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# SOME REMARKS ON MATRIX PENCIL COMPLETION PROBLEMS

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The matrix pencil completion problem introduced in Loiseau et al. [12] is reconsidered and the latest results achieved in that field are discussed.

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## 1. INTRODUCTION

A challenge concerning the Kronecker invariants assignment to a matrix pencil that is completed by rows or columns has been introduced in [12]. This problem, called the matrix pencil completion problem therein, covers many questions of algebra and control theory, especially those describing the situations in which state feedback is used for altering the system dynamics. Some particular cases illustrating this point will be mentioned below.

The aim of the paper is to discuss some results achieved recently and complete the picture of what has been done by some new results, and thus provide the reader with deeper insight into this very interesting problem.

The notation used in the paper is standard; the basic symbols are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}[s]$ that denote the fields of real numbers, complex numbers, and the ring of polynomials (of variable $s$) over $\mathbb{R}$, respectively. Other symbols will be introduced in the text at the place where they are needed.

### 1.1. Kronecker canonical form

*Two matrix pencils $sE_1 - H_1$ and $sE_2 - H_2$, where $E_1, H_1, E_2$, and $H_2$ are $r \times c$ matrices, are said to be strictly pencil equivalent (s.p.e.), or just equivalent if it is clear from the context that the strict pencil equivalence is meant, if there exist nonsingular matrices $Q$ and $P$ such that*

$$sE_1 - H_1 = P[sE_2 - H_2]Q$$

The strict pencil equivalence, denoted by $\sim$, defines an equivalence relation on the set of matrix pencils and the canonical form under this equivalence is the well-known Kronecker canonical form $sE_K - H_K$ [3, 8] that consists of the blocks of the following forms:

$$(1) \begin{bmatrix} s-a_j & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & s-a_j \end{bmatrix} \in \mathbb{R}^{k_{ij} \times k_{ij}} \qquad (2) \begin{bmatrix} 1 & s & & \\ & \ddots & \ddots & \\ & & \ddots & s \\ & & & 1 \end{bmatrix} \in \mathbb{R}^{(n_i+1)\times(n_i+1)}$$

$$(3) \begin{bmatrix} s & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & s & 1 \end{bmatrix} \in \mathbb{R}^{c_i \times (c_i+1)} \qquad (4) \begin{bmatrix} 1 & & & \\ s & \ddots & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & s \end{bmatrix} \in \mathbb{R}^{(r_i+1)\times r_i}$$

where the integers $k_{ij} > 0$, $n_i \geq 0$, $c_i \geq 0$, $r_i \geq 0$ and $a_j \in \mathbb{C}$ is called a finite zero of $sE - H$. The case $c_i = 0$ ($r_i = 0$) for some $i$'s means that there are zero columns (rows) in $sE_K - H_K$. Very frequently the above blocks will also be referred to as $k_{ij}$-, $n_i$-, $c_i$-, and $r_i$-blocks.

Associated with these blocks there are four types of invariants, *the Kronecker invariants*, which are defined by the above blocks, namely

(1) finite elementary divisors (f.e.d.) represented by $k_{ij}$-blocks, i. e. by the integers $k_{ij}$ and complex numbers $a_j$,

(2) infinite elementary divisors (i.e.d.) represented by $n_i$-blocks, i. e. by the integers $n_i$,

(3) column minimal indices (c.m.i.) given by the integers $c_i$,

(4) and row minimal indices (r.m.i.) given by the integers $r_i$.

More features and details concerning the Kronecker canonical form can be found for instance in [3]. It should be noted that the integers $n_i$ are called *the infinite zero orders* in linear system theory and that f.e.d. of a pencil uniquely determine the invariant polynomials of the pencil.

For a given $r \times c$ pencil $sE - H$ there exists another special form, which could be called a standard (or system) form since it reminds of the system matrix of a linear system [20]. Denoting $n := \operatorname{rank} E$ then

$$sE - H \sim sE_S - H_S := \begin{bmatrix} sI_n - A & -B \\ -C & -D \end{bmatrix}$$

where the number of rows of $[-C \;\; -D]$ is $k$, $k + n = r$, while $t$, $n + t = c$, is the number of columns of $[-B^T \;\; -D^T]^T$. Such a form can be achieved, for example, by applying the SVD (singular value decomposition) algorithm to the matrix $E$. Hence, any pencil $sE - H$ defines a corresponding linear time-invariant system, defined by the quadruple $(A, B, C, D)$, governed by the equations

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du. \end{aligned}$$

Following [20], the pencil $sE_S - H_S$ will be referred to as *the system matrix.*

It is further observed that the matrices $(A, B, C, D)$ are of particularly simple forms (having the least number of parameters) if $sE - H$ is already in the Kronecker form. The system $(A, B, C, D)$ is then in the Morse form, [19].

**Remark 1.** Many features of the matrices $A, B, C, D$, and $E$ can be stated in terms of the Kronecker invariants of $sE - H$. The claims below immediately follow, by inspection, from the Kronecker canonical form.

- $n := \operatorname{rank} E = \sum_{i,j} k_{ij} + \sum n_i + \sum c_i + \sum r_i$,
- $sE - H$ is right invertible $\iff$ $sE_K - H_K$ has no $r_i$-blocks,

  $\operatorname{rank} E = r \iff$ there are no $n_i$-blocks and $r_i$-blocks in $sE_K - H_K$,
- $[C \; D] = 0 \iff$ there are no $n_i$-blocks and $r_i = 0 \; \forall i$.

