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THE OPTIMAL CONTROL CHART PROCEDURE

JAROSLAV SKŘIVÁNEK

The moving average (MA) chart, the exponentially weighted moving average (EWMA) chart and the cumulative sum (CUSUM) chart are the most popular schemes for detecting shifts in a relevant process parameter. Any control chart system of span k is specified by a partition of the space $\mathbb{R}^k$ into three disjoint parts. We call this partition as the control chart frame of span k. A shift in the process parameter is signalled at time t by having the vector of the last k sample characteristics fall out of the central part of this frame. The optimal frame of span k is selected in order to maximize the average run length (ARL) if shift in the relevant process parameter is on an acceptable level and to minimize it on a rejectable level. We have proved in this article that the set of all frames of span k with an appropriate metric is a compact space and that the ARL for continuously distributed sample characteristics is continuous as a function of the frame. Consequently, there exists the optimal frame among systems of span k. General attitude to control chart systems is the common platform for universal control charts with the particular point for each sample and variable control limits plotted one step ahead.

Keywords: control chart, frame of span k, average run length, probability distribution, compact metric space

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1. INTRODUCTION

The moving average (MA) chart, the exponentially weighted moving average (EWMA) chart, the cumulative sum (CUSUM) chart and the chart for arithmetic average with warning limits (by the ISO 7873 standard) are the most popular schemes for detecting shift in the process mean. They are described in detail for example in [8]. Some authors (e. g. [9]) gave simulation results that indicate that MA, EWMA and CUSUM charts are competitive among themselves. This control charts combine information from two or more samples in order to improve performance. A presence of signal at time t depends on more sample characteristics $Y_t, Y_{t-1}, \dots$. There are also other charts for controlling parameters other than a normal process mean.

The moving average of span k at time t is

$$M_t = \frac{Y_{t-k+1} + Y_{t-k+2} + \dots + Y_t}{k} \tag{1}$$

for $t = k, k+1, \ldots$. Control limits are on constant levels UCL and LCL. A single point M_t out of control limits at time t in the chart is signal of changes in the relevant process parameter.

Warning and action limits are plotted on constant levels UWL, LWL, UAL and LAL in the control charts for arithmetic average with warning limits of span k. A shift in the process parameter is signalled at time t by having at least k succeeded points $Y_{t-k+1}, Y_{t-k+2}, \ldots, Y_t$ in the sequence of sample characteristics (e.g. sample means) fall outside the warning limits or one point outside the action limits.

There exists a lot of other control chart schemes with numerous modifications and combinations (see [1, 6, 7]). They play the crucial role in systems of Statistical Quality Control (see for example [3, 4, 5, 8]). General attitude to control systems in this text is a platform of universal control charts with one point Y_t for each sample along with variable control limits

$$UCL_t = \sup A_t \quad \text{and} \quad LCL_t = \inf A_t, \tag{2}$$

plotted at time t at most one step ahead, where borders of the acceptance region $A_t \subset \mathbb{R}$ for Y_t depend on the characteristics $Y_{t-1}, Y_{t-2}, \ldots$ before time t.

2. SYSTEMS OF SPAN k

Let $\mathbb{R}^k$ be the set of all ordered k-tuples of real numbers with common topology. A partition $\mathcal{S}$ of the space $\mathbb{R}^k$ into three disjoint parts $\mathcal{C}$, $\mathcal{U}$ and $\mathcal{L}$ with the properties

(i) central part $\mathcal{C}$ is nonempty and closed,

(ii) upper and lower parts $\mathcal{U}$ and $\mathcal{L}$ are open, at least one of them is nonempty,

(iii) if $(x_1, x_2, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_k) \in \mathcal{U}$ and $y < z$ then $(x_1, x_2, \ldots, x_{j-1}, z, x_{j+1}, \ldots, x_k) \in \mathcal{U}$ for each j, (3)

(iv) if $(x_1, x_2, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_k) \in \mathcal{L}$ and $z < y$ then $(x_1, x_2, \ldots, x_{j-1}, z, x_{j+1}, \ldots, x_k) \in \mathcal{L}$ for each j,

is called *the control chart frame of span k*. A shift up in the process parameter will be signalled at time t by having the random vector $(Y_{t-k+1}, Y_{t-k+2}, \ldots, Y_t)$ of sample characteristics fall into the upper part $\mathcal{U}$ of the frame, a shift down will be signalled by having this point fall into the lower part $\mathcal{L}$. The point in the central part $\mathcal{C}$ signifies no signal. The MA chart and the chart for arithmetic average with warning limits are systems of some span k. Examples of frames for this schemes are in Figure 1. On the other hand, the EWMA chart and the CUSUM are systems of unbounded span because the signal is conditioned by information from all samples preceding the actual time.

