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GENERALIZED POLAR VARIETIES AND AN EFFICIENT REAL ELIMINATION PROCEDURE[1]

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Dedicated to our friend František Nožička.

Let V be a closed algebraic subvariety of the n-dimensional projective space over the complex or real numbers and suppose that V is non-empty and equidimensional. In this paper we generalize the classic notion of polar variety of V associated with a given linear subvariety of the ambient space of V. As particular instances of this new notion of generalized polar variety we reobtain the classic ones and two new types of polar varieties, called *dual* and (in case that V is affine) *conic*. We show that for a generic choice of their parameters the generalized polar varieties of V are either empty or equidimensional and, if V is smooth, that their ideals of definition are Cohen–Macaulay. In the case that the variety V is affine and smooth and has a complete intersection ideal of definition, we are able, for a generic parameter choice, to describe locally the generalized polar varieties of V by explicit equations. Finally, we use this description in order to design a new, highly efficient elimination procedure for the following algorithmic task: In case, that the variety V is $\mathbb{Q}$-definable and affine, having a complete intersection ideal of definition, and that the real trace of V is non-empty and smooth, find for each connected component of the real trace of V a representative point.

Keywords: geometry of polar varieties and its generalizations, geometric degree, real polynomial equation solving, elimination procedure, arithmetic circuit, arithmetic network, complexity

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1. INTRODUCTION

Let $\mathbb{P}^n$ denote the n-dimensional projective space over the field of complex numbers $\mathbb{C}$ and let, for $0 \le p \le n$, V be a pure p-codimensional closed algebraic subvariety of $\mathbb{P}^n$. In this paper we introduce the new notion of a *generalized polar variety* of V associated with a given linear subspace K, a given non-degenerate hyperquadric Q and a given hyperplane H of the ambient space $\mathbb{P}^n$, subject to the condition that

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$Q \cap H$ is a non-degenerate hyperquadric of H. We denote this generalized polar variety by $\widehat{W}_K(V)$. It turns out that $\widehat{W}_K(V)$ is either empty or a smooth subvariety of V having pure codimension i in V, if V is smooth and K is a "sufficiently generic", $(n-p-i)$-dimensional, linear subspace of $\mathbb{P}^n$, for $0 \leq i \leq n-p$ (see Corollary 10 and the following comments).

The concept of generalized polar varieties has two instances of particular interest. One instance reproduces the classic polar varieties, which we call *direct*. The other instance produces a certain type of non-classic polar varieties, which we call *dual*.

In this paper we are mainly concerned with the case that H is the hyperplane at infinity of $\mathbb{P}^n$ determining thus an embedding of the complex n-dimensional affine space $\mathbb{A}^n$ into the projective space $\mathbb{P}^n$. Let $S := V \cap H$ be the affine trace of V and suppose S is non-empty. Then S is a pure p-codimensional closed subvariety of the affine space $\mathbb{A}^n$. The affine traces of the direct polar variety of V give rise to two types of polar varieties of the affine variety S, called *conic* and *cylindric*, respectively. A conic polar variety of S is associated with an affine linear subspace of $\mathbb{A}^n$ and a cylindric polar variety is associated with a linear subspace of the hyperplane at infinity of $\mathbb{P}^n$, namely H. The concept of the conic polar varieties seems to be new, whereas the cylindric polar varieties of S are the classic ones.

The affine trace $\widehat{W}_K(S) := \widehat{W}_K(V) \cap \mathbb{A}^n$ is called the *affine* generalized polar variety of S associated with the linear subvariety K and the hyperquadric Q of $\mathbb{P}^n$. The affine generalized polar varieties of S give rise to cylindric (i. e., classic) and dual affine polar varieties. However, the conic polar varieties of S cannot be obtained in this way because of the particular choice of the hyperplane H. Let us denote the field of real numbers by $\mathbb{R}$ and the real n-dimensional projective and affine spaces by $\mathbb{P}^n_{\mathbb{R}}$ and $\mathbb{A}^n_{\mathbb{R}}$, respectively. Assume that V is $\mathbb{R}$-definable and let $V_{\mathbb{R}} := V \cap \mathbb{P}^n_{\mathbb{R}}$ and $S_{\mathbb{R}} := S \cap \mathbb{A}^n_{\mathbb{R}} = V \cap \mathbb{A}^n_{\mathbb{R}}$ be the real traces of the complex algebraic varieties V and S. Similarly, define $H_{\mathbb{R}} := H \cap \mathbb{P}^n_{\mathbb{R}}$. Suppose that the real varieties $V_{\mathbb{R}}$ and $S_{\mathbb{R}}$ are non-empty and that K and Q are $\mathbb{R}$-definable. Then the generalized *real* polar varieties $\widehat{W}_K(V_{\mathbb{R}}) := \widehat{W}_K(V) \cap \mathbb{P}^n_{\mathbb{R}}$ and $\widehat{W}_K(S_{\mathbb{R}}) := \widehat{W}_K(S) \cap \mathbb{A}^n_{\mathbb{R}} = \widehat{W}_K(V) \cap \mathbb{A}^n_{\mathbb{R}}$ are well defined and lead to the corresponding notions of dual polar variety of $V_{\mathbb{R}}$ and $S_{\mathbb{R}}$ and of cylindric polar variety of $S_{\mathbb{R}}$. Suppose that $S_{\mathbb{R}}$ is smooth. Then "sufficiently generic" real dual polar varieties of $S_{\mathbb{R}}$ contain for each connected component of $S_{\mathbb{R}}$ at least one representative point. The same is true for the real cylindric polar varieties if additionally the ideal of definition of S is a complete intersection ideal and if $S_{\mathbb{R}}$ is compact (see Proposition 1 and Proposition 2).

Let $\mathbb{Q}$ be the field of rational numbers, let $X_1, \ldots, X_n$ be indeterminates over $\mathbb{R}$ and let a regular sequence $F_1, \ldots, F_p$ in $\mathbb{Q}[X_1, \ldots, X_n]$ be given such that $(F_1, \ldots, F_p)$ is the ideal of definition of the affine variety S. Then, in particular, S is a $\mathbb{Q}$-definable, complete intersection variety. Suppose that the hyperquadric Q is $\mathbb{Q}$-definable and that $Q \cap H_{\mathbb{R}}$ can be described by the standard, n-variate positive definite quadratic form (inducing on $\mathbb{A}^n_{\mathbb{R}}$ the usual euclidean distance). Assume that the projective linear variety K is spanned by $n-p-i+1$ rational points $(a_{1,0} : \cdots : a_{1,n}), \ldots, (a_{n-p-i+1,0} : \cdots : a_{n-p-i+1,n})$ of $\mathbb{P}^n$ with $a_{j,1}, \ldots, a_{j,n}$ generic for $1 \leq j \leq n-p-i+1$. Thus K has dimension $n-p-i$. Suppose that S is smooth. Then the generalized affine polar variety $\widehat{W}_K(S)$ is either empty or of pure codimension i in S. Moreover, the ideal of definition of $\widehat{W}_K(S)$ in $\mathbb{Q}[X_1, \ldots, X_n]$ is Cohen–Macaulay and generated by $F_1, \ldots, F_p$ and all $(n-i+1)$-minors of the polynomial $((n-i+1) \times n)$-matrix

$$\begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial F_p}{\partial X_1} & \cdots & \frac{\partial F_p}{\partial X_n} \\ a_{1,1} - a_{1,0}X_1 & \cdots & a_{1,n} - a_{1,0}X_n \\ \vdots & \vdots & \vdots \\ a_{n-p-i+1,1} - a_{n-p-i+1,0}X_1 & \cdots & a_{n-p-i+1,n} - a_{n-p-i+1,0}X_n \end{bmatrix}$$

(see Theorem 9). It is even possible to show that $\widehat{W}_K(S)$ is smooth. However, the proof of this fact is considerably more involved and less transparent than the proof of the Cohen–Macaulay property of $\widehat{W}_K(S)$ given in this paper. For details we refer to [2]. The algorithmic applications described below do not require the smoothness of $\widehat{W}_K(S)$, the Cohen–Macaulay property suffices.

In [5] and [4], cylindric (i. e., classic) polar varieties were used in order to design a new generation of efficient algorithms for finding at least one representative point of each connected component of a given smooth, compact hypersurface or complete intersection subvariety of $\mathbb{A}^n_{\mathbb{R}}$. The *dual* polar varieties introduced in this paper can be used for the same algorithmic task in the *non-compact* (but still smooth) case. This leads to a complexity result that represents the basic motivation and (in some sense) the main outcome of this paper: If the real variety $S_{\mathbb{R}}$ is non-empty and smooth and if S is given as before by a regular sequence $F_1, \ldots, F_p$ in $\mathbb{Q}[X_1, \ldots, X_n]$ such that, for any $1 \leq h \leq p$, the ideal generated by $F_1, \ldots, F_h$ is radical, then it is possible to find a (real algebraic) representative point of each connected component of $S_{\mathbb{R}}$ in (polynomial) sequential time $\binom{n}{p} L^2 (nd\delta)^{O(1)}$ (counting arithmetic operations in $\mathbb{Q}$ at unit costs). Here d is an upper bound for the degrees of the polynomials $F_1, \ldots, F_p$, L denotes the (sequential time) *arithmetic circuit complexity* of them and $\delta \leq d^n p^{n-p}$ is the (suitably defined) *degree of the real interpretation* of the polynomial equation system $F_1, \ldots, F_p$ (see Theorem 11). Although this complexity bound is polynomial in δ, it may become exponential with respect to the number of variables n, at least in the worst case. This exponential worst case complexity becomes unavoidable since $S_{\mathbb{R}}$ may contain exponentially many connected components. On the other hand, the elimination problem under consideration is intrinsically of non-polynomial character with respect to the syntactic input length for any reasonable continuous data structure (compare [23] and [13]).

In view of [14] we may conclude that no numerical procedure (based on the bit representation of integers) is able to solve this algorithmic task more efficiently than our symbolic-seminumeric procedure.

On the other hand, we would like to emphasize an important practical outcome of our fairly theoretical contribution: Combining the algorithm described in the proof of Theorem 11 below with the software package "Kronecker" ([37, 50]), designed for the solution of polynomial equations over the complex numbers, it was possible to find the coefficients of suitable one-dimensional wavelet transforms (multiresolution analysis, MRA) for the construction of optimal filters for image compression and decompression (see [39]).

2. INTRINSIC ASPECTS OF POLAR VARIETIES

For two given linear subvarieties A and B of the complex n-dimensional projective space $\mathbb{P}^n$ we denote by $\langle A, B\rangle$ the linear subvariety of $\mathbb{P}^n$ spanned by A and B. We say that A and B intersect transversally (in symbols: $A \pitchfork B$) if $\langle A, B\rangle = \mathbb{P}^n$ holds. In case that A and B do not intersect transversally, we shall write $A \not\pitchfork B$. Let V be a projective subvariety of $\mathbb{P}^n$ and suppose that V is of pure codimension p for some $0 \leq p \leq n$ (this means that all irreducible components of V have the same codimension p). We denote by V_{reg} the set of all regular (smooth) points of V. Observe that V_{reg} is a complex submanifold of $\mathbb{P}^n$ of codimension p and that V_{reg} is Zariski-dense in V. We call $V_{\mathrm{sing}} := V \setminus V_{\mathrm{reg}}$ the singular locus of the projective variety V. Let V and W be two given pure codimensional projective subvarieties of $\mathbb{P}^n$ and let M be a given point of $\mathbb{P}^n$ belonging to the intersection of V_{reg} and W_{reg}. We say that V and W intersect transversally at the point M if the Zariski tangent spaces $T_M V$ and $T_M W$ of the algebraic varieties V and W at the point M intersect transversally (here we interpret $T_M V$ and $T_M W$ as linear subvarieties of the ambient space $\mathbb{P}^n$ that contain the point M).

For the rest of this paper let us fix integers $n \geq 0$, $0 \leq p \leq n$ and a projective subvariety V of $\mathbb{P}^n$ having pure codimension p. Using the projective setting, we first recall in Subsection 2.1 the classic notion of a polar variety of V associated with a given linear subvariety of $\mathbb{P}^n$ (in this paper, we shall call such polar varieties *direct*). Then, in Subsection 2.2 we introduce the new notion of a *generalized* polar variety of V associated with a given linear subspace K, a given non-degenerate hyperquadric Q and a given hyperplane H of the ambient space $\mathbb{P}^n$, subject to the condition that $Q \cap H$ is a non-degenerate hyperquadric of H. The dual polar varieties of V are introduced and the direct polar varieties of V are reobtained as particular instances of generalized polar varieties of V.

We will pay particular attention to the case that H is the hyperplane at infinity of $\mathbb{P}^n$. We may then consider the complex n-dimensional affine space $\mathbb{A}^n$ as embedded in $\mathbb{P}^n$. In this context we may define the *affine* direct (*conic* and *cylindric*), dual and generalized polar varieties of the affine variety $S := V \cap \mathbb{A}^n$, which we suppose to be non-empty. Finally, in Subsection 2.3 we will introduce and discuss the *real* (generalized, direct, dual, affine) *polar varieties* of the real varieties $V_{\mathbb{R}} := V \cap \mathbb{P}^n_{\mathbb{R}}$ and $S_{\mathbb{R}} := S \cap \mathbb{A}^n_{\mathbb{R}}$ (supposing that $V_{\mathbb{R}}$ and $S_{\mathbb{R}}$ are non-empty). We will formulate two sufficient conditions for the non-emptiness of such real polar varieties.

2.1. Classic polar varieties

Let $L \subset \mathbb{P}^n$ be a linear subvariety. The *direct polar variety of V associated with L*, denoted by $W_L(V)$, is defined as the Zariski–closure of the constructible set

$$\{M \in V_{\mathrm{reg}} \setminus L \,|\, T_M V \not\pitchfork \langle M, L\rangle \text{ at } M\}. \tag{1}$$

Remark that the direct polar variety $W_L(V)$ is contained in V. The direct polar varieties occurring in this paper are always organized as a decreasing sequence

$$V = W_{L^0} = \cdots = W_{L^{p-2}} \supset W_{L^{p-1}} \supset \cdots \supset \cdots \supset W_{L^{n-2}} \supset W_{L^{n-1}} = \emptyset$$

associated with a given flag of projective linear subvarieties of the n-dimensional projective space, namely

$$\mathcal{L}: \qquad L^0 \subset L^1 \cdots \subset L^{p-1} \subset \cdots \subset L^{n-2} \subset L^{n-1} \subset \mathbb{P}^n.$$

Here the superscripts indicate the dimension of the respective linear subvariety. In order to simplify notations, we shall write

$$V_i := W_{L^{p+i-2}}(V), \quad 1 \leq i \leq n-p,$$

and we call V_i the *ith direct polar variety* of the subvariety V associated with the flag $\mathcal{L}$. The subscript i reflects the expected codimension of V_i in V. Note that the relevant part of the flag $\mathcal{L}$ leading to non-trivial polar varieties ranges from L^{p-1} to L^{n-2}.

