is widely used, which leads to the biproper compensator $C(s) = [I_l + K(s)T(s)]^{-1} \in \mathbb{R}_p^{l\times l}(s)$.

There are many reasons for which we prefere a feedback realization of a given compensator. For instance, feedback is easier to implement and enables us to realize more tradeoffs between conflicting performance requirements. However, the question under which conditions is the compensator $C(s)$ realizable with a certain type of feedback has not been completely solved yet. Just some partial results are available.

Theorem 3. ([17]) Given a system $T(s) \in \mathbb{R}_p^{p\times l}(s)$ of rank p and a compensator $C(s) \in \mathbb{R}_p^{l\times l}(s)$ with $\operatorname{rank} C(s) = l$, then there exists a dynamic state feedback

$$u(s) = F(s)x(s) + Gv(s), \tag{28}$$

with $F(s) \in \mathbb{R}_p^{l\times n}(s)$ and $G \in \mathbb{R}^{l\times l}$ being nonsingular, such that

$$C(s) = [I_l - F(s)N(s)D^{-1}(s)]^{-1}$$

where $N(s)$ and $D(s)$ form an n.m.f.d. of $(sI_n - A)^{-1}B$, if and only if $C(s)$ is a biproper matrix.

A direct consequence of the above theorem is the condition under which $C(s)$ is realizable by a static state feedback (4).

Corollary 3. ([2]) Using the same notation as in Theorem 3, the compensator $C(s)$ is realizable by a static state feedback (4) if and only if $C(s)$ is biproper and the product $C^{-1}(s)D(s)$ is a polynomial matrix.

As far as the issue of stability is concerned, the problem can be formulated as follows. Find, for a given plant $T(s)$ and a stable model $T_m(s)$, an admissible compensator $C(s)$ such that (26) holds and internal stability, which means that no cancellation of unstable poles and zeros in the product $T(s)\,C(s)$ will occur, is ensured.

One way to tackle the problem lies in prestabilizing the plant $T(s)$ by a state feedback (4). This can always be done so that there is no loss of generality if the plant $T(s)$ is assumed to be an element of $\mathbb{R}_{ps}^{p\times l}(s)$. Then the equation (26) can be viewed as an equation over the ring of proper and stable rational functions $\mathbb{R}_{ps}(s)$, that is to say, $C(s)$ is defined over $\mathbb{R}_{ps}(s)$, too. Mathematically speaking, the problems of model matching and model matching with stability are very similar (as are the properties of $\mathbb{R}_p(s)$ and $\mathbb{R}_p(s)$). From control theoretical point of view it means that the unstable zeros of $T(s)$ has to be kept unchanged to preserve the internal stability.

5. CONCLUSIONS

The problem of exact model matching by different types of feedback has been discussed and some open questions related to this problem have been pointed out. It is believed that further investigation of the problem will give more insight into the structure of linear control systems and help in understanding the properties of basic control laws.

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