Control limits at time t in the universal control chart for a system of span k are

$$\begin{aligned} UCL_t &= \sup\{y \in \mathbb{R}; (Y_{t-k+1}, Y_{t-k+2}, \ldots, Y_{t-1}, y) \in \mathcal{C}\} \\ LCL_t &= \inf\{y \in \mathbb{R}; (Y_{t-k+1}, Y_{t-k+2}, \ldots, Y_{t-1}, y) \in \mathcal{C}\}. \end{aligned} \tag{4}$$

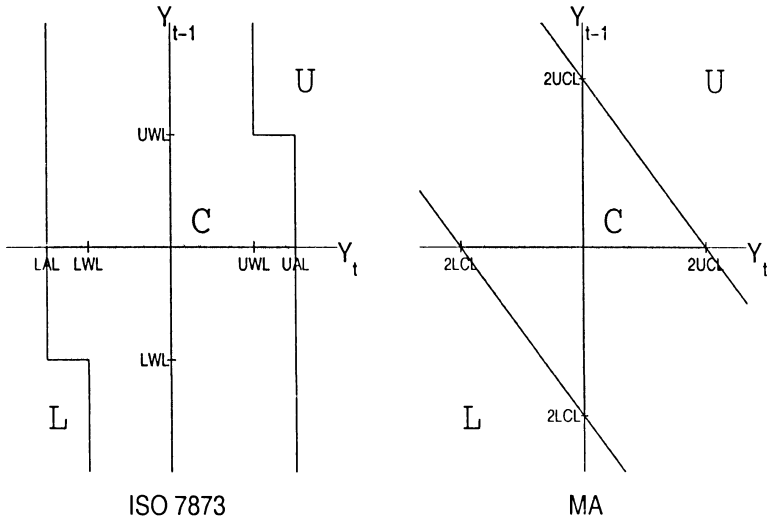

Fig. 1. Frames of span 2.

You can see comparison of the classical and the universal control chart for the MA scheme of span 2 in Figure 2. Little triangles indicate the time moments with signal (the 7th sample).

Let λ_0 be the standard Lebesgue measure on $\mathbb{R}^k$, $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ be an element of $\mathbb{R}^k$ and $\mathbf{X}_k$ be the set of all frames of span k. Let λ be a normalized measure on $\mathbb{R}^k$ with the same zero sets like λ_0. For example

$$\lambda(A) = \lambda_0\left(\{(\Phi(x_1), \Phi(x_2), \ldots, \Phi(x_k)) \in (0;1)^k;\ \mathbf{x} \in A\}\right)$$

for any Lebesgue measurable $A \subset \mathbb{R}^k$ and $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt$. Let the symbol $\dot{-}$ represents the symmetric difference operation.

Lemma 1. Let $\mathcal{S}^{(i)} = (\mathcal{U}^{(i)}, \mathcal{C}^{(i)}, \mathcal{L}^{(i)}) \in \mathbf{X}_k$ for $i \in \{1, 2\}$. Then $\rho_k : \mathbf{X}_k^2 \to \mathbb{R}$, $\rho_k(\mathcal{S}^{(1)}, \mathcal{S}^{(2)}) = \lambda(\mathcal{U}^{(1)} \dot{-} \mathcal{U}^{(2)}) + \lambda(\mathcal{L}^{(1)} \dot{-} \mathcal{L}^{(2)})$, is a metric on $\mathbf{X}_k$.

Proof. We have to prove that

(i) $\rho_k(\mathcal{S}^{(1)}, \mathcal{S}^{(2)}) = 0$ if and only if $\mathcal{S}^{(1)} = \mathcal{S}^{(2)}$,

(ii) $\rho_k(\mathcal{S}^{(1)}, \mathcal{S}^{(2)}) = \rho_k(\mathcal{S}^{(2)}, \mathcal{S}^{(1)})$,

(iii) $\rho_k(\mathcal{S}^{(1)}, \mathcal{S}^{(2)}) + \rho_k(\mathcal{S}^{(2)}, \mathcal{S}^{(3)}) \geq \rho_k(\mathcal{S}^{(1)}, \mathcal{S}^{(3)})$

for every $\mathcal{S}^{(1)}$, $\mathcal{S}^{(2)}$ and $\mathcal{S}^{(3)} \in \mathbf{X}_k$.