Direct polar varieties allow nice affine interpretations. Let us therefore consider the n-dimensional affine space $\mathbb{A}^n$ embedded in the projective space $\mathbb{P}^n$.

We assume now that the variety V is the projective closure of a given closed subvariety S of the affine space $\mathbb{A}^n$ and that S has pure codimension p. We call $S_{\text{reg}} := V_{\text{reg}} \cap \mathbb{A}^n$ and $S_{\text{sing}} := V_{\text{sing}} \cap \mathbb{A}^n$ the set of smooth (regular) points and the singular locus of the affine variety S, respectively. For any smooth point M of the affine variety S we interpret, as usual, the tangent space $T_M\, S$ of S at M as a linear subspace of $\mathbb{A}^n$ passing through the origin. Thus, if we interpret M as a point of the projective variety V, the affine trace of the tangent space $T_M\, V$ of V at M, namely $T_M\, V \cap \mathbb{A}^n$, turns out to be the affine linear subspace of $\mathbb{A}^n$ that is parallel to $T_M\, S$ and passes through M, namely $M + T_M\, S$. In the same sense we write $M + A := \langle M, A\rangle \cap \mathbb{A}^n$ for any linear subvariety A of $\mathbb{P}^n$.

Now we adapt the concept of a direct polar variety to the affine case. For any member L of the flag $\mathcal{L}$ we define $W_L(S)$, the *affine* direct polar variety associated with L, as the affine trace of the projective polar variety $W_L(V)$ introduced above, namely, $W_L(S) := W_L(V) \cap \mathbb{A}^n$. One sees easily that, in terms of the usual notion of (non-)transversality for affine linear subspaces of $\mathbb{A}^n$, the affine polar variety $W_L(S)$ is nothing else but the Zariski–closure (in $\mathbb{A}^n$) of the constructible set

$$\{M \in S_{\text{reg}} \setminus (L \cap \mathbb{A}^n) \,|\, M + T_M\, S;\ \not\pitchfork\ M + L \text{ at } M\}.$$

Again the relevant part of the flag $\mathcal{L}$ leading to non-trivial affine polar varieties ranges from L^{p-1} to L^{n-2}. Similarly as above, we abbreviate

$$S_i := W_{L^{p+i-2}}(S), \ \ 1 \leq i \leq n-p,$$

and we call S_i the ith *affine* direct polar variety of S associated with the flag $\mathcal{L}$. Again, the subscript i denotes the expected codimension of S_i in S.

The following two situations are of particular interest

- L^{n-1} is the hyperplane at infinity with respect to the given embedding of the affine space $\mathbb{A}^n$ in the projective space $\mathbb{P}^n$.
- The single-point variety L^0 is not contained in the hyperplane at infinity of $\mathbb{P}^n$.

The affine direct polar varieties associated with the flag $\mathcal{L}$ are called *cylindric* in the first situation and *conic* in the second one. The cylindric polar varieties are the classic ones, the subject of extensive investigations: Let us mention among others the contributions of J.-V. Poncelet (who introduced the concept of polar varieties), F. Severi, J. A. Todd, S. Kleiman, R. Piene, D. T. Lê, B. Teissier, J.-P. Henry and M. Merle (see e. g. [42] and the references cited therein).

It is evident that any conic polar variety can be transformed into a cylindric one by means of a suitable (linear) automorphism of the projective space.

Suppose now that L^{n-1} is the hyperplane at infinity of $\mathbb{P}^n$. Thus, for $1 \leq j \leq n-1$, we may interpret the affine cone of the projective linear variety L^j as a $(j+1)$-dimensional subspace of $\mathbb{A}^n$. Due to this interpretation the flag $\mathcal{L}$ of projective linear subvarieties becomes a flag of linear subspaces

$$\mathcal{I}: \qquad I^1 \subset I^2 \subset \cdots \subset I^{n-1} \subset \mathbb{A}^n.$$

As above, the superscripts indicate the dimension of the respective linear subspaces of $\mathbb{A}^n$. Observe now that, for any $1 \leq j \leq n-1$, and any regular point M of S, the identity $(M + L^{j-1}) \cap \mathbb{A}^n = M + I^j$ holds. Moreover, the affine linear spaces $M + T_M\, S$ and $M + L^{j-1}$ intersect transversally at M if and only if the linear spaces $T_M\, S$ and I^j intersect transversally. This implies that the affine direct polar variety $W_{L^{j-1}}(S)$ is the Zariski–closure of the constructible set

$$\left\{M \in S_{\text{reg}} \,|\, T_M\, S \not\pitchfork I^j\right\}.$$

Remark that this is just the usual definition of the polar variety of S associated with the linear space I^j.

Thus we have shown that our cylindric polar varieties are exactly the classic polar varieties. In case that S is a smooth closed subvariety of $\mathbb{A}^n$, it is well known that the classic cylindric polar varieties associated with a generic flag $\mathcal{I}$ of linear subspaces of $\mathbb{A}^n$ have the expected, pure codimension in S (see e. g. [48], Corollaire 1.3.2 and Définition 1.4, [36], Proposition 4.1.1 and Théorème 4.1.2 or [4], Theorem 1). Moreover, we shall show in Corollary 10 below that these varieties are Cohen–Macaulay (this fact seems to be folklore).

Therefore, if L^{n-1} is the hyperplane at infinity and if the remaining part of the flag $\mathcal{L}$ is chosen generically, the cylindric polar varieties $S_1, \ldots, S_{n-p}$ are Cohen–Macaulay and of pure codimension $1, \ldots, n-p$ in S. Since any affine direct polar variety can be obtained from a cylindric one by means of an automorphism of the projective space $\mathbb{P}^n$, we conclude that, for any generic flag $\mathcal{L}$ of projective subvarieties of $\mathbb{P}^n$, the corresponding (conic) polar varieties of S are Cohen–Macaulay and have the expected, pure codimension in S.

2.2. Generalized polar varieties

Let Q be a non-degenerate hyperquadric defined in the projective space $\mathbb{P}^n$. For a linear variety $A \subset \mathbb{P}^n$ of dimension a, let $A^\vee$ denote its dual with respect to Q. The dimension of $A^\vee$ is $n - a - 1$.

Further, let H be a hyperplane such that the intersection $Q \cap H$ is a non-degenerate hyperquadric of H (this means that H is not tangent to Q, or equivalently, that H does not belong to the dual hyperquadric of Q). If A is a linear subvariety of $\mathbb{P}^n$ contained in H, we denote by A^* its dual with respect to $Q \cap H$. The dimension of A^* is $n - a - 2$. Observe that the linear varieties A^* and $A^\vee \cap H$ coincide.

We are going to introduce the notion of a generalized polar variety contained in the projective space $\mathbb{P}^n$. Such polar varieties will be associated with a given flag of linear subvarieties, a non-degenerate hyperquadric and a hyperplane of $\mathbb{P}^n$, which is supposed not to be tangent to the hyperquadric. We consider this situation to be represented by a point of a suitable parameter space given as a Zariski open subset of the product of the corresponding flag variety, the space of hyperquadrics and the dual space of $\mathbb{P}^n$. We will denote a current point in this parameter space by $P = (\mathcal{K}, Q, H)$.

In view of subsequent algorithmic applications to real polynomial equation solving, the principal aim of this paper is the proof of suitable smoothness results for generic polar varieties associated with the given projective variety V. For this purpose we will work locally (in the Zariski sense) in the variety V. This allows us to restrict our attention to locally closed conditions in the parameter space (instead of the more general constructible ones).

For a given a point $P = (\mathcal{K}, Q, H)$ we define, for any member K of the flag $\mathcal{K}$, the *generalized polar variety* $\widehat{W}_K(V)$ associated with K as the Zariski–closure of the constructible set

$$\{M \in V_{\mathrm{reg}} \setminus (K \cup H) \,|\, T_M V \not\pitchfork \langle M, (\langle M, K\rangle \cap H)^*\rangle \text{ at } M\}. \tag{2}$$

Note that $\widehat{W}_K(V)$ is contained in V. Let us denote the given flag by

$$\mathcal{K}: \qquad \mathbb{P}^n \supset K^{n-1} \supset K^{n-2} \supset \cdots \supset K^{n-p-1} \supset \cdots \supset K^1 \supset K^0.$$

Then the generalized polar varieties associated with $\mathcal{K}$ are organized as a decreasing sequence as follows:

$$V = \widehat{W}_{K^{n-1}} = \cdots = \widehat{W}_{K^{n-p}} \supset \widehat{W}_{K^{n-p-1}} \supset \cdots \supset \widehat{W}_{K^1} \supset \widehat{W}_{K^0}.$$

In order to simplify notations, we write in the same spirit as in Subsection 2.1:

$$\widehat{V}_i := \widehat{W}_{K^{n-p-i}}, \quad 1 \le i \le n - p.$$

We call $\widehat{V}_i$ the *ith generalized polar variety* of V associated with the parameter point P. The subscript i reflects the expected codimension of $\widehat{V}_i$ in V. Note that the relevant part of the flag $\mathcal{K}$ leading to non-trivial polar varieties ranges from K^{n-p-1}

to K^0. Let K be any member of the flag $\mathcal{K}$ and assume that H is the hyperplane at infinity of $\mathbb{P}^n$ and that V is the projective closure of a given pure p-dimensional closed subvariety S of the affine space $\mathbb{A}^n$. Then we call $\widehat{W}_K(S) := \widehat{W}_K(V) \cap \mathbb{A}^n$ the affine generalized polar variety associated to K.

Two particular choices of the parameter point $P = (\mathcal{K}, H, Q)$ are noteworth. Let us fix a non-degenerate hyperquadric Q and a hyperplane H not tangent to Q. Furthermore, let be given a flag

$$\mathcal{L}: \qquad L^0 \subset L^1 \cdots \subset L^{p-1} \subset \cdots \subset L^{n-2} \subset L^{n-1} \subset \mathbb{P}^n$$

organized as an *increasing* sequence of linear subvarieties of the n-dimensional projective space and suppose that $L^{n-1} = H$ holds.

We associate two new flags of linear subspaces of $\mathbb{P}^n$ with the flag $\mathcal{L}$, both organized as *decreasing* sequences. We call these two flags the internal and the external flag of $\mathcal{L}$ and denote them by $\underline{\mathcal{K}}$ and $\overline{\mathcal{K}}$, respectively.

We write the *internal flag* $\underline{\mathcal{K}}$ as

$$\underline{\mathcal{K}}: \qquad \mathbb{P}^n \supset \underline{K}^{n-1} \supset \underline{K}^{n-2} \supset \cdots \supset \underline{K}^{n-p-1} \supset \cdots \supset \underline{K}^1 \supset \underline{K}^0.$$

For i ranging from 1 to $n-p$, we define the relevant part of $\underline{\mathcal{K}}$ by $\underline{K}^{n-p-i} := (L^{p+i-2})^*$ (observe that the linear variety L^{p+i-2} is contained in the hyperplane H). The irrelevant part $\underline{K}^{n-1} \supset \underline{K}^{n-2} \supset \cdots \supset \underline{K}^{n-p}$ of $\underline{\mathcal{K}}$ may be chosen arbitrarily.

Consider now an arbitrary member $\underline{K}$ of the relevant part of the internal flag $\underline{\mathcal{K}}$. Furthermore, let L be the member of the flag $\mathcal{L}$ determined by the condition $\underline{K} = L^*$, and let M be a point belonging to $V_{\text{reg}} \setminus H$. Taking into account that $\underline{K}$ is contained in H, whereas M does not belong to H, we conclude that

$$\langle M, \underline{K} \rangle \cap H = \underline{K}$$

holds. This implies

$$\langle M, (\langle M, \underline{K} \rangle \cap H)^* \rangle = \langle M, \underline{K}^* \rangle = \langle M, L \rangle.$$

Provided that H does not contain any irreducible component of V, we finally infer from (1) and (2) that

$$\widehat{W}_{\underline{K}}(V) = W_L(V) \tag{3}$$

holds.

As before let H be the hyperplane at infinity of $\mathbb{P}^n$ and let V be the projective closure of a given pure p-codimensional closed subvariety S of the affine space $\mathbb{A}^n$. Then H does not contain any irreducible component of V and from (3) we deduce that the affine generalized polar variety $\widehat{W}_{\underline{K}}(S) = \widehat{W}_{\underline{K}}(V) \cap \mathbb{A}^n$ is exactly the cylindric polar variety $W_L(S)$. Moreover, all cylindric polar varieties of S can be obtained in this way, by a suitable choice of the flag $\mathcal{L}$ with $L^{n-1} = H$.

More generally, choosing the flag $\mathcal{L}$ and the hyperplane H appropriately, one obtains any direct polar variety of V as a generalized polar variety associated with some member of the internal flag of $\mathcal{L}$.

We write the *external flag* $\overline{\mathcal{K}}$ as

$$\overline{\mathcal{K}}: \qquad \mathbb{P}^n \supset \overline{K}^{n-1} \supset \overline{K}^{n-2} \supset \cdots \supset \overline{K}^{n-p-1} \supset \cdots \supset \overline{K}^{1} \supset \overline{K}^{0}.$$

For i ranging from 1 to $n-p$, we define the relevant part of $\overline{\mathcal{K}}$ by $\overline{K}^{n-p-i} := (L^{p+i-1})^{\vee}$. The irrelevant part $\overline{K}^{n-1} \supset \overline{K}^{n-2} \supset \cdots \supset \overline{K}^{n-p}$ of $\overline{\mathcal{K}}$ may be chosen arbitrarily.

Consider now an arbitrary member $\overline{K}$ of the relevant part of the external flag $\overline{\mathcal{K}}$. Further, let L be the member of the flag $\mathcal{L}$ determined by the condition $\overline{K} = L^{\vee}$, and let M be a point belonging to $V_{\text{reg}} \setminus (\overline{K} \cup H)$. From $\overline{K}^0 \subset \overline{K}$ we deduce that $\overline{K}^0$ is contained in $\langle M, \overline{K}\rangle$. Taking into account that $\overline{K}^{0^{\vee}} = L^{n-1} = H$ holds, we conclude that any element of $\langle M, \overline{K}\rangle^{\vee}$ belongs to the hyperplane H. Thus $\langle M, \overline{K}\rangle^{\vee}$ is contained in $(\langle M, \overline{K}\rangle \cap H)^{\vee} \cap H$. A straightforward dimension argument implies now

$$\langle M, \overline{K}\rangle^{\vee} = \left(\langle M, \overline{K}\rangle \cap H\right)^{\vee} \cap H = \left(\langle M\,, \overline{K}\rangle \cap H\right)^{*}.$$

Hence, from (2) we conclude that the generalized polar variety $\widehat{W}_{\overline{K}}(V)$ coincides with the Zariski–closure of the constructible set

$$\left\{M \in V_{\text{reg}} \setminus (\overline{K} \cup H) \,|\, |T_M V \pitchfork \langle M, \langle M, \overline{K}\rangle^{\vee}\rangle \text{ at } M\right\}. \tag{4}$$

We call $\widehat{W}_{\overline{K}}(V)$ the *dual polar variety of* V *associated with* $\overline{K}$.