**Remark 2.** The column minimal indices of $sE - H$ are the c.m.i. of the pencil $[sI_n - A \;\; -B]$, or – as we shall also say – of the pair $(A, B)$. They are also called the controllability indices of $(A, B)$ and can be obtained from a normal external description (n.e.d.) of $(A, B)$, which is defined below.

Let $N(s), D(s)$ be polynomial matrices such that

- $\begin{bmatrix} sI_n - A & -B \end{bmatrix} \begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = 0$,
- $\Pi(sI_n - A)N(s) = 0$ with $\Pi$ being the maximal left annihilator of $B$ (i.e. the rows of $\Pi$ form a basis for the left kernel of $B$),
- $D(s)$ is column reduced ($D(s) = D_{hc} \operatorname{diag}\{s^{\kappa_i}\}+$ terms of lower degrees where $D_{hc}$ is of full rank; see for instance [7, 23] for detail).

Such matrices $N(s)$, $D(s)$ are said to form *a normal (right) external description* (n.e.d.) of $(A, B)$ [26] and the column degrees, $\kappa_i$, of $D(s)$ are equal to $c_i$.

Analogously, the r.m.i., $r_i$, of $\begin{bmatrix} sI_n - A \\ -C \end{bmatrix}$ are the column degrees of an n.e.d. of $(A^T, C^T)$ and are called the observability indices of the pair $(C, A)$.

### 1.2. Matrix pencil completion problem

The $r \times c$ pencil $sE' - H'$ is said to be *a subpencil* of a given $(r+l) \times (c+q)$ $(l, q \geq 0)$ pencil $sE - H$ if

$$sE - H \sim \begin{bmatrix} sE' - H' & \star \\ \star & \star \end{bmatrix}, \tag{1}$$

where $\star$'s stand for unspecified pencils of compatible dimensions.

It is of interest to study the relationships between the pencil $sE - H$ and its subpencil $sE' - H'$. Particularly interesting is the question under which conditions a given pencil $sE' - H'$ can be completed by some other pencils such that the relationship (1) holds, that is to say, the pencil $sE-H$ will have prescribed Kronecker invariants. This problem is known as *the matrix pencil completion problem*; see [12] for detail.

It has already been noted that the formulation of the matrix pencil completion problem was motivated by some control-theoretical questions. As an illustration, consider the problem of invariant polynomials assignment, which may be viewed as one of the basic problems of linear control theory.

**Example 1.** Let a linear time-invariant system $(A, B)$,

$$\dot{x} = Ax + Bu, \quad A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \tag{2}$$

with the state feedback

$$u = Fx + v, \quad F \in \mathbb{R}^{m\times n}, \tag{3}$$

around be given. This gives a closed-loop system $(A + BF, B)$ governed by

$$\dot{x} = (A + BF)x + Bv. \tag{4}$$

The only difference between the equations (2) and (4) is that the matrix $A$ is replaced by $A+BF$. And as the state trajectory of (2) is given in terms of the eigenstructure (a synonym for the eigenvalue structure given by the Jordan form of $A$ and the structure of its right and left eigenvectors including the generalized ones) of the matrix $A$ – see [7] for instance, the relationship (4) shows that the state feedback (3) will be a powerful tool when altering the behaviour of the system (2). Therefore the question to what extent the eigenstructure of $A + BF$ can be changed by $F$ is one of the fundamental questions of control theory. It will now be shown how this question is expressed in terms of the matrix pencil completion problem.

Notice first that $[sI_n - A - BF, -B] \sim [sI_n - A, -B]$ and let $\Pi$ denote the maximal left annihilator of $B$. Then

$$\Pi[sI_n - A - BF] = \Pi[sI_n - A],$$

which implies that the pencil $\Pi[sI_n - A]$ can be completed by rows (by a pencil denoted by $\star$) such that

$$sI_n - A - BF \sim \begin{bmatrix} \Pi(sI_n - A) \\ \star \end{bmatrix}, \tag{5}$$

which is a partial case of the matrix pencil completion problem (1).

The question under what conditions the relationship (5) holds is answered in [20] in the case of controllable systems, and a complete solution, when the system (2) is possibly uncontrollable, can be found in [24].

The relationship (1) implies that any study of the matrix pencil completion problem will involve the Kronecker invariants of $sE' - H'$ and those of $sE - H$, i. e. eight lists of invariants. But there exists a trick using which the number of these lists can be lowered. The trick lies in applying the conformal mapping

$$s = \frac{(1 + aw)}{w}, \tag{6}$$

where $a$ is not a zero of both pencils, to these pencils. The mapping shifts the pencil infinite zeros in the location 0, while keeping all other finite zeros in finite positions. In this way the problem with finite and infinite elementary divisors is reduced to that with finite elementary divisors only, see [16, 26] for detail. However, despite this simplification the problem still remains very complex and difficult.