(i): Obviously $\rho_k(\mathcal{S}^{(1)}, \mathcal{S}^{(2)}) = 0$ if frames $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$ are identical. If $\mathcal{S}^{(1)} \neq \mathcal{S}^{(2)}$ then at least one of the sets $\mathcal{U}^{(1)} - \mathcal{U}^{(2)}$, $\mathcal{U}^{(2)} - \mathcal{U}^{(1)}$, $\mathcal{L}^{(1)} - \mathcal{L}^{(2)}$, $\mathcal{L}^{(2)} - \mathcal{L}^{(1)}$ is nonempty. Let it be $\mathcal{U}^{(1)} - \mathcal{U}^{(2)}$. At the other cases we proceed alike. As $\mathcal{U}^{(1)}$ is open, for arbitrary $\mathbf{t} = (t_1, t_2, \ldots, t_k) \in \mathcal{U}^{(1)} - \mathcal{U}^{(2)}$ there exists such real number $\varepsilon > 0$ that the ε-neighbourhood $O_\varepsilon(\mathbf{t})$ of $\mathbf{t}$ is a subset of $\mathcal{U}^{(1)}$. Put

$$D = \{\mathbf{x} \in O_\varepsilon(\mathbf{t}); (\forall i \in \{1, \ldots, k\})(x_i \leq t_i)\}.$$

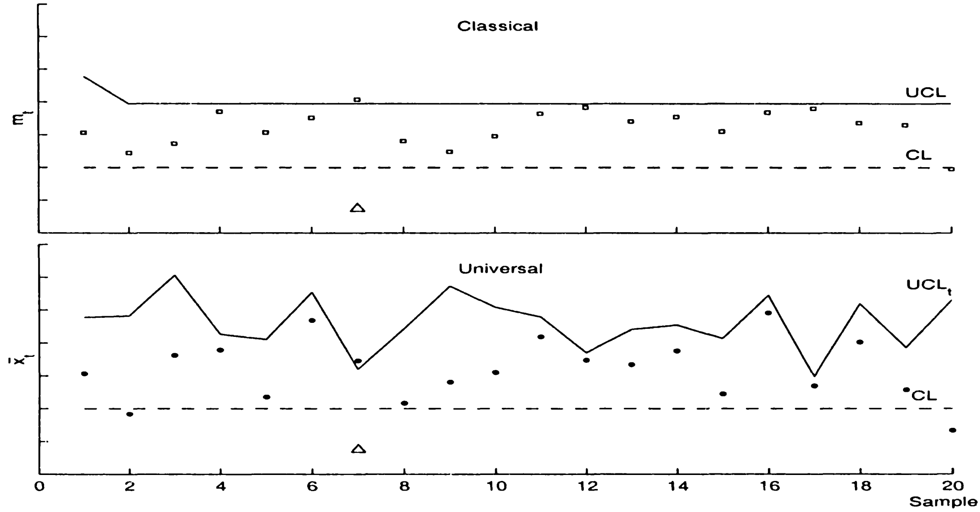

Fig. 2. One-sided MA chart and its universal form.

By the item (iii) of (3) is $D \cap \mathcal{U}^{(2)} = \emptyset$. As $\lambda(D) > 0$ and $D \subset \mathcal{U}^{(1)} - \mathcal{U}^{(2)}$, we have

$$0 < \lambda(\mathcal{U}^{(1)} - \mathcal{U}^{(2)}) \le \rho_k(\mathcal{S}^{(1)}, \mathcal{S}^{(2)}).$$

(ii): The symmetric difference is commutative.

(iii): For arbitrary three sets A_1, A_2 and A_3 is

$$(A_1 \dot{-} A_2) \cup (A_2 \dot{-} A_3) \supset A_1 \dot{-} A_3.$$

Hence

$$\lambda(A_1 \dot{-} A_2) + \lambda(A_2 \dot{-} A_3) \ge \lambda((A_1 \dot{-} A_2) \cup (A_2 \dot{-} A_3)) \ge \lambda(A_1 \dot{-} A_3). \qquad \square$$

Lemma 2. The metric space $(\mathbf{X}_k, \rho_k)$ is complete.