Again, let us assume that the variety V is the projective closure of a given closed subvariety S of the affine space $\mathbb{A}^n$, that S has pure codimension p and that H is the hyperplane at infinity of $\mathbb{P}^n$. We denote by $\widehat{W}_{\overline{K}}(S)$ the *affine* dual polar variety of S associated with $\overline{K}$, defined as the affine trace of the projective dual polar variety, namely $\widehat{W}_{\overline{K}}(S) := \widehat{W}_{\overline{K}}(V) \cap \mathbb{A}^n$.

Now from (4) one easily deduces that the affine dual polar variety $\widehat{W}_{\overline{K}}(S)$ is nothing else but the Zariski–closure (in $\mathbb{A}^n$) of the constructible set

$$\left\{M \in S_{\text{reg}} \setminus (\overline{K} \cap \mathbb{A}^n) \,|\, M + T_M\, S \pitchfork M + \langle M\, \overline{K}\rangle^{\vee} \text{ at } M\right\}. \tag{5}$$

Let M be a regular point of S that does not belong to $\overline{K} \cap \mathbb{A}^n$. Since the linear subvariety $\langle M, \overline{K}\rangle^{\vee}$ is contained in the hyperplane at infinity of $\mathbb{P}^n$, we may interpret the affine cone of $\langle M\,, \overline{K}\rangle^{\vee}$ as a linear subspace $I_{M,\overline{K}}$ of $\mathbb{A}^n$. In the same way we may interpret the affine cone of the linear variety L as a linear subspace I of $\mathbb{A}^n$. Then the linear space $I_{M,\overline{K}}$ consists exactly of those elements of I that are orthogonal to the point M with respect to the bilinear form induced by $Q \cap H$. From (5) one easily deduces that the affine dual polar variety $\widehat{W}_{\overline{K}}(S)$ is the Zariski–closure of the constructible set

$$\left\{M \in S_{\text{reg}} \setminus (\overline{K} \cap \mathbb{A}^n) \,|\, T_M\, S \pitchfork I_{M,\overline{K}}\right\}.$$

In conclusion: Internal flags lead to direct polar varieties that include the classic (cylindric) ones and external flags lead to a new type of polar varieties, namely the dual ones.

The affine interpretation of direct and dual polar varieties plays a fundamental role in the context of semialgebraic geometry, the main subject of this paper. In the next subsection we will discuss *real* polar varieties.

2.3. Real polar varieties

Recall the following notation: $\mathbb{P}^n_{\mathbb{R}}$ and $\mathbb{A}^n_{\mathbb{R}}$ for the real n-dimensional projective and affine spaces. Sometimes, we will also write $\mathbb{P}^n := \mathbb{P}^n_{\mathbb{C}}$ and $\mathbb{A}^n := \mathbb{A}^n_{\mathbb{C}}$ for n-dimensional complex projective and affine spaces.

Let a flag of real linear subvarieties of the projective space $\mathbb{P}^n_{\mathbb{R}}$ be given, namely

$$\mathcal{L}: \qquad L^0 \subset L^1 \subset \cdots \subset L^{n-1} \subset \mathbb{P}^n_{\mathbb{R}}.$$

Let H be the hyperplane at infinity of $\mathbb{P}^n_{\mathbb{C}}$, and let $H_{\mathbb{R}} := H \cap \mathbb{A}^n_{\mathbb{R}}$ be its real trace. Thus $H_{\mathbb{R}}$ fixes an embedding of the real affine space $\mathbb{A}^n_{\mathbb{R}}$ into $\mathbb{P}^n_{\mathbb{R}}$. Furthermore, let an $\mathbb{R}$-definable, non-degenerate hyperquadric Q of $\mathbb{P}^n_{\mathbb{C}}$ be given and suppose that $Q \cap H$ is also non-degenerate, and that $Q \cap H_{\mathbb{R}}$ can be described by means of a positive definite bilinear form. Observe that $Q \cap H_{\mathbb{R}}$ induces a Riemannian structure on the affine space $\mathbb{A}^n_{\mathbb{R}}$ and that $\mathcal{L}$ induces a flag of $\mathbb{R}$-definable linear subvarieties of the complex projective space $\mathbb{P}^n_{\mathbb{C}}$. We call this flag the complexification of $\mathcal{L}$. Suppose that we are given a purely p-codimensional, $\mathbb{R}$-definable closed subvariety S of $\mathbb{A}^n_{\mathbb{C}}$ whose projective closure in $\mathbb{P}^n_{\mathbb{C}}$ is V. We denote by $V_{\mathbb{R}} := V \cap \mathbb{P}^n_{\mathbb{R}}$ and $S_{\mathbb{R}} := S \cap \mathbb{A}^n_{\mathbb{R}}$ the real traces of V and S.

For the given flag $\mathcal{L}$ of linear subvarieties of $\mathbb{P}^n_{\mathbb{R}}$ we define the notion of an internal and an external flag and the notion of a *real* generalized, direct, cylindric, conic and dual polar variety of $V_{\mathbb{R}}$ and of $S_{\mathbb{R}}$ in the same way as in the Subsections 2.1 and 2.2. It turns out that these polar varieties are the real traces of their complex counterparts given by V, S and the complexification of $\mathcal{L}$ and its internal and external flag. All our comments on direct and dual affine polar varieties made in the Subsections 1.1 and 1.2 are valid mutatis mutandis in the real case. Again we denote the (real) internal and external flag associated with $\mathcal{L}$ by $\underline{\mathcal{K}}$ and $\overline{\mathcal{K}}$, respectively. For any member L of the flag $\mathcal{L}$, $\underline{K}$ of the flag $\underline{\mathcal{K}}$ and $\overline{K}$ of the flag $\overline{\mathcal{K}}$, we denote the corresponding real polar variety by

$$W_L(V_{\mathbb{R}}),\ \ W_L(S_{\mathbb{R}}),\ \ \widehat{W}_{\underline{K}}(V_{\mathbb{R}}),\ \ \widehat{W}_{\underline{K}}(S_{\mathbb{R}}),\ \ \widehat{W}_{\overline{K}}(V_{\mathbb{R}}) \text{ and } \widehat{W}_{\overline{K}}(S_{\mathbb{R}}).$$

Let us now assume that $L^{n-1} = H_{\mathbb{R}}$ and let us consider the real affine polar varieties associated with the internal and external flags $\underline{\mathcal{K}}$ and $\overline{\mathcal{K}}$ of $\mathcal{L}$.

Let us first consider the case of the internal flag $\underline{\mathcal{K}}$. As we have seen in Subsection 2.1, the flag $\mathcal{L}$ of linear subvarieties of $\mathbb{P}^n_{\mathbb{R}}$ induces a flag of linear subspaces of $\mathbb{A}^n_{\mathbb{R}}$, say

$$\mathcal{I}: \qquad I^1 \subset I^2 \subset \cdots \subset I^{n-1} \subset \mathbb{A}^n_{\mathbb{R}}.$$

Let now L be any member of the relevant part of the given flag $\mathcal{L}$, let I be the member of the flag $\mathcal{I}$ representing L and let $\underline{K}$ be the member of the internal flag $\underline{\mathcal{K}}$

defined by $\underline{K} := L^*$. Observe that $\underline{K}$ is contained in the hyperplane at infinity $H_{\mathbb{R}}$. From our considerations in the Subsections 1.1 and 1.2 we deduce that

$$\widehat{W}_{\underline{K}}(S_{\mathbb{R}}) = \widehat{W}_{\underline{K}}(V_{\mathbb{R}}) \cap \mathbb{A}^n_{\mathbb{R}} = W_L(V_{\mathbb{R}}) \cap \mathbb{A}^n_{\mathbb{R}} = W_L(S_{\mathbb{R}})$$

holds and that the real cylindric polar variety $W_L(S_{\mathbb{R}})$ is the Zariski–closure of the semialgebraic set

$$\{M \in (S_{\mathbb{R}})_{\text{reg}} \,|\, T_M S_{\mathbb{R}} \not\pitchfork I\}$$

in $\mathbb{A}^n_{\mathbb{R}}$.

Observe that the affine cone of the real linear subvariety $\underline{K}$ of $\mathbb{P}^n_{\mathbb{R}}$ corresponds to the orthogonal complement of I in $\mathbb{A}^n_{\mathbb{R}}$ (here we refer to orthogonality with respect to the Riemannian structure induced by Q on $\mathbb{A}^n_{\mathbb{R}}$). In this sense, the real polar variety $\widehat{W}_{\underline{K}}(S_{\mathbb{R}})$ is of cylindric type and orthogonal to the directions of $\underline{K}$ defining it.

In principle, the cylindric real polar variety $\widehat{W}_{\underline{K}}(S_{\mathbb{R}})$ may be empty, even in case that S contains real smooth points. However, under certain circumstances, we may conclude that $\widehat{W}_{\underline{K}}(S_{\mathbb{R}})$ is non-empty. This is the content of the following statement:

Proposition 1. Suppose that S is a pure p-codimensional complete intersection variety given as the set of common zeros of p polynomials $F_1, \ldots, F_p \in \mathbb{R}[X_1, \ldots, X_n]$, where $X_1, \ldots, X_n$ are indeterminates over the reals. Suppose that the ideal generated by $F_1, \ldots, F_p$ is radical and that $S_{\mathbb{R}}$ is a smooth and compact real variety. Then $\widehat{W}_{\underline{K}}(S_{\mathbb{R}})$ contains at least one point of each connected component of $S_{\mathbb{R}}$.

Proposition 1 is an easy consequence of the arguments used in [4], Section 2.4, which will not be repeated here.

Let us now consider the external flag $\overline{\mathcal{K}}$. Observe that $\overline{K}^0$ is a zero-dimensional linear subvariety of $\mathbb{P}^n_{\mathbb{R}}$, namely the origin of $\mathbb{A}^n$. Therefore any member of the external flag $\overline{\mathcal{K}}$ has a non-empty intersection with $\mathbb{A}^n_{\mathbb{R}}$. Assume now that the Riemannian metric of $\mathbb{A}^n$ induced by the hyperquadric Q is the ordinary euclidean distance. These assumptions lead to the following proposition:

Proposition 2. Suppose that $S_{\mathbb{R}}$ is a smooth, pure p-codimensional real variety. Let $\overline{K}$ be any member of the external flag $\overline{\mathcal{K}}$ and suppose that $\overline{K} \cap \mathbb{A}^n_{\mathbb{R}}$ is not contained in $S_{\mathbb{R}}$. Then, the real affine dual polar variety $\widehat{W}_{\overline{K}}(S_{\mathbb{R}})$ is non-empty and contains at least one point of each connected component of $S_{\mathbb{R}}$.

Proof. Since $\overline{K} \cap \mathbb{A}^n_{\mathbb{R}}$ is not contained in $S_{\mathbb{R}}$, there exists a point P of $\overline{K} \cap \mathbb{A}^n$ that does not belong to $S_{\mathbb{R}}$. Consider now an arbitrary connected component C of $S_{\mathbb{R}}$. Then C is a smooth, closed subvariety of $\mathbb{A}^n_{\mathbb{R}}$ whose distance to the point P is realized by a point M of C. Since P does not belong to $S_{\mathbb{R}}$, one has $M - P \neq 0$.

The square of the euclidean distance of any point X of $\mathbb{A}^n_{\mathbb{R}}$ to the point P is a real valued polynomial function defined on $\mathbb{A}^n_{\mathbb{R}}$ whose gradient in X is $2(X - P)$. Applying now the Lagrangian Multiplier Theorem (see e. g. [47]) to this function and the polynomial equations defining $S_{\mathbb{R}}$ we deduce that $M - P$ belongs to the orthogonal complement of the real tangent space $T_M(S_{\mathbb{R}})$ (observe that M is a smooth point of $S_{\mathbb{R}}$). The real trace $I_{M,\overline{K}} \cap \mathbb{A}^n_{\mathbb{R}}$ of the linear space $I_{M,\overline{K}}$ introduced in Subsection 2.2 consists of all elements of the orthogonal complement of $\overline{K} \cap \mathbb{A}^n_{\mathbb{R}}$ that are also orthogonal to M. Observe now that the linear space $T_M(S_{\mathbb{R}}) + (I_{M,\overline{K}} \cap \mathbb{A}^n_{\mathbb{R}})$ is strictly contained in $\mathbb{A}^n_{\mathbb{R}}$, since otherwise any point of $\mathbb{A}^n_{\mathbb{R}}$ would be orthogonal to $M - P$. On the other hand, $T_M(S_{\mathbb{R}}) + (I_{M,\overline{K}} \cap \mathbb{A}^n_{\mathbb{R}}) \neq \mathbb{A}^n_{\mathbb{R}}$ implies that $T_M(S_{\mathbb{R}}) \not\pitchfork (I_{M,\overline{K}} \cap \mathbb{A}^n_{\mathbb{R}})$ holds. From (5) we finally deduce that the point M belongs to the real affine dual polar variety $\widehat{W}_{\overline{K}}(S_{\mathbb{R}}) = \widehat{W}_{\overline{K}}(S) \cap \mathbb{A}^n_{\mathbb{R}}$ and therefore we have $C \cap \widehat{W}_{\overline{K}}(S_{\mathbb{R}}) \neq \emptyset$. $\square$

Observe that the statement of Proposition 2 becomes trivial for $\overline{K}$ belonging to the irrelevant part of $\overline{\mathcal{K}}$, since in this case $\widehat{W}_{\overline{K}}(S_{\mathbb{R}}) = S_{\mathbb{R}}$ holds.

3. EXTRINSIC ASPECTS OF POLAR VARIETIES

In this section we will describe more closely the generalized polar varieties of a closed subvariety S of $\mathbb{A}^n$, which is given by a system of polynomial equations. We suppose that these polynomial equations form a regular sequence and generate the ideal of definition of S. Let K be a "sufficiently generic" linear subvariety of $\mathbb{P}^n$ of dimension at most $n-p$. We will show that the polar variety $\widehat{W}_K(S)$ of S is either empty or equidimensional of expected codimension in S. We will describe $\widehat{W}_K(S)$ locally by transversal intersections of explicitly given hypersurfaces of $\mathbb{A}^n$ and, in case that S is smooth, globally by explicit polynomial equations, which generate the ideal of definition of $\widehat{W}_K(S)$.

3.1. Explicit description of affine polar varieties

Let $\mathbb{P}^n$ and $\mathbb{A}^n$ be the n-dimensional projective or affine space over $\mathbb{C}$ or $\mathbb{R}$, according to the context. As above, we consider $\mathbb{A}^n$ to be embedded in $\mathbb{P}^n$ in the usual way. For given complex or real numbers $x_0, \dots, x_n$ that are not all zero, $x := (x_0 : x_1 : \dots : x_n)$ denotes the corresponding point of the projective space $\mathbb{P}^n$. Moreover, for $x_0 = 1$ we denote the corresponding point of the affine space $\mathbb{A}^n$ by $(x_1, \dots, x_n) := (1 : x_1 : \dots : x_n)$. Let $X_0, \dots, X_n$ be indeterminates over $\mathbb{C}$ (or $\mathbb{R}$).