## 2. COLUMN COMPLETION OF RIGHT INVERTIBLE PENCILS

For the time being the most advanced results concerning the matrix completion problem are established in the case of right invertible pencils (see Remark 1) where just column completion is considered. More precisely, given right invertible pencils $sE - H$ and $sE' - H'$, the pencil $sE' - H'$ is to be completed in such a way that

$$sE - H \sim [sE' - H',\ \star\,]. \tag{7}$$

In the light of the conformal mapping introduced above, it can be seen that conditions under which there exists a solution to this problem will be based on the solvability conditions for the pencils without infinite elementary divisors. Such conditions will just comprise the c.m.i. and f.e.d. of transformed pencils and this will enable us to derive conditions for the original pencils.

### 2.1. Conditions for pencils of the form $[sI_n - A, -B]$

It is natural to start our discussion with the results established in [1] for pencils of the form $[sI_n - A,\ -B]$ since these pencils are clearly right invertible, without i.e.d., and therefore described by f.e.d. and c.m.i. only. Thus, let $[sI_n - A, -B]$ and $[sI_n - A', -B']$, where $A, A' \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times(m+q)}$ with $\operatorname{rank} B = m + q$, and $B' \in \mathbb{R}^{n\times m}$ with $\operatorname{rank} B' = m$, be given. It easily follows from the form of the pencils that the pencil $[sI_n - A', -B']$ can be completed just by constant (containing real numbers only) columns.

**Remark 3.** The reader familiar with linear control theory can recognize in this case an application of the nonregular state feedback (a state feedback described by $u = Fx + Gv$ with $G \in \mathbb{R}^{(m+q)\times m}$, $\operatorname{rank} G = m$) to the system (2). More facts on the use of nonregular state feedback in linear control can be found for instance in [18] and references therein. In this terminology, $A' = A + BF$ and $B' = BG$.

Let further $c_1 \geq c_2 \geq \cdots \geq c_{m+q}$, $c'_1 \geq c'_2 \geq \cdots \geq c'_m$ and $\alpha_1(s), \alpha_2(s), \cdots, \alpha_n(s)$, $\alpha'_1(s)$, $\alpha'_2(s), \cdots, \alpha'_n(s)$ denote the c.m.i. and invariant polynomials of $[sI_n - A, -B]$ and $[sI_n - A', -B']$, respectively. It is also assumed that the invariant polynomials are *non-increasingly* ordered, i.e. $\alpha_n(s)| \cdots |\alpha_2(s)|\alpha_1(s)$, where $\alpha_{i+1}(s)|\alpha_i(s)$ means that $\alpha_{i+1}(s)$ divides $\alpha_i(s)$ (similarly for the polynomials $\alpha'_i(s)$), and that $N(s), D(s)$ and $N'(s), D'(s)$ stand for normal external descriptions of $(A, B)$ and $(A', B')$.

With this notation (and that introduced above) we can now introduce the following four formulations of the matrix pencil completion problem. On the basis of these formulations the results known until know will be presented, which is a starting point for further considerations.

**Proposition 1.** Given pencils $[sI_n - A, -B]$ and $[sI_n - A', -B']$ having $c_i$, $\alpha_i(s)$ and $c'_j$, $\alpha'_j(s)$ as their column minimal indices and invariant polynomials, respectively, the following statements are equivalent.

(a) There exist an $(m+q) \times n$ matrix $F$ and an $(m+q) \times m$ matrix $G$, $\operatorname{rank} G = m$, such that $c'_1, c'_2, \cdots, c'_m$ and $\alpha'_1(s), \alpha'_2(s), \cdots, \alpha'_n(s)$ are the c.m.i. and invariant polynomials of the pencil $[sI_n - A - BF, -BG]$.

(b) There exist an $n \times q$ matrix over $\mathbb{R}$, denoted by $\star$, such that

$$[sI_n - A, -B] \;\sim\; [sI_n - A', -B', \;\star]$$

(c) There exist an integer $k$ and polynomial matrices $W(s) \in \mathbb{R}^{m\times q}[s]$, $X(s) \in \mathbb{R}^{q\times q}[s]$, $Y(s) \in \mathbb{R}^{q\times k}[s]$, and $Z(s) \in \mathbb{R}^{k\times k}[s]$ with invariant factors $\alpha_1(s), \alpha_2(s), \ldots, \alpha_k(s)$ such that

(1) the matrix

$$\begin{bmatrix} D'(s) & W(s) \\ 0 & X(s) \end{bmatrix}, \tag{8}$$

when column reduced, has column degrees $c_1, c_2, \cdots, c_{m+q}$,

(2) and the matrix

$$\begin{bmatrix} X(s) & Y(s) \\ 0 & Z(s) \end{bmatrix} \tag{9}$$

has invariant polynomials $\alpha'_1(s), \alpha'_2(s), \cdots, \alpha'_{k+q}(s)$.

(d) There exists an $q \times n$ matrix pencil, denoted again by $\star$, such that

$$\begin{bmatrix} \star \\ \Pi(sI_n - A) \end{bmatrix} \sim \Pi'[sI_n - A'] \tag{10}$$

where $\Pi$ and $\Pi'$ denote the maximal annihilators of $B$ and $B'$, respectively.

Proof of Proposition 1. The assertion (a) means exactly that $[sI_n - A - BF, -BG] \sim [sI_n - A', -B']$. Since $G$ is of full column rank, there exists an invertible matrix, say $H \in \mathbb{R}^{(m+q)\times(m+q)}$, such that $G = H[I_m, 0_{q\times m}]^T$. Then it can readily be verified that

$$[sI_n - A, B] \sim [sI_n - A - BF, -BG, \star] \sim [sI_n - A', -B', \star]$$

and reversely, which establishes the equivalence between (a) and (b).