Proof. Put $\mathcal{S}^{(n)} = (\mathcal{U}^{(n)}, \mathcal{C}^{(n)}, \mathcal{L}^{(n)}$. Let $(\mathcal{S}^{(n)})_{n=1}^{\infty}$ be Cauchy sequence in $\mathbf{X}_k$. Hence

$$(\forall \varepsilon > 0)(\exists l \in \mathbb{N})(\forall i,\, j \in \mathbb{N})(i,\, j > l \Rightarrow \rho_k(\mathcal{S}^{(i)}, \mathcal{S}^{(j)}) < \varepsilon),$$

where $\mathbb{N}$ is the set of all natural numbers. We are going to show the limit of this sequence is the frame $\mathcal{S} = (\mathcal{U}, \mathcal{C}, \mathcal{L})$, where $\mathcal{U} = \operatorname{int}\left(\bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} \mathcal{U}^{(j)}\right)$, $\mathcal{L} = \operatorname{int}\left(\bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} \mathcal{L}^{(j)}\right)$ and $\mathcal{C} = \mathbb{R}^k - (\mathcal{U} \cup \mathcal{L})$. Here int is the interior operator in the standard metric on $\mathbb{R}^k$. Let us now prove the sets $\bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} \mathcal{U}^{(j)} \supset \bigcup_{i=1}^{\infty} \bigcap_{j=i}^{\infty} \mathcal{U}^{(j)}$ differ only in boundary points. As both these sets have property (iii) of (3), they consequently differ on a set of measure zero.

Suppose there exists $\mathbf{t} \in \bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} \mathcal{U}^{(j)} - \bigcup_{i=1}^{\infty} \bigcap_{j=i}^{\infty} \mathcal{U}^{(j)}$, an internal point of $\bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} \mathcal{U}^{(j)}$. Then there exists such $\delta > 0$ that

$$\mathbf{t} - \delta = (t_1 - \delta, \ldots, t_k - \delta) \in \bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} \mathcal{U}^{(j)} - \bigcup_{i=1}^{\infty} \bigcap_{j=i}^{\infty} \mathcal{U}^{(j)}.$$

We would like to show that this leads to a contradiction. The set $\bigcap_{i=1}^{\infty}\bigcup_{j=i}^{\infty}\mathcal{U}^{(j)}$ contains all the points of $\mathbb{R}^k$ that are elements of infinitely many upper parts $\mathcal{U}^{(j)}$ while elements of $\bigcup_{i=1}^{\infty}\bigcap_{j=i}^{\infty}\mathcal{U}^{(j)}$ are included in almost all parts $\mathcal{U}^{(j)}$. It means that

$$(\forall\, l \in \mathbb{N})(\exists\, m,\, n \in \mathbb{N})\left(m,\, n > l \;\&\; \mathbf{t} - \delta \in \mathcal{U}^{(n)} \;\&\; \mathbf{t} \notin \mathcal{U}^{(m)}\right).$$

Hence for $M = \{\mathbf{x} \in \mathbb{R}^k;\, (\forall\, i \in \{1,\dots,k\})(t_i - \delta \le x_i \le t_i)\}$ is

$$\rho_k(\mathcal{S}^{(m)}, \mathcal{S}^{(n)}) \ge \lambda(M) > 0,$$

contradictory to the fact that the sequence $(\mathcal{S}^{(n)})_{n=1}^{\infty}$ is Cauchy. So, the sets

$$\bigcap_{i=1}^{\infty}\bigcup_{j=i}^{\infty}\mathcal{U}^{(j)} \supset \bigcup_{i=1}^{\infty}\bigcap_{j=i}^{\infty}\mathcal{U}^{(j)} \supset \operatorname{int}\left(\bigcap_{i=1}^{\infty}\bigcup_{j=i}^{\infty}\mathcal{U}^{(j)}\right)$$

differ only on zero sets. Because it is true for lower parts of frames too, we have

$$(\forall \varepsilon > 0)(\exists\, l \in \mathbb{N})(\forall n \in \mathbb{N})\left(n > l \Rightarrow \lambda(\mathcal{U}^{(n)''} - \mathcal{U}^{(n)'}) + \lambda(\mathcal{L}^{(n)''} - \mathcal{L}^{(n)'}) < \frac{\varepsilon}{2}\right),$$

where $\mathcal{U}^{(n)'} = \bigcap_{j=n}^{\infty}\mathcal{U}^{(j)}$ and $\mathcal{U}^{(n)''} = \bigcup_{j=n}^{\infty}\mathcal{U}^{(j)}$. Moreover, $\mathcal{U}^{(n)'} \subset \mathcal{U}^{(n)} \subset \mathcal{U}^{(n)''}$ and $\mathcal{U}^{(n)'} \subset \mathcal{U} \subset \mathcal{U}^{(n)''}$ almost everywhere and consequently $\lambda(\mathcal{U}^{(n)''} - \mathcal{U}^{(n)}) \le \lambda(\mathcal{U}^{(n)''} - \mathcal{U}^{(n)'})$ and $\lambda(\mathcal{U}^{(n)''} - \mathcal{U}) \le \lambda(\mathcal{U}^{(n)''} - \mathcal{U}^{(n)'})$. It holds similarly for $\mathcal{L}$'s. Hence