As of now we suppose that the given projective, purely p-codimensional variety V is defined by p nonzero forms $f_1, \dots, f_p$ over $\mathbb{C}$ (or $\mathbb{R}$) in the variables $X_0, \dots, X_n$. In other words, we suppose

$$V := V(f_1, \dots, f_p),$$

where $V(f_1, \dots, f_p)$ denotes the set of common zeros of $f_1, \dots, f_p$ in $\mathbb{P}^n$. Therefore, the homogeneous polynomials $f_1, \dots, f_p$ form a regular sequence in the polynomial ring $\mathbb{C}[X_0, , \dots, X_n]$ (or $\mathbb{R}[X_0, , \dots, X_n]$). Let $S := V \cap \mathbb{A}^n$ and assume that S is non-empty. The dehomogenizations of $f_1, \dots, f_p$ are denoted by

$$F_1 := f_1(1, X_1, \dots, X_n), \dots, F_p := f_p(1, X_1, \dots, X_n).$$

Observe that $F_1, \dots, F_p$ are nonzero polynomials in the variables $X_1, \dots, X_n$ over $\mathbb{C}$ (or $\mathbb{R}$). Thus we have

$$S = V \cap \mathbb{A}^n = V(F_1, \dots, F_p),$$

where $V(F_1, \dots, F_p)$ denotes the set of common zeros of $F_1, \dots, F_p$ in $\mathbb{A}^n$. Note that the polynomials $F_1, \dots, F_p$ form a regular sequence in $\mathbb{C}[X_1, , \dots, X_n]$ (or in $\mathbb{R}[X_1, , \dots, X_n]$).

The projective Jacobian of $f_1, \dots, f_p$ is denoted by

$$J(f_1, \dots, f_p) := \left[\frac{\partial f_j}{\partial X_k}\right]_{\substack{1 \leq j \leq p \\ 0 \leq k \leq n}} .$$

For any point x of $\mathbb{P}^n$ we write

$$J(f_1, \dots, f_p)(x) := \left[\frac{\partial f_j}{\partial X_k}(x)\right]_{\substack{1 \leq j \leq p \\ 0 \leq k \leq n}}$$

for the projective Jacobian of the polynomials $f_1, \dots, f_p$ at the point x. Similarly we denote the affine Jacobian of the polynomials $F_1, \dots, F_p$ by

$$J(F_1, \dots, F_p) := \left[\frac{\partial F_j}{\partial X_k}\right]_{\substack{1 \leq j \leq p \\ 1 \leq k \leq n}} ,$$

and we write for any point x of $\mathbb{A}^n$:

$$J(F_1, \dots, F_p)(x) := \left[\frac{\partial F_j}{\partial X_k}(x)\right]_{\substack{1 \leq j \leq p \\ 1 \leq k \leq n}} .$$

A point x of V (or of $V \cap \mathbb{A}^n$) is called $(f_1, \dots, f_p)$-regular (or $(F_1, \dots, F_p)$-regular) if the Jacobian $J(f_1, \dots, f_p)(x)$ (or $J(F_1, \dots, F_p)(x)$) has maximal rank p. Note that the $(f_1, \dots, f_p)$-regular points of V are always smooth points of V, but not vice-versa. For the sake of simplicity, we shall therefore suppose from now on that all smooth points of V are $(f_1, \dots, f_p)$-regular. In other words, we suppose that $f_1, \dots, f_p$ (and hence $F_1, \dots, F_p$) generate a radical ideal of its ambient polynomial ring. Any smooth point of S is therefore $(F_1, \dots, F_p)$-regular. On the other hand, by assumption, the polynomials $F_1, \dots, F_p$ form a regular sequence in $\mathbb{C}[X_1, \dots, X_n]$. Therefore, we conclude that the coordinate ring $\mathbb{C}[S]$ of the affine variety is Cohen–Macaulay.

Suppose for rest of this section that our ground field is $\mathbb{C}$. Next, we will generate local equations for the generalized polar varieties of the affine complete intersection variety S. To this end (and having in mind the algorithmic applications of our geometric considerations to real affine polar varieties in Section 4) we may restrict our attention to the case where H is the hyperplane at infinity of $\mathbb{P}^n$ (defined by the equation $X_0 = 0$) and where the given non-degenerate hyperquadric Q is defined by a quadratic form R, which can be represented as follows:

$$R(X_0, \dots, X_n) := {X_0}^2 + \sum_{k=1}^{n} 2c_k X_0 X_k + \sum_{k=1}^{n} {X_k}^2$$

with $c_1, \dots, c_n$ belonging to $\mathbb{C}$ or $\mathbb{R}$, according to the context. Observe that this representation of R implies the hyperquadrics Q and $Q \cap H$ to be non-degenerate in $\mathbb{P}^n$ and H, respectively. Further, observe that $Q \cap H$ is defined by the quadratic form $R_0(X_1, \dots, X_n) := \sum_{k=1}^n X_k^2 \in \mathbb{R}[X_1, \dots, X_n]$. Therefore, in particular, $Q \cap H_{\mathbb{R}}$ is represented by a positive definite quadratic form that induces the usual euclidean distance on $\mathbb{A}^n_{\mathbb{R}}$. Let us note that the special shape of R (and hence, of the quadratic form R_0 representing $Q \cap H_{\mathbb{R}}$) does not limitate the generality of the arguments which will follow. These may be applied mutatis mutandis to any non-degenerate hyperquadric whose intersection with the hyperplane at infinity H is still non-degenerate.

Fix now $1 \leq i \leq n-p$ and choose for each $1 \leq j \leq n-p-i+1$ a point $A_j = (a_{j,0} : \dots : a_{j,n})$ of $\mathbb{P}^n$ with $a_{j,0} = 0$ or $a_{j,0} = 1$ and $a_{j,1}, \dots, a_{j,n}$ generic (our genericity conditions will become evident in the sequel). By this choice, we may assume that the points $A_1, \dots, A_{n-p-i+1}$ span an $(n-p-i)$-dimensional linear subvariety $K := K^{n-p-i}$ of the projective space $\mathbb{P}^n$.

Let us consider an $(f_1, \dots, f_p)$-regular point $M = (x_0 : \dots : x_n)$ of V with $x_0 \neq 0$ and $M \notin K$. Then one easily sees that the $(n-p-i)$-dimensional linear subvariety $\langle M, K \rangle \cap H$ is spanned by the $n-p-i+1$ linearly independent points

$$x_0\, A_1 - a_{1,0}\, M, \dots, x_0\, A_{n-p-i+1} - a_{n-p-i+1,0} M.$$

Let $Y_1, \dots, Y_n$ be new indeterminates and let $\Theta := \sum_{k=1}^n X_k Y_k$, $\Theta \in \mathbb{R}[X_1, \dots, X_n, Y_1, \dots, Y_n]$, denote the (polarized) bilinear form associated with the hyperquadric $Q \cap H$. For $1 \leq j \leq n-p-i+1$, let $\ell_j \in \mathbb{C}[X_1, \dots, X_n]$ be defined by

$$\ell_j := \ell_j^{(x_0, \dots, x_n)} := \Theta(x_0\, a_{j,1} - a_{j,0}\, x_1, \dots, x_0\, a_{j,n} - a_{j,0}\, x_n,\ X_1, \dots, X_n)$$

and $G_j \in \mathbb{C}[X_0, X_1, \dots, X_n]$ by

$$G_j := G_j^{(x_0, \dots, x_n)} := x_0\, \ell_j^{(x_0, \dots, x_n)}(X_1, \dots, X_n) - X_0\, \ell_j^{(x_0, \dots, x_n)}(x_1, \dots, x_n).$$

Then the linear forms $\ell_1, \dots, \ell_{n-p-i+1}$ define the $(p+i-2)$-dimensional linear variety $(\langle M, K \rangle \cap H)^*$ in H and are therefore linearly independent. Moreover, the linear forms $G_1, \dots, G_{n-p-i+1}$ vanish at M and at any point of $(\langle M, K \rangle \cap H)^*$. Hence, they vanish at any point of the $(p+i-1)$-dimensional linear variety $\langle M, (\langle M, K \rangle \cap H)^* \rangle$. From the linear independence of $\ell_1, \dots, \ell_{n-p-i+1}$ one easily deduces the linear independence of the linear forms $G_1, \dots, G_{n-p-i+1}$. Therefore $G_1, \dots, G_{n-p-i+1}$ describe the linear variety $\langle M, (\langle M, K \rangle \cap H)^* \rangle$ used in (2) to define the generalized polar variety $\widehat{W}_K(V)$ (see Subsection 2.2).

Observe now that for any $1 \leq j \leq n-p-i+1$ the linear form $G_j^{(x_0, \dots, x_n)}$ can be written as

$$\begin{aligned} G_j^{(x_0, \dots, x_n)} &= -(X_0 - x_0)\, \ell_j^{(x_0, \dots, x_n)}(x_1, \dots, x_n) \\ &\qquad + x_0\, \ell_j^{(x_0, \dots, x_n)}(X_1 - x_1, \dots, X_n - x_n) \\ &= -(X_0 - x_0)\, \ell_j^{(x_0, \dots, x_n)}(x_1, \dots, x_n) + x_0 \sum_{k=1}^{n} (x_0\, a_{j,k} - a_{j,0}\, x_k)(X_k - x_k). \end{aligned}$$

Without loss of generality suppose that $x_0 = 1$ holds. Then $x := (x_1, \dots, x_n)$ is an $(F_1, \dots, F_p)$-regular point of $S = V \cap \mathbb{A}^n$ and the polynomial $G_j^{(1,x_1,\dots,x_n)}$ depends only on the variables $X_1, \dots, X_n$. Therefore, it makes sense to consider the Jacobian

$$T^{(i)} := T^{(i)}(X_1, \dots, X_n) := J(F_1, \dots, F_p, G_1^{(1,x_1,\dots,x_n)}, \dots, G_{n-p-i+1}^{(1,x_1,\dots,x_n)}),$$

whose entries belong to the polynomial ring $\mathbb{C}[X_1, \dots, X_n]$. Observe that the polynomial matrix $T^{(i)}$ is of the following explicit form, namely

$$T^{(i)} = \begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial F_p}{\partial X_1} & \cdots & \frac{\partial F_p}{\partial X_n} \\ a_{1,1} - a_{1,0}X_1 & \cdots & a_{1,n} - a_{1,0}X_n \\ \vdots & \vdots & \vdots \\ a_{n-p-i+1,1} - a_{n-p-i+1,0}X_1 & \cdots & a_{n-p-i+1,n} - a_{n-p-i+1,0}X_n \end{bmatrix}$$

with $a_{1,0}, \dots, a_{n-p-i+1,0}$ being elements of the set $\{1, 0\}$.

Moreover, observe that the condition

$$T_M(V) \not\pitchfork \langle M, (\langle M, K\rangle \cap H)^*\rangle$$

from (2) is equivalent to the vanishing of all $(n-i+1)$-minors of the $((n-i+1)\times n)$-matrix $T^{(i)}$ at the point x. Therefore the polynomials $F_1, \dots, F_p$ and the $(n-i+1)$-minors of $T^{(i)}$ define the generalized affine polar variety $\widehat{W}_K(S)$ outside of the locus S_{sing} (recall that by assumption all smooth points of S are $(F_1, \dots, F_p)$-regular). Let W be the closed subvariety of $\mathbb{A}^n$ defined by these equations. Then any irreducible component of $\widehat{W}_K(S)$ is an irreducible component of W. In particular, we have $\widehat{W}_K(S) \cap S_{\text{reg}} = W \cap S_{\text{reg}}$, and $\widehat{W}_K(S) = W$ if the affine variety S is smooth. Note, that i is the expected codimension of $\widehat{W}_K(S) = \widehat{W}_{K^{n-p-i}}(S)$ in S. These considerations lead to the following conclusion:

Lemma 3. Any irreducible component of $\widehat{W}_K(S) = \widehat{W}_{K^{n-p-i}}(S)$ has codimension at most i in S.

The proof is given in [3].

In the further analysis of the generalized affine polar variety $\widehat{W}_K(S)$ we shall distinguish from time to time two cases, namely the case that the linear projective variety $K = K^{n-p-i}$, spanned by the given points $A_1, \dots, A_{n-p-i+1}$ of $\mathbb{P}^n$, is contained in the hyperplane at infinity H of $\mathbb{P}^n$, and the case that K is not contained in H. If K is contained in H, we have $a_{1,0} = \cdots = a_{n-p-i+1,0} = 0$ and if K is not contained in H, we may suppose without loss of generality that $a_{n-p-i+1,0} = 1$ holds.

Let us now discuss the particular case that $K = K^{n-p-i}$ is contained in the hyperplane at infinity H of $\mathbb{P}^n$. Let $\overline{S}$ be the Zariski closure of the affine variety S in the projective space $\mathbb{P}^n$ and let $L := K^*$. Thus L is a $(p+i-2)$-dimensional linear projective subvariety of H, the projective variety $\overline{S}$ is of pure codimension p in $\mathbb{P}^n$ and none of the irreducible components of $\overline{S}$ is contained in H. Furthermore, we have $K = L^*$ and $S = \overline{S} \cap \mathbb{A}^n$. From (3) we deduce now that $\widehat{W}_K(\overline{S}) = W_L(\overline{S})$ holds. This implies

$$\widehat{W}_K(S) = \widehat{W}_K(\overline{S}) \cap \mathbb{A}^n = W_L(\overline{S}) \cap \mathbb{A}^n = W_L(S).$$

Therefore $\widehat{W}_K(S)$ is the *cylindric* polar variety associated with the $(p + i - 2)$-dimensional linear subvariety L of the hyperplane at infinity H of $\mathbb{P}^n$.

We now return to the analysis of the general situation. Let be given a complex $((n - p - i + 1) \times (n + 1))$-matrix

$$b := \begin{bmatrix} b_{1,0} & \cdots & b_{1,n} \\ \vdots & \vdots & \vdots \\ b_{n-p-i,0} & \cdots & b_{n-p-i,n} \\ b_{n-p-i+1,0} & \cdots & b_{n-p-i+1,n} \end{bmatrix}$$

with $b_{n-p-i+1,0} = a_{n-p-i+1,0}, \dots, b_{n-p-i+1,n} = a_{n-p-i+1,n}$ and with $b_{1,0}, \dots, b_{n-p-i,0}$ being elements of the set $\{1, 0\}$ and suppose that b has maximal rank $n - p - i + 1$ and that the entries $a_{n-p-i+1,n-i+1}, \dots, a_{n-p-i+1,n}$ are generic with respect to the other entries of b (e.g., $a := (a_{j,k})_{\substack{1 \le j \le n-p-i+1 \\ 0 \le k \le n}}$ is such a $((n - p - i + 1) \times (n + 1))$-matrix).