Similarly, if $BG = B'$, then there exists a matrix $P$ of full row rank such that $\Pi = P\Pi'$, which implies

$$P\Pi'[sI_n - A'] = \Pi[sI_n - A - BF] = \Pi[sI_n - A].$$

Further, since $P$ is of full row rank, there exists an invertible matrix $Q \in \mathbb{R}^{(n-m)\times(n-m)}$ such that $P = [0_{(n-m-q)\times q}, I_{n-m-q}]Q$ and (d) follows.

Conversely, if there exist invertible matrices $Q \in \mathbb{R}^{(n-m)\times(n-m)}$ and $T \in \mathbb{R}^{n\times n}$ such that

$$\begin{bmatrix} \star \\ \Pi(sI_n - A) \end{bmatrix} = Q\Pi'[sI_n - A']T$$

for some matrix pencil $\star$, then $\Pi = [0_{(n-m-q)\times q}, I_{n-m-q}]Q\Pi' T$, which implies that there exists a matrix $G \in \mathbb{R}^{(m+q)\times m}$, $\operatorname{rank} G = m$, such that $B' = TBG$ and

$$\Pi A = [0_q, I_{n-m-q}]Q\Pi' A' T = \Pi T^{-1} A' T.$$

This gives that $A' = T[A + BF]T^{-1}$, $F \in \mathbb{R}^{(m+q)\times n}$, and the equivalence between (a) and (d) follows.

Finally, the equivalence between (a) and (c) is proved in [15]. □

Each of the above formulations has some strong points that suggest how the problem could be approached. At the first glance the statement (c) seems to be most useful. Indeed, it reveals that the whole problem consists of the subproblems (c1) and (c2) which are mutually related since the same matrix $X(s)$ appears in (8) as well as in (9). We shall first pay attention to the subproblem (c1). In terms of matrix pencils this completion problem was studied in [1], where necessary and sufficient conditions of solvability were established, and then (later on and using the polynomial matrix approach) was reconsidered in [14].

Here, alternative conditions derived from those in [5] are presented. These conditions are in certain sense simpler than those established in [1, 16] and moreover provide a natural generalization of Rosenbrock's and Heymann's results on the invariant polynomials and controllability indices assignment; see Remark 4 below.

**Lemma 1.** When $k = 0$, the problems defined in Proposition 1 are solvable if and only if

$$\alpha_i(s) = 1 \quad \text{for } i = 1, 2, \dots, n, \tag{11}$$

$$\sum_{j \,|\, c'_j \leq i} c'_j \leq \sum_{j \,|\, c_j \leq i} c_j, \quad i = 1, 2, \dots, n, \tag{12}$$

$$\sum_{j=1}^{i} \delta_j \geq \sum_{j=1}^{i} c_j, \quad i = 1, 2, \dots, m+q, \tag{13}$$

where equality holds for $i = m+q$ and $\delta_1 \geq \delta_2 \geq \dots \geq \delta_{m+q}$ is the non-increasingly ordered list $\left\{\deg \alpha'_j(s)\right\}_n \cup \{c'_i\}_m$.

For the proof of Lemma 1 see [5].

**Remark 4.** It is worth pointing out that the solvability conditions (12) and (13) reduce to simpler forms in some interesting particular cases. For instance, if $m = 0$, the condition (12) vanishes and (13) becomes

$$\sum_{j=1}^{i} \deg \alpha'_j(s) \geq \sum_{j=1}^{i} c_j, \quad i = 1, 2, \dots, n \tag{14}$$

where equality holds for $j = n$ and, by convention, $c_i = 0$ for $i > q$. These conditions are well-known (see e. g. [20]) and have been discussed many times in the control literature; see for example [9, 10, 24, 26] and references therein.

Next, if the list $\{\alpha'_i(s)\}_q$ is not specified, the condition (13) reduces to

$$\sum_{j=1}^{i} c'_j \geq \sum_{j=1}^{i} c_j \ , \ i = 1, \dots, m+q$$

These conditions are implied by (12) and are necessary and sufficient for this particular case. They were given in [6] for the controllability indices assignment by non-regular state feedback; see also [1, 5, 10, 13, 18] and references therein.

The second subproblem gives rise to another kind of completion problems that is considered in [21, 22].

**Lemma 2.** Let $Z(s) \in \mathbb{R}^{k \times k}[s]$ be as in Proposition 1, i. e. with the invariant polynomials $\alpha_1(s), \alpha_2(s), \cdots, \alpha_k(s)$. Then there exist matrices $X(s)$ and $Y(s)$, as in Proposition 1, such that the matrix (8) has $\alpha'_1(s), \alpha'_2(s), \cdots, \alpha'_{k+q}(s)$ as its invariant polynomials if and only if

$$\alpha'_{i+q}(s) \,|\, \alpha_i(s) \,|\, \alpha'_i(s), \quad i = 1, 2, \dots, k \tag{15}$$

The only problem now is whether the conditions stated in Lemma 1 and Lemma 2 can be tied together. This query is answered in [15].