$$\rho_k(\mathcal{S}^{(n)}, \mathcal{S}) \le \lambda(\mathcal{U}^{(n)''} - \mathcal{U}^{(n)}) + \lambda(\mathcal{U}^{(n)''} - \mathcal{U})$$
$$+ \lambda(\mathcal{L}^{(n)''} - \mathcal{L}^{(n)}) + \lambda(\mathcal{L}^{(n)''} - \mathcal{L}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \qquad \square$$

Lemma 3. The metric space $(\mathbf{X}_k, \rho_k)$ is totally bounded.

Proof. We try to prove that for every $\varepsilon > 0$ there exists ε-net A_ε in the metric space $\mathbf{X}_k$. It is such finite subset of $\mathbf{X}_k$ that for every element $\mathcal{S} \in \mathbf{X}_k$ there exists such $\operatorname{apr}_\varepsilon(\mathcal{S}) \in A_\varepsilon$ that $\rho_k(\mathcal{S}, \operatorname{apr}_\varepsilon(\mathcal{S})) < \varepsilon$.

Let C_n be a partition of $\mathbb{R}^k$ on the n^k same cubes in sense of the measure λ. Let $\mathcal{I}_n = \{I \subset \mathbb{R};\, (\exists\, l \in \{1,2,\dots,n\})(I = \Phi^{-1}(\langle\frac{l-1}{n}, \frac{l}{n}\rangle))\}$. The measure λ of each element of the set

$$\mathrm{C}_n = \{I_1 \times I_2 \times \ldots \times I_k \subset \mathbb{R}^k;\, (I_1, I_2, \dots, I_k) \in \mathcal{I}_n^k\}$$

is $\frac{1}{n^k}$. Put

$$A_\varepsilon = \{(\mathcal{U}, \mathcal{C}, \mathcal{L}) \in \mathbf{X}_k;\, (\exists\, B \subset \mathrm{C}_{n(\varepsilon)})(\mathcal{C} = \bigcup B)\}$$

where $\frac{2k}{\varepsilon}+1 \ge n(\varepsilon) > \frac{2k}{\varepsilon}$. Really, A_ε is sought ε-net. If $\mathcal{S}^{(0)} = (\mathcal{U}^{(0)}, \mathcal{C}^{(0)}, \mathcal{L}^{(0)}) \in \mathbf{X}_k$, let us select for $\operatorname{apr}_\varepsilon(\mathcal{S}^{(0)})$ such element of A_ε that its central part is

$$\operatorname{apr}_\varepsilon(\mathcal{C}^{(0)}) = \bigcap\{\mathcal{C} \subset \mathbb{R}^k;\, \mathcal{C} \supset \mathcal{C}^{(0)} \& (\mathcal{U}, \mathcal{C}, \mathcal{L}) \in A_\varepsilon\}.$$

Then $\mathcal{U}^{(0)}$ and $\text{apr}_\varepsilon(\mathcal{U}^{(0)})$ (similarly $\mathcal{L}^{(0)}$ and $\text{apr}_\varepsilon(\mathcal{L}^{(0)})$) differ only in interiors at most $k \cdot n(\varepsilon)^{k-1}$ elements of C_n. Hence

$$\rho_k(\mathcal{S}^{(0)}, \text{apr}_\varepsilon(\mathcal{S}^{(0)})) \leq 2k \cdot n(\varepsilon)^{k-1} \cdot \frac{1}{n(\varepsilon)^k} = \frac{2k}{n(\varepsilon)} < \varepsilon. \qquad \square$$

Theorem 1. The metric space $(\mathbf{X}_k, \rho_k)$ is compact.

Proof. Immediate consequence of Lemmas 2 and 3 by claims of General Topology (see for example [2]). □

3. AVERAGE RUN LENGTH

Let L be the number of succeeded samples with no signal coming from a nonextendible passing section of the process. Average run length (ARL) is the mean value of the random variable L. The criterion ARL depends on properties of the process $(Y_t)_{t=-\infty}^{\infty}$ of sample characteristics and on used control chart system. Settings of control chart procedure are determined economically (see [10, 11]). They are usually selected in order to maximize ARL if shift in a relevant process parameter is on an acceptable level Δ_a (usually zero) and to minimize it on a rejectable level Δ_r. Two acceptable and rejectable levels are distinguished for two-sided asymmetrical control chart system. Let $\Delta_r^l < \Delta_a^l \leq 0 \leq \Delta_a^u < \Delta_r^u$ be lower rejectable and acceptable along with upper acceptable and rejectable levels for the process parameter shift from a target value in two-sided control chart system. There is often $\Delta_r^l = -\Delta_r^u$ and $\Delta_a^l = -\Delta_a^u$ in symmetrical systems. The idea is now to show that a control chart design can be interpreted as a problem to find maximum or minimum of continuous function on $\mathbf{X}_k$.