Let $K(b)$ be the linear subvariety of $\mathbb{P}^n$ spanned by the $n - p - i + 1$ projective points

$$(b_{1,0} : \cdots : b_{1,n}), \dots, (b_{n-p-i+1,0} : \cdots : b_{n-p-i+1,n}).$$

Observe that $K(b)$ is $(n - p - i)$-dimensional and that $K(a) = K^{n-p-i}$ holds. For the sake of notational succinctness let us use, for $1 \le j \le n-p-i+1$ and $1 \le k \le n$, the abbreviation

$$r_{j,k}^{(b)}(X_k) := b_{j,k} - b_{j,0} X_k.$$

We consider now the polynomial $((n - i + 1) \times n)$-matrix

$$T_b^{(i)} = \begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial F_p}{\partial X_1} & \cdots & \frac{\partial F_p}{\partial X_n} \\ r_{1,1}^{(b)}(X_1) & \cdots & r_{1,n}^{(b)}(X_n) \\ \vdots & \vdots & \vdots \\ r_{n-p-i+1,1}^{(b)}(X_1) & \cdots & r_{n-p-i+1,n}^{(b)}(X_n) \end{bmatrix}.$$

Observe that $T_a^{(i)} = T^{(i)}$ holds.

Let $s \in \{n - i, n - i + 1\}$. For any ordered sequence $(k_1, \dots, k_s)$ of different elements of the set $\{1, \dots, n\}$ we denote by $M^{(b)}(\{k_1, \dots, k_s\}) := M^{(b)}(k_1, \dots, k_s)$

the minor that corresponds to the first s rows and to the columns $k_1, \dots, k_s$ of the matrix $T_b^{(i)}$.

Let us fix an ordered sequence I of $n-i$ different elements of the set $\{1, \dots, n\}$, say $I := (1, \dots, n-i)$, and let us consider the upper $(n-i)$-minor

$$m^{(b)} := M^{(b)}(I) := \det \begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_{n-i}} \\ \vdots & \vdots & \vdots \\ \frac{\partial F_p}{\partial X_1} & \cdots & \frac{\partial F_p}{\partial X_{n-i}} \\ r_{1,1}^{(b)}(X_1) & \cdots & r_{1,n}^{(b)}(X_{n-i}) \\ \vdots & \vdots & \vdots \\ r_{n-p-i,1}^{(b)}(X_1) & \cdots & r_{n-p-i,n}^{(b)}(X_{n-i}) \end{bmatrix}$$

of the matrix $T_b^{(i)}$.

Note that $m^{(b)}$ depends only on the entries $b_{j,k}$, $1 \le j \le n-p-i$, $0 \le k \le n-i$, of the matrix b. In what follows we will assume that b satisfies the additional condition $m^{(b)} \neq 0$. Let us assume without loss of generality that the polynomial $(p \times p)$-matrix

$$\begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_p} \\ \vdots & \vdots & \vdots \\ \frac{\partial F_p}{\partial X_1} & \cdots & \frac{\partial F_p}{\partial X_p} \end{bmatrix}$$

is non-singular. Then, in particular, the genericity of the entries $a_{j,k}$ of the $((n-p-i+1) \times (n+1))$-matrix a implies that $m^{(a)}$ is a nonzero element of the polynomial ring $\mathbb{C}[X_1, \dots, X_n]$. Therefore the matrix a satisfies this condition.

The Exchange Lemma of [4] implies that, for any ordered sequence $(k_1, \dots, k_{n-i+1})$ of different elements of the set $\{1, \dots, n\}$, the identity

$$m^{(b)} M^{(b)}\left(k_1, \dots, k_{n-i+1}\right)$$

$$= \sum_{l \in \{k_1, \dots, k_{n-i+1}\} \setminus \{1, \dots, n-i\}} \mu_l \, M^{(b)}(\{k_1, \dots, k_{n-i+1}\} \setminus \{l\}) \, M^{(b)}(1, \dots, n-i, l) \quad (6)$$

holds with $\mu_l \in \{-1, 0, 1\}$, for any index $l \in \{k_1, \dots, k_{n-i+1}\} \setminus \{1, \dots, n-i\}$.

Let us abbreviate $M_{n-i+1}^{(b)} := M^{(b)}(1, \dots, n-i+1)$, $M_{n-i+2}^{(b)} := M^{(b)}(1, \dots, n-i, n-i+2), , \dots, M_n^{(b)} := M^{(b)}(1, \dots, n-i, n)$. Assume now that there is given a point x of S satisfying the conditions $m^{(b)}(x) \neq 0$ and

$$M_{n-i+1}^{(b)}(x) = \cdots = M_n^{(b)}(x) = 0. \quad (7)$$

Then we infer from (6) that $M^{(b)}(k_1, \dots, k_{n-i+1})(x) = 0$ holds for any ordered sequence $(k_1, \dots, k_{n-i+1})$ of different elements of the set $\{1, \dots, n\}$. This means that all $(n-i+1)$-minors of the matrix $T_b^{(i)}$ vanish at the point x. Since $m^{(b)}(x) \neq 0$ implies $x \in S_{\mathrm{reg}}$, we conclude that x belongs to the polar variety $\widehat{W}_{K(b)}(S)$. On the other hand, any point x of $\widehat{W}_{K(b)}(S)$ satisfies the condition (7). Therefore, the polar variety $\widehat{W}_{K(b)}(S)$ is defined by the equations $F_1, \ldots, F_p, M_{n-i+1}^{(b)}, \ldots, M_n^{(b)}$ outside of the locus $V(m^{(b)})$.

Let $Z_{n-i+1}, \ldots, Z_n$ be new indeterminates and consider the $((n-i+1) \times n)$-matrix

$$\begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_{n-i}} & \frac{\partial F_1}{\partial X_{n-i+1}} & \cdots & \frac{\partial F_1}{\partial X_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial F_p}{\partial X_1} & \cdots & \frac{\partial F_p}{\partial X_{n-i}} & \frac{\partial F_p}{\partial X_{n-i+1}} & \cdots & \frac{\partial F_p}{\partial X_n} \\ r_{1,1}^{(b)}(X_1) & \cdots & r_{1,n-i}^{(b)}(X_{n-i}) & r_{1,n-i+1}^{(b)}(X_{n-i+1}) & \cdots & r_{1,n}^{(b)}(X_n) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ r_{n-p-i,1}^{(b)}(X_1) & \cdots & r_{n-p-i,n-i}^{(b)}(X_{n-i}) & r_{n-p-i,n-i+1}^{(b)}(X_{n-i+1}) & \cdots & r_{n-p-i,n}^{(b)}(X_n) \\ r_{n-p-i+1,1}^{(b)}(X_1) & \cdots & r_{n-p-i+1,n-i}^{(b)}(X_{n-i}) & Z_{n-i+1} - b_{n-p-i+1,0}X_{n-i+1} & \cdots & Z_n - b_{n-p-i+1,0}X_n \end{bmatrix}.$$

Let $\widetilde{M}_{n-i+1}^{(b)}, \widetilde{M}_{n-i+2}^{(b)}, \ldots, \widetilde{M}_n^{(b)}$ denote the $(n-i+1)$-minors of this matrix obtained by successively selecting the columns $1, \ldots, n-i$, $n-i+1$, then $1, \ldots, n-i, n-i+2$, up to, finally, the columns $1, \ldots, n-i, n$. Let $U_b := \mathbb{A}^n \setminus V(m^{(b)})$ and observe that U_b is non-empty since $m^{(b)}$, by assumption, is a non-zero polynomial.
Now we consider the following morphism of smooth, affine varieties

$$\Phi_i^{(b)} : U_b \times \mathbb{A}^i \to \mathbb{A}^p \times \mathbb{A}^i,$$

defined by

$$\Phi_i^{(b)}(x,z) := \Big(F_1(x), \ldots, F_p(x), \widetilde{M}_{n-i+1}^{(b)}(x,z), \ldots, \widetilde{M}_n^{(b)}(x,z) \Big)$$

for any pair of points $x \in U_b$, $z \in \mathbb{A}^i$. Analyzing now the Jacobian $J(\Phi_i^{(b)})(x,z)$ of $\Phi_i^{(b)}$ at an arbitrary point (x,z) of $(\Phi_i^{(b)})^{-1}(0, \ldots, 0)$ with $x \in U_b$ and $z \in \mathbb{A}^i$ one concludes the next Lemma.

Lemma 4. The origin $(0, \ldots, 0)$ of the affine space $\mathbb{A}^p \times \mathbb{A}^i$ is a regular value of the morphism $\Phi_i^{(b)}$.

Applying now the Weak–Transversality–Theorem of Thom–Sard (see e.g. [17]) to $\Phi_i^{(b)}$, we deduce from Lemma 4 that there exists a residual dense set Ω of $\mathbb{A}^i$ such that, for any point $z \in \Omega$, the polynomials

$$F_1, \ldots, F_p, \ \widetilde{M}_{n-i+1}^{(b)}(X_1, \ldots, X_n, z), \ldots, \widetilde{M}_n^{(b)}(X_1, \ldots, X_n, z)$$

of $\mathbb{C}[X_1, \ldots, X_n]$ intersect transversally in any of their common zeros outside of the positive codimensional, Zariski closed locus $\mathbb{A}^n \setminus U_b$ (we call the subset Ω of

$\mathbb{A}^i$ *residual dense* if Ω contains with respect to the Hermitian topology of $\mathbb{A}^i$ the intersection of a countable family of open dense sets). From the genericity of the entries $b_{n-p-i+1,n-i+1} = a_{n-p-i+1,n-i+1}, \dots, b_{n-p-i+1,n} = a_{n-p-i+1,n}$ of the matrix $T_b^{(i)}$ we deduce that we may assume without loss of generality that the point $\alpha := (a_{n-p-i+1,n-i+1}, \dots, a_{n-p-i+1,n})$ belongs to the set Ω. Observing now that $M_{n-i+1}^{(b)} = \widetilde{M}_{n-i+1}^{(b)}(X_1, \dots, X_n, \alpha), \dots, M_n^{(b)} = \widetilde{M}_n^{(b)}(X_1, \dots, X_n, \alpha)$ holds, we conclude that the equations $F_1, \dots, F_p, M_{n-i+1}^{(b)}, \dots, M_n^{(b)}$ intersect transversally at any point of $\widehat{W}_{K(b)}(S)$ not belonging to the locus $V(m^{(b)})$ and that such points exist. We have therefore shown the following statement:

Lemma 5. Let the notations and assumptions be as before. Then the polynomial $((n-i+1) \times n)$-matrix $T_b^{(i)}$ satisfies the following condition:
The equations $F_1, \dots, F_p$, $M_{n-i+1}^{(b)}, \dots, M_n^{(b)}$ define the generalized polar variety $\widehat{W}_{K(b)}(S)$ outside of the locus $V(m^{(b)})$ and intersect transversally in any point of the affine variety $\widehat{W}_{K(b)}(S) \setminus V(m^{(b)})$. In particular, $\widehat{W}_{K(b)}(S) \setminus V(m^{(b)})$ is either empty or a smooth, complete intersection variety of dimension $n-p-i$.

Observe that all upper $(n-i)$-minors of $T^{(i)}$ vanish at a given $(F_1, \dots, F_p)$-regular point x of S if and only if x belongs to the polar variety $\widehat{W}_{K^{n-p-i-1}}(S)$ which is contained in $\widehat{W}_K(S) = \widehat{W}_{K^{n-p-i}}(S)$. Applying now Lemma 5 to any upper $(n-i)$-minor of the matrix $T^{(i)} = T_a^{(i)}$ we conclude:

Proposition 6. For any $(F_1, \dots, F_p)$-regular point x of $\widehat{W}_{K^{n-p-i}}(S) \setminus \widehat{W}_{K^{n-p-i-1}}(S)$ there exist indices $1 \leq k_1 < \dots < k_{n-i} \leq n$ with the following property:
Let $m := M(\{k_1, \dots, k_{n-i}\})$ be the upper $(n-i)$-minor of the polynomial $((n-i+1) \times n)$-matrix $T^{(i)}$ determined by the columns $(k_1, \dots, k_{n-i})$, let $\{1, \dots, n\} \setminus \{k_1, \dots, k_{n-i}\} = \{k_{n-i+1}, \dots, k_n\}$ and let $M_{n-i+1} := M(\{k_1, \dots, k_{n-i}, k_{n-i+1}\})$, $M_{n-i+2} := M(\{k_1, \dots, k_{n-i}, k_{n-i+2}\}), \dots, M_n := M(\{k_1, \dots, k_{n-i}, k_n\})$. Then the minor m does not vanish at the point x and the equations $F_1, \dots, F_p, M_{n-i+1}, \dots, M_n$ intersect transversally at x. Moreover, the polynomials $F_1, \dots, F_p$, $M_{n-i+1}, \dots, M_n$ define the polar variety $\widehat{W}_{K^{n-p-i}}(S)$ outside of the locus $V(m)$.

Fix for the moment $1 \leq j \leq n-p-i+1$ and let E_j be the $(n-p-i-1)$-dimensional linear projective subvariety of $\mathbb{P}^n$ spanned by the points $A_1, \dots, A_{j-1}, A_{j+1}, \dots, A_{n-p-i+1}$. In particular, we have $E_{n-p-i+1} = K^{n-p-i-1}$.

From the generic choice of the complex numbers $a_{j,k}$, $1 \leq j \leq n-p-i+1$, $1 \leq k \leq n$ we infer that Proposition 6 remains still valid if we replace in its statement the upper $(n-i)$-minor $m = M(k_1, \dots, k_{n-i})$ by the $(n-i)$-minor of $T^{(i)}$ given by the rows $1, \dots, p+j-1, p+j+1, \dots, n-p-i+1$ and the columns $k_1, \dots, k_{n-i}$ and the polar variety $\widehat{W}_{K^{n-p-i-1}}(S)$ by $\widehat{W}_{E_j}(S)$.

Let $\Delta_i := \bigcap_{1 \leq j \leq n-p-i+1} \widehat{W}_{E_j}(S)$. Then Δ_i is contained in $\widehat{W}_{K^{n-p-i+1}}(S)$ and Proposition 6 implies that, outside of the locus Δ_i, the polar variety $\widehat{W}_{K_{n-p-i}}(S)$ is smooth and of pure codimension i in S. It is not too difficult to deduce from Proposition 6 that the codimension of Δ_i in S is at least $2i+1$. Hence, for $\frac{n-p-1}{2} < i \leq n-p$, the algebraic variety Δ_i is empty and therefore, the polar variety $\widehat{W}_{K^{n-p-i}}(S)$ is smooth in any of its $(F_1, \dots, F_p)$-regular points. In the next subsection we will show this property of $\widehat{W}_{K^{n-p-i}}(S)$ for any $0 \leq i \leq n-p$ (see Theorem 9 below).