**Lemma 3.** If there exists a solution to the problems of Proposition 1, then the invariant polynomials of the matrix $X(s)$, say $\phi_1(s), \phi_2(s), \ldots, \phi_q(s)$, satisfy the condition

$$\prod_{i=1}^{j} \sigma_i(s) \text{ is divided by } \prod_{i=1}^{j} \phi_i(s), \quad j = 1, 2, \ldots, q \tag{16}$$

with equality for $j = q$, and where

$$\sigma_i(s) = \frac{\beta_1^i(s)\beta_2^i(s)\ldots\beta_{n+i}^i(s)}{\beta_1^{i-1}(s)\beta_2^{i-1}(s)\ldots\beta_{n+i-1}^{i-1}(s)}, \quad i = 1, 2, \ldots, q \tag{17}$$

and

$$\beta_j^i(s) = \operatorname{lcm}(\alpha_j(s), \alpha'_{j+q-i}(s)), \quad i = 0, 2, \ldots, q, \ j = 1, 2, \ldots, n+i \tag{18}$$

where $\alpha_i(s) = 1$, $\alpha'_i(s) := 1$ for $i > n$.

With the help of Lemmas 1 – 3 we are now able to establish new solvability conditions for the problems stated in Proposition 1.

**Theorem 1.** The problems stated in Proposition 1 have a solution if and only if the following conditions hold.

$$\alpha'_{i+q}(s) \mid \alpha_i(s) \mid \alpha'_i(s) \ , \ i = 1, \ldots, n \tag{19}$$

where by convention $\alpha'_i(s) := 1$ for $i > n$,

$$\sum_{j \mid c'_j \le i} c'_j \le \sum_{j \mid c_j \le i} c_j \ , \ i = 1, 2, \ldots, n \tag{20}$$

and

$$\sum_{j=1}^{i} \delta'_j \ge \sum_{j=1}^{i} c_j \ , \ i = 1, 2, \ldots, m+q, \tag{21}$$

with equality holding for $i = m + q$ and where $\{\delta'_i\}_{m+q}$ denotes the non-increasingly reordered list $\{c'_i\}_m \cup \{\deg \sigma_i\}_q$. The polynomials $\sigma_i(s)$ and $\beta_j^i(s)$ are defined in (17) and (18).

Proof of Theorem 1. (A sketch). The conditions (19) and (20) are conditions established in Lemma 1 and Lemma 2 for the subproblems that are *a fortiori* solved in Proposition 1 implying that they must hold in this case, too. The conditions (21) are based on the conditions (13) where $\{\delta_i\}_{m+q}$ is now given by the reordered list $\{c'_i\}_m \cup \{\deg \phi_i\}_q$ where, as in Lemma 3, $\{\phi_i(s)\}_q$ is the list of the invariant factors of $X(s)$. Observe that, since $\{\delta'_i\}_{m+q}$ is the reordered list $\{c'_i\}_m \cup \{\deg \sigma_i\}_q$, conditions (16) imply that

$$\sum_{j=1}^{i} \delta'_j \ge \sum_{j=1}^{i} \delta_j \ , \ i = 1, 2, \ldots, m+q.$$

This leads to the necessity of conditions (21).

The sufficiency of the conditions (19) – (21) follows from a constructive procedure that is similar to that in [5, 15]. □

**Remark 5.** The conditions (21) can also be written in the following form.

$$\sum_{j=1}^{n+t_i} \deg \beta_j^{t_i}(s) + \sum_{j=1}^{i-t_i} c'_j \geq \sum_{j=1}^{i} c_j + \sum_{j=1}^{n} \deg \alpha_j(s) \ , \ i = 1, 2, \ldots, m+q \tag{22}$$

where $t_i$ is the number of elements of the list $\{\sigma_j\}_q$ in the sublist $\{\delta'_j\}_i$ of $\{\delta'_j\}_{m+q}$. These inequalities were already derived in [16]. They avoid the calculation of $\sigma_i(s)$, $i = 1, \ldots, q$, or at least their degrees, and hence they could be more convenient from the computational point of view.

### 2.2. Conditions for right invertible pencils $sE - H$

Going back to the original problem, i. e. the column completion of right invertible pencils, it can be seen that finding solvability conditions is just a matter of applying the conformal mapping (6) to the pencils. The proposition below exactly describes how the Kronecker invariants are transformed.

**Lemma 4.** Let $(sE-H)$ be a right invertible matrix pencil with $E, H \in \mathbb{R}^{n\times(n+m+q)}$ whose Kronecker invariants are column minimal indices $c_1, c_2, \ldots, c_{m+q}$, invariant polynomials $\alpha_1(s), \alpha_2(s), \ldots, \alpha_n(s)$, and infinite zero orders $n_1, n_2, \ldots, n_p$ that are, by convention, non-increasingly ordered. The mapping $\mathcal{C}$ defined by

$$sE - H \longmapsto \mathcal{C}(sE-H) = w\widetilde{E} - \widetilde{H} \ ,$$

where $\widetilde{E} = aE - H$ and $\widetilde{H} = -E$, is a one-to-one correspondence on $\mathbb{R}^{n\times(n+m+q)} \times \mathbb{R}^{n\times(n+m+q)}$. If $a$ is not a zero of $sE-H$, then the Kronecker invariants of $w\widetilde{E}-\widetilde{H}$ are column minimal indices $c_1, c_2, \ldots, c_{m+q}$ and invariant factors $\tilde{\alpha}_1(w), \tilde{\alpha}_2(w), \ldots, \tilde{\alpha}_n(w)$,

$$\tilde{\alpha}_i(w) = \alpha_i\left(\frac{1+aw}{w}\right) w^{\deg \alpha_i(s)} w^{n_i} \ , \tag{23}$$

where $n_i = 0$ for $i > p$ by convention.