Theorem 2. Let ARL be the average run length of a zero-one signal process $(S_t)_{t=-\infty}^{\infty}$. Then

$$P(S_t = 1 | S_{t-1} = 0) = 1/ARL.$$

Proof. Let L be the length of a passing maximal process section with no signal (zeros). The mean value of L is $ARL = \sum_{j=1}^{\infty} j \cdot P(L = j)$. Let $A_{n,t}$ be the event "random time point t is a member of a signalless process section of length n", $B_t \supset A_{n,t}$ be the event "no signal is present at time t" ($S_t = 0$) and $\bar{B}_t$ be the complement of this event. The conditional probability $P(A_{n,t}|B_t)$ is proportional only to the probability $P(L = n)$ and to the integer n. Hence

$$P(A_{n,t}|B_t) = \frac{n \cdot P(L = n)}{\sum_{j=1}^{\infty} j \cdot P(L = j)} \quad \text{and} \quad P(\overline{B}_t|A_{i,t-1}) = \frac{1}{i}.$$

By the total probability formula we have

$$\begin{aligned} P(\overline{B}_t|B_{t-1}) &= \sum_{i=1}^{\infty} P(\overline{B}_t|A_{i,t-1}) \cdot P(A_{i,t-1}|B_{t-1}) \\ &= \sum_{i=1}^{\infty} \frac{1}{i} \cdot \frac{i \cdot P(L = i)}{ARL} = \frac{1}{ARL}. \end{aligned} \qquad \square$$

Let $ARL_\Delta(\mathcal{S})$ be the average run length in a control chart system of span k with a frame $\mathcal{S} = (\mathcal{U}, \mathcal{C}, \mathcal{L}) \in \mathbf{X}_k$ over a process $(Y_t)_{t=-\infty}^{\infty}$ of independent sample characteristics Y_t, each of them with the same probability distribution at a relevant parameter shift of Δ. Evidently

$$ARL_\Delta(\mathcal{S}) = \frac{\pi_{1,\Delta}(\mathcal{S})}{\pi_{1,\Delta}(\mathcal{S}) - \pi_{2,\Delta}(\mathcal{S})}, \tag{5}$$

where conditional probabilities

$$\pi_{1,\Delta}(\mathcal{S}) = P((Y_{t-k}, Y_{t-k+1}, \dots, Y_{t-1}) \in \mathcal{C} | \text{shift} = \Delta) \tag{6}$$

and

$$\pi_{2,\Delta}(\mathcal{S}) = P((Y_{t-k}, Y_{t-k+1}, \dots, Y_{t-1}) \in \mathcal{C} \& (Y_{t-k+1}, Y_{t-k+2}, \dots, Y_t) \in \mathcal{C} | \text{shift} = \Delta)$$

do not depend on t.

4. OPTIMAL SYSTEM

The next theorem is crucial for existence of the optimal system of span k in a good deal of design problems.

Theorem 3. Let members of the process $(Y_t)_{t=-\infty}^{\infty}$ of sample characteristics be independent and alike continuously distributed. Then $\pi_{1,\Delta}(\mathcal{S})$ and $\pi_{2,\Delta}(\mathcal{S})$ of (6), considered as functions of variable $\mathcal{S}$, are uniformly continuous on the metric space $(\mathbf{X}_k, \rho_k)$.