Finally, let us consider the case $i := n-p$. Observe that $T^{(n-p)}$ is a $((p+1) \times n)$-matrix which contains the Jacobian $J(F_1, \dots, F_p)$ as its first p rows. Thus, for any $(F_1, \dots, F_p)$-regular point x of $\widehat{W}_{K^0}(S)$, there exists an upper p-minor m of $T^{(n-p)}$ with $m(x) \neq 0$. Therefore, we define $\widehat{W}_{K^{-1}}$ as the empty set. Thus, in particular, Δ_{n-p} is empty and this implies that $\widehat{W}_{K^0}(S)$ is smooth and of pure codimension $(n-p)$ outside of the locus S_{sing}. This leads us to the following statement which is shown in [3].

Lemma 7. The generalized polar variety $\widehat{W}_{K^0}(S)$ is either empty or of (expected) codimension $n-p$ in S (i.e., $\widehat{W}_{K^0}(S)$ contains at most finitely many points). Moreover, $\widehat{W}_{K^0}(S)$ is contained in S_{reg}.

3.2. Geometric conclusions

The geometric main outcome of this section is Theorem 9 below, which is a basic result for generalized affine polar varieties in the reduced complete intersection case. The proof of this result requires two technical statements, namely Lemma 7 and Proposition 8 below.

Let the assumptions and notations be as before. Proposition 6 and Lemma 7 imply our next result whose proof is contained in [3].

Proposition 8. Suppose that the generalized affine polar variety $\widehat{W}_{K^{n-p-i}}(S)$ is non-empty. Then $\widehat{W}_{K^{n-p-i}}(S)$ is of pure codimension i in S (and therefore, the codimension of $\widehat{W}_{K^{n-p-i}}(S)$ in S coincides with the expected one). Moreover, for each irreducible component C of $\widehat{W}_{K^{n-p-i}}(S)$ there exists an upper $(n-i)$-minor m of $T^{(i)}$ such that m does not vanish identically on C. In particular, no irreducible component of $\widehat{W}_{K^{n-p-i}}(S)$ is contained in $\widehat{W}_{K^{n-p-i-1}}(S)$.

Let us remark that, for a generic choice of the parameters $a_{j,k}$, $1 \leq j \leq n-p$, $1 \leq k \leq n$, Propositions 6 and 8 yield a local description of the generalized polar varieties of a given complete intersection variety by polynomial equations.

Theorem 9. Let the assumptions and notations be as at the beginning of Section 3. Suppose furthermore that any point of S is $(F_1, \dots, F_p)$-regular and that the affine polar variety $\widehat{W}_{K^{n-p-i}}(S)$ is non-empty. Then the (radical) ideal of definition of $\widehat{W}_{K^{n-p-i}}(S)$ in $\mathbb{C}[X_1, \dots, X_n]$ is Cohen–Macaulay and generated by $F_1, \dots, F_p$ and all $(n-i+1)$-minors of $T^{(i)}$.

Proof. Let $\mathfrak{a}$ be the determinantal ideal of $\mathbb{C}[S]$ induced by the $(n-i+1)$-minors of $T^{(i)}$. Since, by assumption, any point of S is $(F_1, \ldots, F_p)$-regular, the ideal $\mathfrak{a}$ defines the (non-empty) polar variety $\widehat{W}_{K^{n-p-i}}(S)$ in S. From Proposition 8 we infer that any isolated prime component of the ideal $\mathfrak{a}$ has height i. Therefore, since $\mathbb{C}[S]$ is a Cohen–Macaulay ring, the grade of the ideal $\mathfrak{a}$ coincides with its height i. Observe that $\mathfrak{a}$ is the ideal generated by the maximal minors of the $((n-i+1) \times n)$-matrix induced by $T^{(i)}$ in $\mathbb{C}[S]$.

From [9], Theorem 2.7 and Proposition 16.19 we conclude now that the determinantal ideal $\mathfrak{a}$ is Cohen–Macaulay (compare [18, 19], and [20], Section 18.5 for the general context of determinantal ideals in a Cohen–Macaulay ring). In particular, the ideal $\mathfrak{a}$ has no embedded associated primes and the ideal generated in $\mathbb{C}[X_1, \ldots, X_n]$ by $F_1, \ldots, F_p$ and all $(n-i+1)$-minors of $T^{(i)}$ is Cohen–Macaulay.

Let $\mathfrak{p}$ be an arbitrary prime component of the ideal $\mathfrak{a}$. Then $\mathfrak{p}$ is an isolated component of $\mathfrak{a}$ and has height i. From Proposition 8 we infer that there exists an upper $(n-i)$-minor m of $T^{(i)}$ that does not vanish identically on the irreducible component of $\widehat{W}_{K^{n-p-i}}(S)$ defined by $\mathfrak{p}$ in S. Considering the coordinate ring $\mathbb{C}[S]$ as a $\mathbb{C}[X_1, \ldots, X_n]$-module, we may localize $\mathbb{C}[S]$ and the ideals $\mathfrak{a}$ and $\mathfrak{p}$ by the non-zero polynomial m, obtaining thus non-trivial ideals $\mathfrak{a}_m$ and $\mathfrak{p}_m$ of $\mathbb{C}[S]_m$. Since the variety S is smooth by assumption, we deduce from Lemma 5 that $(\mathbb{C}[S]/\mathfrak{a})_m$ is a regular ring. This implies that $\mathfrak{p}_m$ is a primary component of the ideal $\mathfrak{a}_m$, and therefore, $\mathfrak{p}$ is also a primary component of the ideal $\mathfrak{a}$. Since $\mathfrak{p}$ was chosen as an arbitrary prime component of $\mathfrak{a}$, we conclude that the ideal $\mathfrak{a}$ is radical. Therefore $\mathfrak{a}$ is the ideal of definition of the affine variety $\widehat{W}_{K^{n-p-i}}(S)$ in S. We conclude now that the polynomial ideal generated by $F_1, \ldots, F_p$ and all $(n-i+1)$-minors of $T^{(i)}$ is Cohen–Macaulay and the ideal of definition of $\widehat{W}_{K^{n-p-i}}(S)$ in $\mathbb{C}[X_1, \ldots, X_n]$. $\square$

In the case of classic, cylindric affine polar varieties (i. e., in the case $K^{n-p-i} \subset H$) Propositions 6 and 8 and Theorem 9 are nothing else but a careful reformulation of [4], Theorem 1. In terms of standard algebraic geometry, Theorem 9 implies the following result:

Corollary 10. Let S be a smooth, pure p-dimensional closed subvariety of $\mathbb{A}^n$. Let K be a linear, projective subvariety of $\mathbb{P}^n$ of dimension $(n-p-i)$ with $1 \leq i \leq n-p$. Suppose that K is generated by $n-p-i+1$ many points $A_1 = (a_{1,0} : \cdots : a_{1,n}), \ldots, A_j = (a_{j,0} : \cdots : a_{j,n}), \ldots, A_{n-p-i+1} = (a_{n-p-i+1,0} : \cdots : a_{n-p-i+1,n})$ of $\mathbb{P}^n$ with $a_{j,0} = 0$ or $a_{j,0} = 1$ and $a_{j,1}, \ldots, a_{j,n}$ generic for any $1 \leq j \leq n-p-i+1$. Then $\widehat{W}_K(S)$ is either empty or a Cohen–Macaulay variety of pure codimension i in S.

In the case of classic, cylindric affine polar varieties (i. e., in the case $K^{n-p-i} \subset H$), Corollary 10 seems to be folklore (compare [48], Corollaire 1.3.2 and Définition 1.4).

Observe that Corollary 10 remains mutatis mutandis true if we replace in its formulation the affine variety S by the projective variety V and if V is smooth.

4. REAL POLYNOMIAL EQUATION SOLVING

The geometric and algebraic results of Section 2 allow us to enlarge the range of applications of the new generation of elimination procedures for real algebraic varieties introduced in [4] and [5].

Let S be a pure p-codimensional and $\mathbb{Q}$-definable, closed algebraic subvariety of the n-dimensional, complex, affine space $\mathbb{A}^n_{\mathbb{C}}$ and suppose that S is given by p polynomial equations $F_1, \dots, F_p$ of degree at most d, forming a regular sequence in $\mathbb{Q}[X_1, \dots, X_n]$. Assume that, for any $1 \leq k \leq p$, $F_1, \dots, F_k$ generate a radical ideal. Moreover, suppose that the real algebraic variety $S_{\mathbb{R}} := S \cap \mathbb{A}^n_{\mathbb{R}}$ is non-empty and smooth.

In this section we will describe an elimination procedure that finds a representative point for each connected component of $S_{\mathbb{R}}$. The complexity of this algorithm will be of *intrinsic* type, depending on the maximal geometric degree of the dual polar varieties of S that are associated with the external flag of a generic, $\mathbb{Q}$-definable flag contained in the hyperplane at infinity H of the n-dimensional, projective space $\mathbb{P}^n_{\mathbb{C}}$.

In order to explain this algorithm, let us first discuss these polar varieties and then the data structure and the algorithmic model we will use.

Let us choose a rational point $u = (u_1, \dots, u_n)$ of $\mathbb{A}^n \setminus S_{\mathbb{R}}$ with generic coordinates $u_1, \dots, u_n$ and, generically in the hyperplane at infinity H, a flag $\mathcal{L}$ of $\mathbb{Q}$-definable, linear subvarieties of $\mathbb{P}^n_{\mathbb{C}}$, namely

$$\mathcal{L}: \qquad L^0 \subset L^1 \subset \cdots \subset L^{p-1} \subset \cdots \subset L^{n-2} \subset L^{n-1} \subset \mathbb{P}^n_{\mathbb{C}}$$

with $L^{n-1} = H$. Let Q_u be the hyperquadric of $\mathbb{P}^n_{\mathbb{C}}$ defined by the quadratic form

$$R_u(X_0, X_1, \dots, X_n) := X_0^2 - 2 \sum_{1 \leq k \leq n} u_k X_0 X_k + \sum_{1 \leq k \leq n} X_k^2.$$

Observe that the hyperquadrics Q_u and $Q_u \cap H$ are non-degenerate in $\mathbb{P}^n_{\mathbb{C}}$ and H, respectively, and that $Q_u \cap H_{\mathbb{R}}$ is represented by the positive definite quadratic form $R_0(X_1, \dots, X_n) = \sum_{1 \leq k \leq n} X_k^2$ that introduces the usual euclidean distance on $\mathbb{A}^n_{\mathbb{R}}$. One verifies immediately that the point $(1 : u_1 : \cdots : u_n) \in \mathbb{P}^n$ spans, with respect to the hyperquadric Q_u, the dual space of $L^{n-1} = H$.

Let us consider the external flag $\overline{\mathcal{K}}$ associated with $\mathcal{L}$, namely

$$\overline{\mathcal{K}}: \qquad \mathbb{P}^n_{\mathbb{C}} \supset \overline{K}^{n-1} \supset \overline{K}^{n-2} \supset \cdots \supset \overline{K}^{n-p-1} \supset \cdots \supset \overline{K}^1 \supset \overline{K}^0,$$

with $\overline{K}^{n-p-i} := (L^{p+i-1})^{\vee}$, for $1 \leq i \leq n-p$, and with an arbitrarily chosen irrelevant part

$$\overline{K}^{n-1} \supset \overline{K}^{n-2} \supset \cdots \supset \overline{K}^{n-p}.$$

Observe that $\overline{K}^0$ consists of the rational point $(1 : u_1 : \cdots : u_n) \in \mathbb{P}^n$.
Let $1 \leq i \leq n-p$ and recall that the $(p+i-1)$-dimensional, $\mathbb{Q}$-definable, linear subvariety L^{p+i-1} was chosen generically in the hyperplane at infinity H of $\mathbb{P}^n_{\mathbb{C}}$. Therefore, $\overline{K}^{n-p-i}$ is an $(n-p-i)$-dimensional, $\mathbb{Q}$-definable, linear subvariety of $\mathbb{P}^n_{\mathbb{C}}$, which we may imagine to be spanned by $n-p-i+1$ rational points

$$A_1 = (a_{1,0} : \cdots : a_{1,n}), \dots, A_{n-p-i+1} = (a_{n-p-i+1,0} : \cdots : a_{n-p-i+1,n})$$

of $\mathbb{P}^n_{\mathbb{C}}$ with $a_{1,1} = u_1, \ldots, a_{1,n} = u_n$ and $a_{j,1}, \ldots, a_{j,n}$ generic, for $2 \leq j \leq n-p-i+1$, and $a_{1,0} = 1, a_{2,0} = \cdots = a_{n-p-i,0} = 0$. Observe that the point u belongs to $\overline{K}^{n-p-i} \cap \mathbb{A}^n$ and is not contained in $S_{\mathbb{R}}$. Thus Proposition 2 implies that the real affine dual polar variety $\widehat{W}_{\overline{K}^{n-p-i}}(S_{\mathbb{R}})$ contains at least one representative point of each connected component of $S_{\mathbb{R}}$.

In particular, the complex affine dual polar variety $\widehat{W}_{\overline{K}^{n-p-i}}(S)$ is not empty. From the generic choice of the point u and of the flag $\mathcal{L}$ we deduce now that Proposition 6, Lemma 7, Proposition 8 and Theorem 9 are applicable to the generalized, affine, polar variety $\widehat{S}_i := \widehat{W}_{\overline{K}^{n-p-i}}(S)$. Observe that $\widehat{S}_i$ is $\mathbb{Q}$-definable and of pure codimension i in S. According to the terminology introduced in Section 1, we call $\widehat{S}_i$ the ith affine polar variety of S associated with the flag $\overline{\mathcal{K}}$. Observe that $\widehat{S}_i$ is non-empty and intersects each connected component of the real variety $S_{\mathbb{R}}$.

Thus, in particular, $\widehat{S}_{n-p}$ is a $\mathbb{Q}$-definable, zero-dimensional, algebraic variety that contains a representative point for any connected component of $S_{\mathbb{R}}$.