Assuming now that $[sE - H, \ \star] \sim sM - N$, it readily follows that $[w\widetilde{E} - \widetilde{H}, \ \widetilde{\star}] \sim [w\widetilde{M} - \widetilde{N}]$ and reversely. Hence, the problem of column completion of right invertible pencils comes down to the problem of completing a pencil that has no infinite elementary divisors, the case that is treated in Theorem 1. And as the conformal mapping $\mathcal{C}$ is a one-to-one correspondence between matrix pencils, it is also a one-to-one correspondence between their Kronecker invariants. This implies that the necessary and sufficient conditions stated below can directly be deduced from Theorem 1.

**Theorem 2.** Given a right invertible pencil $sE' - H' \in \mathbb{R}^{n\times(n+m)}[s]$ with invariant polynomials $\alpha'_i(s)$, $i = 1,2,\dots,n$, column minimal indices $c'_i$, $i = 1,2,\dots,m$, and infinite zero orders $n'_i$, $i = 1,2,\dots,p$ and a right invertible pencil $sE - H \in \mathbb{R}^{n\times(n+m+q)}$ with invariant polynomials $\alpha_i(s)$, $i = 1,2,\dots,n$, column minimal indices $c_i$, $i = 1,2,\dots,m+q$, and infinite zero orders $n_i$, $i = 1,2,\dots,p$, then there exists an $n \times q$ pencil $\star$ such that

$$[sE' - H',\ \star] \sim sE - H$$

if and only if

$$\alpha'_{i+q}(s) \mid \alpha_i(s) \mid \alpha'_i(s)\ ,\ i = 1,\dots,n \tag{24}$$

where by convention $\alpha'_i(s) := 1$ for $i > n$,

$$n'_{i+q} \le n_i \le n'_i\ ,\ i = 1,\dots,p \tag{25}$$

where $n'_i := 0$ for $i > p$,

$$\sum_{j \mid c'_j \le i} c'_j \le \sum_{j \mid c_j \le i} c_j\ ,\ i = 1,2,\dots,n, \tag{26}$$

$$\sum_{j=1}^{i} \delta''_j \ge \sum_{j=1}^{i} c_j\ ,i = 1,2,\dots,m+q \tag{27}$$

with equality holding for $i = m + q$, where $\{\delta''_j\}_{m+q}$ denotes the reordered list $\{c'_i\}_m \cup \{\deg(\tilde{\sigma}_i(w))\}_q$ and

$$\tilde{\sigma}_i(w) = \frac{\tilde{\beta}^i_1(w)\tilde{\beta}^i_2(w)\dots\tilde{\beta}^i_{n+i}(w)}{\tilde{\beta}^{i-1}_1(w)\tilde{\beta}^{i-1}_2(w)\dots\tilde{\beta}^{i-1}_{n+i-1}(w)},\ i = 1,\dots,q,$$

$$\tilde{\beta}^i_j(w) = \operatorname{lcm}\,(\tilde{\alpha}_j(w), \tilde{\alpha}'_{j+q-i}(w)),\ i = 0,\dots,q,\ j = 1,\dots,n+i$$

with $\tilde{\alpha}_i(w)$ and $\tilde{\alpha}'_i(w)$ being defined in (22).

**Remark 6.** Similarly, as in Remark 5, denoting $\tilde{t}_i$ the cardinality of the list $\{\tilde{\sigma}_j\}_q$ in the sublist $\{\delta''_j\}_i$ of $\{\delta''_j\}_{m+q}$, the inequalities (27) can be rewritten in the form

$$\sum_{j=1}^{n+\tilde{t}_i} \{\deg\beta^{\tilde{t}_i}_j(s) + \max(n_j, n'_{j+q-\tilde{t}_i})\} + \sum_{j=1}^{i-\tilde{t}_i} c'_j \ge \sum_{j=1}^{i} c_j + \sum_{j=1}^{n} \deg\alpha_j(s) + \sum_{j=1}^{p} n_i$$

for $i = 1,2,\dots,m+q$ where $\beta^i_j(s)$ is defined in (18).

**Remark 7.** The fact that the pencil $sE' - H'$ as well as the completed pencil $sE - H$ are right invertible implies that $p \le p' \le p + q$. An interesting particular case is when the number of infinite elementary divisors is not modified. In the case when $p = p'$, the completion can be performed with a constant matrix $\star$, and reversely.

## 3. ROW COMPLETION OF RIGHT INVERTIBLE PENCILS

The statement (d) of Proposition 1 reveals another particular case of the matrix pencil completion problem that is called *the row completion problem.* The most general case of this problem is described by the following relationship.

$$\begin{bmatrix} sE' - H' \\ \star \end{bmatrix} \sim sE - H. \tag{28}$$

In words, given pencils $sE' - H' \in \mathbb{R}^{(n+m)\times n}[s]$ and $sE - H \in \mathbb{R}^{(n+m+q)\times n}[s]$, find conditions under which the pencil $sE' - H'$ can be completed by another pencil, denoted by $\star$, such that the relationship (28) holds.