Proof. Let f_Δ be the probability density function of Y_t at the relevant process parameter shift of Δ, $\hat{\mathbf{x}}_j = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_k)$ for each $j \in \{1, 2, \dots, k\}$,

$$inf_j(\hat{\mathbf{x}}_j) := \inf\{x_j \in \mathbb{R}; \ \mathbf{x} \in \mathcal{C}\},$$

$$sup_j(\hat{\mathbf{x}}_j) := \sup\{x_j \in \mathbb{R}; \ \mathbf{x} \in \mathcal{C}\}$$

and $\Gamma_{\Delta,j}(\hat{\mathbf{x}}_j) = \int_{inf_j(\hat{\mathbf{x}}_j)}^{sup_j(\hat{\mathbf{x}}_j)} f_\Delta(x_j)\,\mathrm{d}x_j$. Then

$$\pi_{1,\Delta}(\mathcal{S}) = \int_{\mathcal{C}} \prod_{i=1}^{k} f_\Delta(x_i)\,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^{k-1}} \Gamma_{\Delta,j}(\hat{\mathbf{x}}_j) \cdot \prod_{i=1, i\neq j}^{k} f_\Delta(x_i)\,\mathrm{d}\hat{\mathbf{x}}_j \tag{7}$$

for arbitrary $j \in \{1, 2, \dots, k\}$ and

$$\begin{aligned} \pi_{2,\Delta}(\mathcal{S}) &= \int_{(x_2,\dots,x_{k+1}) \in \mathcal{C} \& \mathbf{x} \in \mathcal{C}} \prod_{i=1}^{k+1} f_\Delta(x_i)\,\mathrm{d}(x_1, x_2, \dots, x_{k+1}) = \\ &= \int_{\mathbb{R}^{k-1}} \Gamma_{\Delta,1}(\hat{\mathbf{x}}_k) \cdot \Gamma_{\Delta,k}(\hat{\mathbf{x}}_k) \cdot \prod_{i=1}^{k-1} f_\Delta(x_i)\,\mathrm{d}\hat{\mathbf{x}}_k. \end{aligned} \tag{8}$$

Suppose $\varepsilon > 0$. Then there exists such bounded measurable set $B \subset \mathbb{R}^k$ that $\int\limits_{\mathbb{R}^k - B} \prod_{i=1}^k f_\Delta(x_i)\,\mathrm{d}\mathbf{x} < \varepsilon/3$ and the function $\prod_{i=1}^k f_\Delta(x_i)$ is bounded on B. Let $a > 0$ be such constant that $a \cdot \lambda_0(A) \le \lambda(A)$ for every measurable $A \subset B$. Put $b = \sup\limits_{\mathbf{x}\in B} \prod_{i=1}^k f_\Delta(x_i)$ and $\delta = \varepsilon a/(3b)$. If $\rho_k(\mathcal{S}^{(1)}, \mathcal{S}^{(2)}) < \delta$ for some $\mathcal{S}^{(1)}, \mathcal{S}^{(2)} \in \mathbf{X}_k$ then

$$|\pi_{1,\Delta}(\mathcal{S}^{(1)}) - \pi_{1,\Delta}(\mathcal{S}^{(2)})| = \left| \int\limits_{\mathcal{C}^{(1)}} \prod_{i=1}^k f_\Delta(x_i)\,\mathrm{d}\mathbf{x} - \int\limits_{\mathcal{C}^{(2)}} \prod_{i=1}^k f_\Delta(x_i)\,\mathrm{d}\mathbf{x} \right|$$

$$\le \int\limits_{\mathcal{C}^{(1)} \dot{-} \mathcal{C}^{(2)}} \prod_{i=1}^k f_\Delta(x_i)\,\mathrm{d}\mathbf{x} \le \int\limits_{\mathcal{U}^{(1)} \dot{-} \mathcal{U}^{(2)}} \prod_{i=1}^k f_\Delta(x_i)\,\mathrm{d}\mathbf{x} + \int\limits_{\mathcal{L}^{(1)} \dot{-} \mathcal{L}^{(2)}} \prod_{i=1}^k f_\Delta(x_i)\,\mathrm{d}\mathbf{x}$$

$$< \frac{2\varepsilon}{3} + \int\limits_{(\mathcal{U}^{(1)} \dot{-} \mathcal{U}^{(2)})\cap B} \prod_{i=1}^k f_\Delta(x_i)\,\mathrm{d}\mathbf{x} + \int\limits_{(\mathcal{L}^{(1)} \dot{-} \mathcal{L}^{(2)})\cap B} \prod_{i=1}^k f_\Delta(x_i)\,\mathrm{d}\mathbf{x}$$

$$\le \frac{2\varepsilon}{3} + b \cdot \left[\lambda_0((\mathcal{U}^{(1)} \dot{-} \mathcal{U}^{(2)}) \cap B) + \lambda_0((\mathcal{L}^{(1)} \dot{-} \mathcal{L}^{(2)}) \cap B)\right]$$

$$= \frac{2\varepsilon}{3} + \frac{a\varepsilon}{3\delta} \cdot \left[\lambda_0((\mathcal{U}^{(1)} \dot{-} \mathcal{U}^{(2)}) \cap B) + \lambda_0((\mathcal{L}^{(1)} \dot{-} \mathcal{L}^{(2)}) \cap B)\right]$$

$$\le \frac{2\varepsilon}{3} + \frac{\varepsilon}{3\delta} \cdot \left[\lambda((\mathcal{U}^{(1)} \dot{-} \mathcal{U}^{(2)}) \cap B) + \lambda((\mathcal{L}^{(1)} \dot{-} \mathcal{L}^{(2)}) \cap B)\right] < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3\delta} \cdot \delta = \varepsilon.$$