We will now analyse the polar variety $\widehat{S}_i$ more closely. For $2 \leq j \leq n-p$ let $\Lambda_j := \sum_{1 \leq l \leq n} a_{j,l} X_l$ and, for $1 \leq k \leq n$, choose $\zeta_k = (\zeta_{k,1}, \ldots, \zeta_{k,n}) \in \mathbb{Q}^n$ such that $\zeta_1, \ldots, \zeta_{p+1}$ are zeros of $\Lambda_2, \ldots, \Lambda_{n-p}$ and that $\zeta_1, \ldots, \zeta_n$ form a $\mathbb{Q}$-vector space basis of $\mathbb{Q}^n$ (recall that the coefficients of the forms $\Lambda_2, \ldots, \Lambda_{n-p}$ are generic). Let B be the transposed matrix of $(\zeta_{j,k})_{1 \leq j,\, k \leq n}$. For $1 \leq k \leq n$, let $Z_k = \sum_{1 \leq j \leq n} \tilde{\zeta}_{k,j} X_j$, where $(\tilde{\zeta}_{k,1}, \ldots, \tilde{\zeta}_{k,n})$ is the kth row of the inverse of the transposed matrix of B. Let $Z := (Z_1, \ldots, Z_n)$. As in Section 3, consider now the polynomial $((n-i+1) \times n)$-matrix

$$T^{(i)} = \begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial F_p}{\partial X_1} & \cdots & \frac{\partial F_p}{\partial X_n} \\ a_{1,1} - a_{1,0}X_1 & \cdots & a_{1,n} - a_{1,0}X_n \\ \vdots & \vdots & \vdots \\ a_{n-p-i+1,1} - a_{n-p-i+1,0}X_1 & \cdots & a_{n-p-i+1,n} - a_{n-p-i+1,0}X_n \end{bmatrix}.$$

Observe that $T^{(i)}B$ is of the following form:

$$T^{(i)}B = \begin{bmatrix} \multicolumn{2}{c}{J(F_1(Z), \ldots, F_p(Z))} \\ b_1 - c_1X_1 \cdots b_{p+i} - c_{p+i}X_{p+i} & b_{p+i+1} - c_{p+i+1}X_{p+i+1} \cdots b_n - c_nX_n \\ O_{n-p-i,\, p+i} & (\,*\,)_{n-p-i,\, n-p-i} \end{bmatrix},$$

where $b_1, \ldots, b_n$ and the entries of $(\,*\,)_{n-p-i,\, n-p-i}$ are all generic rational numbers and where $c_1, \ldots, c_n$ belong to $\mathbb{Q} \setminus \{0\}$. For the sake of simplicity we shall suppose that $c_1 = \cdots = c_n = 1$ (this assumption does not change the following argumentation substantially).

Thus the $(n-i+1)$-minors of the matrix $T^{(i)}B$, which are not identically zero, are scalar multiples of the $(p+1)$-minors selected among the columns $1, \ldots, p+i$ of the $((p+1) \times n)$-matrix

$$\theta := \begin{bmatrix} \multicolumn{3}{c}{J(F_1(Z), \ldots, F_p(Z))} \\ b_1 - X_1 & \cdots & b_n - X_n \end{bmatrix}$$

and vice versa.

Consider now an arbitrary p-minor m of the Jacobian $J(F_1(Z), \dots, F_p(Z))$. For the sake of definiteness let us suppose that m is given by the columns $1, \dots, p$. For $p+1 \le j \le p+i$, let M_j be the $(p+1)$-minor of the matrix θ given by the columns $1, \dots, p, j$.

Then we deduce from the Exchange Lemma of [4] that, for any point x of S with $m(x) \neq 0$, the condition $M_{p+1}(x) = \cdots = M_{p+i}(x) = 0$ is satisfied if and only if all $(p+1)$-minors of θ vanish at x.

Taking into account that $m(x) \neq 0$ implies the $(F_1, \dots, F_p)$-regularity of the point $x \in S$, we conclude that the equations $F_1, \dots, F_p, M_{p+1}, \dots, M_{p+i}$ define the polar variety $\widehat{S}_i$ outside of the locus $V(m)$.

Moreover, from Theorem 9 and its proof we deduce that the polynomials $F_1, \dots, F_p$, $M_{p+1}, \dots, M_{p+i}$ generate the radical ideal of definition of the affine variety $S_i \setminus V(m)$ in $\mathbb{Q}[X_1, \dots, X_n]_m$.

For $1 \le h \le p$, let S_h be the affine variety defined by the equations $F_1, \dots, F_h$. Denote by $\deg S_h$ the geometric *degree* of S_h in the set-theoretic sense introduced in [28] (see also [21] and [49]). Thus, in particular, we do not take into account multiplicities and components at infinity for our notion of geometric degree. (Since by assumption the polynomials $F_1, \dots, F_h$ form a regular sequence in $\mathbb{Q}[X_1, \dots, X_n]$, it turns out that the geometric degree of the $(n-h)$-dimensional algebraic variety S_h is the number of points by cutting S_h with $(n-h)$ generic affine linear hyperplanes). We call

$$\delta := \max\{\max\{\deg S_h | 1 \le h \le p\}, \max\{\deg \widehat{S}_i | 1 \le i \le n-p\}\}$$

the *degree of the real interpretation of the polynomial equation system* $F_1, \dots, F_p$.

From Proposition 6, Proposition 8 and the genericity of $\mathcal{L}$ in H we deduce that δ does not depend on the choice of the particular flag $\mathcal{L}$.

Since, by assumption, the degrees of the polynomials $F_1, \dots, F_p$ are bounded by d, we infer from the Bézout–Inequality of [28] the degree estimates $\deg S \le d^p$ and $\deg S_h \le d^h \le d^p$, for any $1 \le h \le p$.

Let $1 \le i \le n-p$ and recall from the beginning of Subsection 3.1 that each irreducible component of the polar variety $\widehat{S}_i = W_{\overline{K}^{n-p-i}}(S)$ is a $(n-p-i)$-dimensional irreducible component of the closed subvariety of $\mathbb{A}^n$ defined by the vanishing of $F_1, \dots, F_p$ and of all $(n-i+1)$-minors of the polynomial $((n-i+1) \times n)$-matrix $T^{(i)}$. Taking generic linear combinations of these minors, one deduces easily from the Bézout–Inequality that $\deg \widehat{S}_i$ is bounded by

$$(\deg S) \cdot (p(d-1)+1)^i \le d^{p+i}\, p^i \le d^n\, p^{n-p}.$$

This implies the extrinsic estimate $\delta \le d^n\, p^{n-p}$.

We will now introduce a data structure for the representation of polynomials of $\mathbb{Q}[X_1, \dots, X_n]$ and describe our algorithmic model and complexity measures.

Our elimination procedure will be fomulated in the algorithmic model of (division-free) arithmetic circuits and networks (arithmetic–boolean circuits) over the rational numbers $\mathbb{Q}$.

Roughly speaking, a division-free arithmetic circuit β over $\mathbb{Q}$ is an algorithmic device that supports a step by step evaluation of certain (output) polynomials belonging to $\mathbb{Q}[X_1, \ldots, X_n]$, say $F_1, \ldots, F_p$. Each step of β corresponds either to an input from $X_1, \ldots, X_n$, to a constant (circuit parameter) from $\mathbb{Q}$ or to an arithmetic operation (addition/subtraction or multiplication). We represent the circuit β by a labelled *directed acyclic graph (dag)*. The size of this dag measures the sequential time requirements of the evaluation of the output polynomials $F_1, \ldots, F_p$ performed by the circuit β.

A (division-free) arithmetic network over $\mathbb{Q}$ is nothing else but an arithmetic circuit that additionally contains decision gates comparing rational values or checking their equality, and selector gates depending on these decision gates.

Arithmetic circuits and networks represent non-uniform algorithms, and the complexity of executing a single arithmetic operation is always counted at unit cost. Nevertheless, by means of well known standard procedures our algorithms will always be transposable to the uniform *random* bit model and they will be implementable in practice as well. All this can be done in the spirit of the general asymptotic complexity bounds stated in Theorem 11 below.

Let us also remark that the depth of an arithmetic circuit (or network) measures the *parallel* time of its evaluation, whereas its size allows an alternative interpretation as "number of processors". In this context we would like to emphasize the particular importance of counting only *nonscalar* arithmetic operations (i. e., only essential multiplications), taking $\mathbb{Q}$-linear operations (in particular, additions/subtractions) for cost-free. This leads to the notion of nonscalar size and depth of a given arithmetic circuit or network β. It can be easily seen that the nonscalar size determines essentially the total size of β (which takes into account all operations) and that the nonscalar depth dominates the logarithms of degree and height of the intermediate results of β.

For more details on our complexity model and its use in the elimination theory we refer to [10, 22, 29, 34, 40], and, in particular, to [26] and [37] (where also the implementation aspect is treated).

Now we are ready to formulate the algorithmic main result of this paper.

Theorem 11. Let n, p, d, δ, L and ℓ be natural numbers with $d \geq 2$ and $p \leq n$. Let $X_1, \ldots, X_n, Y$ be indeterminates over $\mathbb{Q}$. There exists an arithmetic network $\mathcal{N}$ over $\mathbb{Q}$ of size $\binom{n}{p}^2 L^2 (nd\delta)^{O(1)}$ and nonscalar depth $O(n(\ell + \log nd) \log \delta)$ with the following property:

Let $F_1, \ldots, F_p$ be a family of polynomials in the variables $X_1, \ldots, X_n$ of a degree at most d and assume that $F_1, \ldots, F_p$ are given by a division-free arithmetic circuit β in $\mathbb{Q}[X_1, \ldots, X_n]$ of size L and nonscalar depth ℓ. Suppose that the polynomials $F_1, \ldots, F_p$ form a regular sequence in $\mathbb{Q}[X_1, \ldots, X_n]$ and that $F_1, \ldots, F_h$ generate a radical ideal for any $1 \leq h \leq p$. Moreover, suppose that the polynomials $F_1, \ldots, F_p$

define a closed, affine subvariety S of $\mathbb{A}^n_{\mathbb{C}}$ such that $S_{\mathbb{R}}$ is non-empty and smooth. Assume that the degree of the real interpretation of the polynomial equation system is bounded by δ. Then the algorithm represented by the arithmetic network $\mathcal{N}$ starts from the circuit β as input and computes the coefficents of $n+1$ polynomials $P, P_1, \ldots, P_n$ in $\mathbb{Q}[Y]$ satisfying the following conditions:

— P is monic and separable,

— $1 \leq \deg P \leq \delta$,

— $\max\{\deg P_k \mid 1 \leq k \leq n\} < \deg P$,

— the cardinality $\#\widehat{S}$ of the (non-empty) affine variety

$$\widehat{S} := \{(P_1(y), \ldots, P_n(y)) \mid y \in \mathbb{C},\ P(y) = 0\}$$

is at most $\deg P$, the affine variety $\widehat{S}$ is contained in S and at least one point of each connected component of $S_{\mathbb{R}}$ belongs to $\widehat{S}$.

Moreover, using sign gates the network $\mathcal{N}$ produces at most $\#\widehat{S}$ sign sequences of elements $\{-1, 0, 1\}$ such that these sign conditions encode the real zeros of the polynomial P "à la Thom" ([15]).

In this way, namely by means of the Thom encoding of the real zeros of P and by means of the polynomials $P_1, \ldots, P_n$, the arithmetic network $\mathcal{N}$ describes the finite, non-empty set

$$\widehat{S} \cap \mathbb{R}^n = \{(P_1(y), \ldots, P_n(y)) \mid y \in \mathbb{R},\ P(y) = 0\},$$

which contains at least one representative point for each connected component of the real variety $S_{\mathbb{R}}$.

Proof. We will freely use the notation introduced at the beginning of this section. Let $F_1, \ldots, F_p$ be polynomials of $\mathbb{Q}[X_1, \ldots, X_n]$ satisfying the assumptions in the statement of the theorem. Let S be the closed, affine subvariety of $\mathbb{A}^n_{\mathbb{C}}$ defined by these polynomials. For $1 \leq j, k \leq n$, let U_k and $U_{j,k}$ be indeterminates over $\mathbb{C}$ and let $U := (U_1, \ldots, U_n, U_{1,1}, \ldots, U_{n,n})$. Furthermore, for $1 \leq l \leq n$, let $Z_l := \sum_{1 \leq k \leq n} U_{l,k} X_k$. We write $Z := (Z_1, \ldots, Z_n)$. Let $\mathfrak{C}$ be an algebraic closure of $\mathbb{C}(U)$ and fix a real closure $\mathfrak{R}$ of $\mathbb{R}(U)$ in $\mathfrak{C}$. Denote by $\mathbb{A}^n(\mathfrak{C})$ and $\mathbb{A}^n(\mathfrak{R})$ the n-dimensional, affine spaces over $\mathfrak{C}$ and $\mathfrak{R}$, respectively. Further, for any $\mathbb{Q}$-definable, closed, algebraic subvariety W of $\mathbb{A}^n_{\mathbb{C}}$ denote by $W(\mathfrak{C})$ and by $W(\mathfrak{R})$ the closed, algebraic subvarieties of $\mathbb{A}^n(\mathfrak{C})$ resp. $\mathbb{A}^n(\mathfrak{R})$ given by an arbitrary set of defining equations of W in $\mathbb{Q}[X_1, \ldots, X_n]$.

Observe that the irreducible and semialgebraically connected components of $S(\mathfrak{C})$ and $S(\mathfrak{R})$ correspond bijectively to the irreducible and connected component of S and $S_{\mathbb{R}}$, respectively.

Consider the $((p+1) \times n)$-matrix

$$T := \begin{bmatrix} & J(F_1(Z), \ldots, F_p(Z)) & \\ U_1 - X_1 & \cdots & U_n - X_n \end{bmatrix}.$$

The entries of T are polynomials belonging to $\mathfrak{R}[X_1, \ldots, X_n]$.

For any choice of p columns $1 \leq i_1 < \ldots < i_p \leq n$ and any index $j \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_p\}$ we denote by $m^{(i_1, \ldots, i_p)}$ the p-minor of $J(F_1(Z), \ldots, F_p(Z))$ given by the columns $i_1, \ldots, i_p$ and by $M^{(i_1, \ldots, i_p, j)}$ the $(p+1)$-minor of T given by the columns $i_1, \ldots, i_p, j$.

Let $\widehat{S}_{n-p}(\mathbb{C})$ be the Zariski closure of the set of all $(F_1, \dots, F_p)$-regular points of $S(\mathbb{C})$, at which all $(p+1)$-minors of T vanish.

Observe that $\widehat{S}_{n-p}(\mathbb{C})$ is the generalized, affine, polar variety of $S(\mathbb{C})$ associated with the zero-dimensional, linear, projective subvariety of the n-dimensional, projective space $\mathbb{P}^n(\mathbb{C})$ over $\mathbb{C}$ that is spanned by the point $(1 : U_1 : \cdots : U_n)$. Let $H(\mathbb{C})$ be the hyperplane at infinity of $\mathbb{P}^n(\mathbb{C})$ and let Q_U be the hyperquadric of $\mathbb{P}^n(\mathbb{C})$ defined by the quadratic form $R_U(X_0, \dots, X_n) := X_0^2 - \sum_{k=1}^n 2U_k X_0 X_k + \sum_{k=1}^n X_k^2$. One verifies immediately that with respect to the hyperquadric Q_U of $\mathbb{P}^n(\mathbb{C})$, the point $(1 : U_1 : \cdots : U_n)$ spans the dual space of $H(\mathbb{C})$ in $\mathbb{P}^n(\mathbb{C})$. Thus $\widehat{S}_{n-p}(\mathbb{C})$ is a dual polar variety with respect to the hyperquadric Q_U of $\mathbb{P}^n(\mathbb{C})$.