It is easy to see that, under the condition that the above pencils are left invertible, the row completion problem (28) is just the dual version (taking transposition) of the column completion problem solved in Theorem 2. Thus, the solvability conditions for this problem are an obvious analog of the conditions (24) – (27).

**Corollary 1.** Given a left invertible pencil $sE' - H' \in \mathbb{R}^{(n+m)\times n}[s]$ with invariant polynomials $\alpha_i'(s)$, $i = 1, 2, \dots, n$, row minimal indices $r_i'$, $i = 1, 2, \dots, m$, and infinite zero orders $n_i'$, $i = 1, 2, \dots, p$, and a left invertible pencil $sE - H \in \mathbb{R}^{(n+m+q)\times n}$ with invariant polynomials $\alpha_i(s)$, $i = 1, 2, \dots, n$, row minimal indices $r_i$, $i = 1, 2, \dots, m+q$, and infinite zero orders $n_i$, $i = 1, 2, \dots, p$, then there exists an $q \times n$ matrix pencil $\star$ such that

$$\begin{bmatrix} sE' - H' \\ \star \end{bmatrix} \sim sE - H$$

if and only if

$$\alpha_{i+q}'(s) \mid \alpha_i(s) \mid \alpha_i'(s)\ , \ i = 1, \dots, n$$

where by convention $\alpha_i'(s) := 1$ for $i > n$,

$$n_{i+q}' \le n_i \le n_i'\ , \ i = 1, \dots, p$$

where $n_i' := 0$ for $i > p$,

$$\sum_{j \,|\, r_j' \le i} r_j' \le \sum_{j \,|\, r_j \le i} r_j\ , \ i = 1, 2, \dots, n,$$

$$\sum_{j=1}^{i} \delta_j'' \ge \sum_{j=1}^{i} (r_j + 1)\ , \ i = 1, 2, \dots, m+q$$

$$\sum_{j=1}^{m+q} \delta_j'' \ge \sum_{j=1}^{m+q} (r_j + 1)\ ,$$

where $\{\delta_i''\}_{m+q}$ denotes the reordered list $\{\eta_i'\}_m \cup \{\deg(\tilde{\sigma}_i(w))\}_q$,

$$\tilde{\sigma}_i(w) = \frac{\tilde{\beta}_1^i(w)\tilde{\beta}_2^i(w)\dots\tilde{\beta}_{n+i}^i(w)}{\tilde{\beta}_1^{i-1}(w)\tilde{\beta}_2^{i-1}(w)\dots\tilde{\beta}_{n+i-1}^{i-1}(w)}\ , \ i = 1, \dots, q,$$

and

$$\tilde{\beta}_j^i(w) = \text{lcm}\ (\tilde{\alpha}_j(w), \tilde{\alpha}'_{j+q-i}(w))\ ,\ i = 0, \dots, q,\ j = 1, \dots, n+i,$$

with $\tilde{\alpha}_i(w)$ and $\tilde{\alpha}'_i(w)$ defined analogously to (6).

But there exists another relationship between the row and column completion problems, which is based on the equivalence of the statements (b) and (d) of Proposition 1. To this end, suppose that the pencil $s\mathcal{E} - \mathcal{H}$ is already in its system form given by the matrices $A, B$, and $C$, i. e.

$$s\mathcal{E} - \mathcal{H} = \begin{bmatrix} sI_n - A & -B \\ -C & 0 \end{bmatrix},$$

and consider the pencil

$$s\bar{\mathcal{E}} - \bar{\mathcal{H}} := \begin{bmatrix} \Pi(sI_n - A) \\ -C \end{bmatrix}$$

with $\Pi$ being the maximal left annihilator of $B$. The pencil $s\bar{\mathcal{E}} - \bar{\mathcal{H}}$ will be called *the reduced pencil* of $s\mathcal{E} - \mathcal{H}$. Thus, according to Remark 1, we consider just pencils the $n_i$-blocks of which are of sizes $n_i \geq 1$.

**Remark 8.**

(s1) If the pencil $s\mathcal{E} - \mathcal{H}$ is right invertible, then the corresponding reduced pencil $s\bar{\mathcal{E}} - \bar{\mathcal{H}}$ is right invertible, too.

(s2) If there exists another pencil

$$s\mathcal{E}' - \mathcal{H}' = \begin{bmatrix} sI_n - A' & -B' \\ -C' & 0 \end{bmatrix},$$

that can be completed by columns such that $[s\mathcal{E} - \mathcal{H},\ \star] \sim s\mathcal{E}' - \mathcal{H}'$, then the pencil $s\bar{\mathcal{E}}' - \bar{\mathcal{H}}'$ can be completed by rows such that

$$\begin{bmatrix} s\bar{\mathcal{E}}' - \bar{\mathcal{H}}' \\ \star \end{bmatrix} \sim s\bar{\mathcal{E}} - \bar{\mathcal{H}}$$

(s3) If the reduced pencil $s\bar{\mathcal{E}} - \bar{\mathcal{H}}$ of $s\mathcal{E} - \mathcal{H}$ is described by the invariant factors $\alpha_i(s)$, column minimal indices $c_i$, infinite zero orders $n_i$, and row minimal indices $r_i$, then the pencil $s\mathcal{E} - \mathcal{H}$ has the same invariant polynomials $\alpha_i(s)$ and row minimal indices $r_i$, while its column minimal indices and infinite zero orders are given by $c_i + 1$ and $n_i + 1$, respectively.