Observe now that the hyperquadric $Q_U \cap H(\mathbb{C})$ of $H(\mathbb{C})$ is defined by the quadratic form $R_0(X_1, \dots, X_n) := \sum_{k=1}^n X_k^2$ and that the point $(U_1, \dots, U_n)$ of $\mathbb{A}^n(\mathfrak{R})$ does not belong to $S(\mathfrak{R})$. Thus we may deduce from Proposition 2 and the Transfer Principle for real closed fields (see e.g. [8]) that the real polar variety $\widehat{S}_{n-p}(\mathfrak{R})$ is non-empty. Thus $\widehat{S}_{n-p}(\mathbb{C})$ is non-empty, too. Now, Proposition 8 (or alternatively Lemma 7) implies that $\widehat{S}_{n-p}(\mathbb{C})$ is zero-dimensional and consists of $(F_1, \dots, F_p)$-regular points of $S(\mathbb{C})$. Thus $\widehat{S}_{n-p}(\mathbb{C})$ is of pure codimension $n-p$ in $S(\mathbb{C})$. From the generic choice of the entries of U we deduce that the geometric degree (i.e., the cardinality) of $\widehat{S}_{n-p}(\mathbb{C})$ is at most δ.

Let us consider an arbitrary point x of $\widehat{S}_{n-p}(\mathbb{C})$. Since x is $(F_1, \dots, F_p)$-regular, there exist indices $1 \le i_1 < \dots < i_p \le n$ such that $m^{(i_1, \dots, i_p)}(x) \ne 0$ holds. Let $i_{p+1}, \dots, i_n$ be an enumeration of the set $\{1, \dots, n\} \setminus \{i_1, \dots, i_p\}$.

Taking into account the generic choice of the entries of U, we deduce from Proposition 6 and Proposition 8 that the equations $F_1, \dots, F_p$, $M^{(i_1, \dots, i_{p+1})}, M^{(i_1, \dots, i_p, i_{p+2})}$ $, \dots, M^{(i_1, \dots, i_p, i_n)}$ intersect transversally at the point x and that they define the algebraic variety $\widehat{S}_{n-p}(\mathbb{C})$ outside of the locus $V(m^{(i_1, \dots, i_p)})$ defined by the equation $m^{(i_1, \dots, i_p)}$ in $\mathbb{A}^n(\mathbb{C})$.

Therefore, the polynomials $F_1, \dots, F_p, M^{(i_1, \dots, i_{p+1})}, M^{(i_1, \dots, i_p, i_{p+2})}, \dots,$ $M^{(i_1, \dots, i_p, i_n)}$ form a regular sequence in $\mathbb{Q}(U)[X_1, \dots, X_n]_{m^{(i_1, \dots, i_p)}}$.

Moreover, for any $1 \le j \le n-p$, the polynomials $(F_1, \dots, F_p)$, $M^{(i_1, \dots, i_{p+1})}$, $M^{(i_1, \dots, i_p, i_{p+2})}, \dots, M^{(i_1, \dots, i_p, i_{p+j})}$ generate a radical ideal in $\mathbb{Q}(U)[X_1, \dots, X_n]_{m^{(i_1, \dots, i_p)}}$.

From the genericity of the entries of U and the considerations at the beginning of this section we deduce that the Zariski closure of

$$V(F_1, \dots, F_p, M^{(i_1, \dots, i_{p+1})}, M^{(i_1, \dots, i_p, i_{p+2})}, \dots, M^{(i_1, \dots, i_p, i_{p+j})}) \setminus V(m^{(i_1, \dots, i_p)})$$

in $\mathbb{A}^n(\mathbb{C})$ is a pure $(p+j)$-codimensional variety of geometric degree at most δ.

We are now able to apply the elimination algorithm described in the proof of [24], Proposition 18 (and improved by [25], Theorem 31) to the following system of polynomial equations and inequations:

$$F_1 = \cdots = F_p = M^{(i_1, \dots, i_p, i_{p+1})} = \cdots = M^{(i_1, \dots, i_p, i_n)} = 0, \; m^{(i_1, \dots, i_p)} \ne 0. \tag{8}$$

Observe that the degree of this system (in the sense of loc.cit.) is at most δ. Moreover, the n-variate polynomials of the system are of degree at most pd and they can be evaluated by a division-free arithmetic circuit of size $O(Lnp^4)$ and non-scalar depth $O(\ell + \log p)$ over the function field $\mathbb{Q}(U)$. The mentioned elimination algorithm is represented by an arithmetic network over $\mathbb{Q}(U)$, whose size and non-scalar depth are $L(nd\delta)^{O(1)}$ and $O(n(\ell + \log(nd)) \log \delta)$, respectively.

For the given input system (8) this network evaluates the coefficients of certain univariate polynomials $P^{(i_1,\dots,i_p)}, P_1^{(i_1,\dots,i_p)}, \dots, P_n^{(i_1,\dots,i_p)}$ in $\mathbb{Q}(U)[Y]$ that satisfy the following conditions:

$P^{(i_1,\dots,i_p)}$ is monic and separable with respect to the variable Y,

$\deg_Y P^{(i_1,\dots,i_p)} = \#(\widehat{S}_{n-p}(\mathbb{C}) \setminus V(m^{(i_1,\dots,i_p)})) \leq \delta$,

$\max\{\deg_Y P_1^{(i_1,\dots,i_p)}, \dots, \deg_Y P_n^{(i_1,\dots,i_p)}\} < \deg_Y P^{(i_1,\dots,i_p)}$,

$\widehat{S}_{n-p}(\mathbb{C}) \setminus V(m^{(i_1,\dots,i_p)})$

$\quad = \{(P_1^{(i_1,\dots,i_p)}(y), \dots, P_n^{(i_1,\dots,i_p)}(y)) \mid y \in \mathbb{C},\ P^{(i_1,\dots,i_p)}(y) = 0\}$.

Now we repeat this procedure for each index set $\{i_1, \dots, i_p\}$ with $1 \leq i_1 < \dots < i_p \leq n$, thus obtaining an arithmetic network $\mathcal{N}_1$ over $\mathbb{Q}(U)$ that computes the coefficients of all polynomials $P^{(i_1,\dots,i_p)}, P_1^{(i_1,\dots,i_p)}, \dots, P_n^{(i_1,\dots,i_p)} \in \mathbb{Q}(U)[Y]$ for the given input system (8).

The network $\mathcal{N}_1$ has size $\binom{n}{p} L(nd\delta)^{O(1)}$ and non-scalar depth $O(n(\ell + \log nd)\log\delta)$. From these data we compute, for the given input systen (8), the coefficients of certain polynomials $\widetilde{P}, \widetilde{P}_1, \dots, \widetilde{P}_n \in \mathbb{Q}(U)[Y]$ that satisfy the conditions:

$\widetilde{P}$ is monic and separable with respect to the variable Y,

$\deg_Y \widetilde{P} = \#\widehat{S}_{n-p}(\mathbb{C}) \leq \delta$,

$\max\{\deg_Y \widetilde{P}_1, \dots, \deg_Y \widetilde{P}_n\} < \deg_Y \widetilde{P}$,

$\widehat{S}_{n-p}(\mathbb{C}) = \{\widetilde{P}_1(y), \dots, \widetilde{P}_n(y)) \mid y \in \mathbb{C},\ \widetilde{P}(y) = 0\}$.

This computation can be realized by an extension $\mathcal{N}_2$ of the network $\mathcal{N}_1$, such that $\mathcal{N}_2$ has asymptotically the same size and non-scalar depth as $\mathcal{N}_1$.

Without loss of generality we may consider the arithmetic network $\mathcal{N}_2$ to be division-free, representing rational functions by polynomial numerators and a common denominator. We choose now a correct test sequence $\gamma_1, \dots, \gamma_N \in \mathbb{Z}^{n^2+m}$ for the polynomials of $\mathbb{Q}[U]$ whose circuit size is bounded by the size of $\mathcal{N}_2$. From [33], Theorem 4.4 (see also [34]) we deduce that such a correct test sequence of length $N = \binom{n}{p} L(nd\delta)^{O(1)}$ exists (observe that the argumentation in [33] and [34] is based on the non-scalar complexity model and that we have to perform a slight adaption of the proof). Let $\mathcal{N}_3$ be the arithmetic network over $\mathbb{Q}$, which we obtain by specializing the vector U of inputs of $\mathcal{N}_2$ to the integer points $\gamma_1, \dots, \gamma_N$ and concatenating the resulting arithmetic networks over $\mathbb{Q}$.

Observe that the arithmetic network $\mathcal{N}_3$ is of size $\binom{n}{p}^2 L^2(nd\delta)^{O(1)}$ and of non-scalar depth $O(n(\ell + \log nd)\log\delta)$. For the given input system (8) there exists an index $1 \leq k \leq N$ such that no denominator vanishes on $u = (u_1, \dots, u_n, u_{1,1}, \dots, u_{n,n}) := \gamma_k$ in the computation of the coefficients of the polynomials $\widetilde{P}, \widetilde{P}_1, \dots, \widetilde{P}_n \in \mathbb{Q}(U)[Y]$ by the arithmetic network $\mathcal{N}_2$ and such that $(u_1, \dots, u_n)$ does not belong to S.

Let $P := \widetilde{P}(u)(Y), P_1 := \widetilde{P}_1(u)(Y), \dots, P_n := \widetilde{P}_n(u)(Y)$ and let $\widehat{S}$ be the generalized, affine, polar variety of S associated with the zero-dimensional, projective subvariety K^0 of $\mathbb{P}^n_{\mathbb{C}}$, which is spanned by the point $(1 : u_1 : \dots : u_n)$. In other words, let $\widehat{S} := W_{K^0}(S)$. Consider the hyperquadric Q_u of the projective space $\mathbb{P}^n_{\mathbb{C}}$ defined by the quadratic form $R_u(X_0, \dots, X_n) := X_0^2 - \sum_{k=1}^n 2u_k X_0 X_k + \sum_{k=1}^n X_k^2$ and observe that the hyperquadric $Q_u \cap H$ of H is given by the quadratic form $R_0(X_1, \dots, X_n) := \sum_{k=1}^n X_k^2$ and that, with respect to the hyperquadric Q_u, the point $(1 : u_1 : \dots : u_n)$ spans the dual space of H in $\mathbb{P}^n_{\mathbb{C}}$. Thus $\widehat{S}$ is a dual polar variety with respect to the hyperquadric Q_u of $\mathbb{P}^n_{\mathbb{C}}$. From Proposition 2 we infer now that $\widehat{S}_{\mathbb{R}}$ contains at least one representative point of each connected component of $S_{\mathbb{R}}$. In particular, $\widehat{S}$ is non-empty. Furthermore, the polynomials $P, P_1, \dots, P_n$ belong to $\mathbb{Q}[Y]$. From the choice of u we deduce that $\widehat{S}$ is a $\mathbb{Q}$-definable, zero-dimensional variety (i. e., $\widehat{S}$ is of pure codimension $n-p$ in S) with $\widehat{S} = \{(P_1(y), \ldots, P_n(y)) \mid y \in \mathbb{C}, P(y) = 0\}$ and $\#\widehat{S} \leq \delta$. Moreover, P is monic and separable, and we have $\max\{\deg P_1, \ldots, \deg P_n\} < \deg P = \#\widehat{S} \leq \delta$.

We apply now any of the known, well parallelizable Computer Algebra algorithms for the determination of all real zeros of a given univariate polynomial, where these zeros are thought to be encoded "à la Thom" (see e. g. [15]), to the polynomial $P \in \mathbb{Q}[Y]$. This subroutine may be realized by an arithmetic network $\mathcal{N}$ over $\mathbb{Q}$, which uses sign gates and extends the network $\mathcal{N}_3$. The size and non-scalar depth of $\mathcal{N}$ are asymptotically the same as those of $\mathcal{N}_3$, namely $\binom{n}{p} L^2 (nd\delta)^{O(1)}$ and $O(n(\ell + \log nd)\log\delta)$, respectively. □

Observe that the algorithm described in the proof of Theorem 11 is based on a generic transformation of the variables $X_1, \ldots, X_n$ and on the generic choice of a point in $\mathbb{A}^n_{\mathbb{R}}$, namely $(u_1, \ldots, u_n)$, outside of the variety $S_{\mathbb{R}}$. Indeed, the projective point $(1 : u_1 : \cdots : u_n)$ spans a zero-dimensional linear subvariety K^0 of $\mathbb{P}^n$ which determines the polar variety $\widehat{S} = W_{K^0}(S)$. The fact that $\widehat{S}$ is a zero-dimensional algebraic variety for a generic choice of a point $u = (u_1, \ldots, u_n) \in \mathbb{A}^n_{\mathbb{R}} \setminus S$ is implicitely used in [45] and [1] for the purpose to find for any connected component of $S_{\mathbb{R}}$ a representative point. However, the algorithm developed in loc.cit. is rewriting based, lacks a rigorous complexity analysis and is much less efficient than ours.

Let us finally mention that a variant of the elimination algorithm described in the proof of Theorem 11 can be obtained by chosing a rational (but possibly non-generic) point of $\mathbb{A}^n_{\mathbb{R}} \setminus S$ and chosing the hyperquadric Q of $\mathbb{P}^n$ generically, subject to the condition that $Q \cap H_{\mathbb{R}}$ is defined by a positive quadratic form. We do not go into the details of this algorithmic variant and its geometric foundations, which require only a suitable adaption of Proposition 2 and Lemma 4.

Remark 12. A more precise estimate for the size of the network $\mathcal{N}$ of Theorem 11, namely $\tilde{O}(\binom{n}{p} L^2 n^8 p^4 d^4 \delta^4)$, can be obtained by choosing more carefully in the proof of Theorem 11 the correct test sequence $\gamma_1, , \ldots, \gamma_N$ and by replacing the elimination algorithm of [24] and [25] by a refined version of it, which is described in [29] and [26] (here the $\tilde{O}$-notation indicates that we neglect polylogarithmic factors in the complexity estimate).

A uniform, probabilistic version of the algorithm described in the proof of Theorem 11 can be realized by a network of size $\tilde{O}(\binom{n}{p} n^4 L p^2 d^2 \delta^2)$ and non-scalar depth $O(n(\ell + \log nd)\log\delta)$, which depends on certain randomly chosen parameters.

On the other hand, taking into account the extrinsic estimate $\delta \leq d^n p^{n-d}$ of the beginning of this section and the straightforward estimates $L \leq d^{n+1}$ and $\ell \leq \log d$, we obtain the worst case bounds $\binom{n}{p}(np^{n-p}d^n)^{O(1)}$ and $O((n \log nd)^2)$ for the size and non-scalar depth of the network $\mathcal{N}$ of Theorem 11. Thus, our worst case sequential time complexity bound meets the standards of todays most efficient $d^{O(n)}$-time procedures for the problem under consideration (compare [6, 7] and also [11, 12, 16, 27, 30, 31, 32, 43, 44]).

In the particular case that the real variety $S_{\mathbb{R}}$ is compact, our method produces the algorithmic main result of [4].

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