(s4) Any matrix pencil $sE - H$ is equivalent to the reduced pencil $s\bar{\mathcal{E}} - \bar{\mathcal{H}}$ that is defined by a triple $A, B, C$.

Remark 8 summarizes the facts enabling to solve the matrix pencil completion problem (28) for the right invertible pencils. Based on (s2) and (s4) of Remark 7, this can be done by applying Theorem 2 to the pencils $s\mathcal{E}' - \mathcal{H}'$ and $s\mathcal{E} - \mathcal{H}$. The resulting conditions can finally, using (s3), be rewritten in terms of the Kronecker invariants of $s\bar{\mathcal{E}} - \bar{\mathcal{H}}$ and $s\bar{\mathcal{E}}' - \bar{\mathcal{H}}'$.

**Theorem 3.** Given a right invertible pencil $sE' - H' \in \mathbb{R}^{n\times(n+m)}[s]$ with invariant polynomials $\alpha_i'(s)$, $i = 1,2,\dots,n$, column minimal indices $c_i'$, $i = 1,2,\dots,m$, and infinite zero orders $n_i'$, $i = 1,2,\dots,p'$ and a right invertible pencil $sE - H \in \mathbb{R}^{(n+q)\times(n+m)}$ with invariant polynomials $\alpha_i(s)$, $i = 1,2,\dots,n$, column minimal indices $c_i$, $i = 1,2,\dots,m-q$, and infinite zero orders $n_i$, $i = 1,2,\dots,p$, then there exists an $q \times (n+m)$ pencil, denoted by $\star$, such that

$$\begin{bmatrix} sE' - H' \\ \star \end{bmatrix} \sim sE - H$$

if and only if

$$\alpha_{i+q}(s) \,|\, \alpha_i'(s) \,|\, \alpha_i(s)\ ,\ i = 1,\dots,n,$$

where by convention $\alpha_i'(s) := 1$ for $i > n$,

$$n_{i+q} \le n_i' \le n_i,\ i = 1,\dots,p',$$

with $n_i := 0$ for $i > p$,

$$\sum_{j\,|\,c_j\le i} (c_j+1) \le \sum_{j\,|\,c_j'\le i} (c_j'+1),\ i = 1,2,\dots,n,$$

$$\sum_{j=1}^{i} \delta_j'' \ge \sum_{j=1}^{i} c_j',\ i = 1,2,\dots,m,$$

and

$$\sum_{j=1}^{m} \delta_j'' = \sum_{j=1}^{m} c_j',$$

where $\{\delta_i''\}_{m+q}$ denotes the reordered list $\{c_i+1\}_{m-q} \cup \{\deg(\tilde{\sigma}_i(w))\}_q$,

$$\tilde{\sigma}_i(w) := \frac{\tilde{\beta}_1^i(w)\tilde{\beta}_2^i(w)\dots\tilde{\beta}_{n+i}^i(w)}{\tilde{\beta}_1^{i-1}(w)\tilde{\beta}_2^{i-1}(w)\dots\tilde{\beta}_{n+i-1}^{i-1}(w)},\ i = 1,\dots,q,$$

$$\tilde{\beta}_j^i(w) := \operatorname{lcm}(\tilde{\alpha}_j(w), \tilde{\alpha}_{j+k-i}'(w)),\ i = 0,\dots,q,\ j = 1,\dots,n+i,$$

where $\tilde{\alpha}_i(w)$ and $\tilde{\alpha}_i'(w)$ are defined by

$$\tilde{\alpha}_i(w) := \alpha_i\left(\frac{1+aw}{w}\right) w^{\deg \alpha_i(s)} w^{n_i+1}$$

and

$$\tilde{\alpha}_i'(w) := \alpha_i'\left(\frac{1+aw}{w}\right) w^{\deg \alpha_i'(s)} w^{n_i'+1}\ .$$

**Remark 9.** As the pencils $sE' - H'$ and $sE - H$ are right invertible, it follows that $p' \leq p \leq p' + q$. This implies, in case the number of infinite zero orders is not modified, that $p = p'$, and the conditions of Theorem 3 are satisfied, i. e. the completion can be realized with a $q \times (n+m)$ constant matrix only.

Analogously, when $sE - H$ and $sE' - H'$ are left invertible pencils, necessary and sufficient conditions for the existence of a column completion such that (7) holds can be obtained by a "dualization" of Theorem 3.

## 4. CONCLUSIONS

Several results concerning the matrix pencil completion problem, which were achieved during the last five years, are discussed and summarized in the paper. The basic results on which the paper is built up are introduced in Proposition 1, subsequent lemmas, and Theorem 1. A generalization to the right invertible pencils is then achieved in Theorem 2. The second line of generalization (row completion of right invertible pencils) is based on the assertion (d) of Proposition 1. This approach is somewhat novel and completes the picture about the right/left invertible pencils.

The matrix pencil completion problem is still unsolved in its full generality and the authors of the paper believe that the reader interested in that problem will find items of useful information in the above text.

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