We have proved that the function $\pi_{1,\Delta}$ is uniformly continuous. Hence by (7) for each $\varepsilon > 0$ there exists such $\delta > 0$ that

$$\left| \int\limits_{\mathbb{R}^{k-1}} \left(\Gamma^{(1)}_{\Delta,1}(\hat{\mathbf{x}}_k) - \Gamma^{(2)}_{\Delta,1}(\hat{\mathbf{x}}_k)\right) \prod_{i=1}^{k-1} f_\Delta(x_i)\,\mathrm{d}\hat{\mathbf{x}}_k \right| < \frac{\varepsilon}{2}$$

and

$$\left| \int\limits_{\mathbb{R}^{k-1}} \left(\Gamma^{(1)}_{\Delta,k}(\hat{\mathbf{x}}_k) - \Gamma^{(2)}_{\Delta,k}(\hat{\mathbf{x}}_k)\right) \prod_{i=1}^{k-1} f_\Delta(x_i)\,\mathrm{d}\hat{\mathbf{x}}_k \right| < \frac{\varepsilon}{2}$$

if $\rho_k(\mathcal{S}^{(1)}, \mathcal{S}^{(2)}) < \delta$. But then

$$|\pi_{2,\Delta}(\mathcal{S}^{(1)}) - \pi_{2,\Delta}(\mathcal{S}^{(2)})| = \left| \int\limits_{\mathbb{R}^{k-1}} \left(\Gamma^{(1)}_{\Delta,1}(\hat{\mathbf{x}}_k)\Gamma^{(1)}_{\Delta,k}(\hat{\mathbf{x}}_k)\right.\right.$$

$$\left.\left. -\Gamma^{(2)}_{\Delta,1}(\hat{\mathbf{x}}_k)\Gamma^{(2)}_{\Delta,k}(\hat{\mathbf{x}}_k)\right) \prod_{i=1}^{k-1} f_\Delta(x_i)\,\mathrm{d}\hat{\mathbf{x}}_k \right| = \left| \int\limits_{\mathbb{R}^{k-1}} \left[\left(\Gamma^{(1)}_{\Delta,k}(\hat{\mathbf{x}}_k) - \Gamma^{(2)}_{\Delta,k}(\hat{\mathbf{x}}_k)\right) \cdot \Gamma^{(2)}_{\Delta,1}(\hat{\mathbf{x}}_k)\right.\right.$$

$$+ \left(\Gamma^{(1)}_{\Delta,1}(\hat{\mathrm{x}}_k) - \Gamma^{(2)}_{\Delta,1}(\hat{\mathrm{x}}_k)\right) \cdot \Gamma^{(1)}_{\Delta,k}(\hat{\mathrm{x}}_k)\Big] \prod_{i=1}^{k-1} f_\Delta(x_i)\,\mathrm{d}\hat{\mathrm{x}}_k\Bigg|$$

$$\leq \left| \int_{\mathbb{R}^{k-1}} \left(\Gamma^{(1)}_{\Delta,k}(\hat{\mathrm{x}}_k) - \Gamma^{(2)}_{\Delta,k}(\hat{\mathrm{x}}_k)\right) \prod_{i=1}^{k-1} f_\Delta(x_i)\,\mathrm{d}\hat{\mathrm{x}}_k \right|$$

$$+ \left| \int_{\mathbb{R}^{k-1}} \left(\Gamma^{(1)}_{\Delta,1}(\hat{\mathrm{x}}_k) - \Gamma^{(2)}_{\Delta,1}(\hat{\mathrm{x}}_k)\right) \prod_{i=1}^{k-1} f_\Delta(x_i)\,\mathrm{d}\hat{\mathrm{x}}_k \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \qquad \square$$

We have proved using Theorems 2 and 3 that $ARL_\Delta(\mathcal{S})$, as a function of variable $\mathcal{S}$, is continuous on $\mathbf{X}_k$. Moreover, $ARL_\Delta(\mathcal{S})$ is decreasing with respect to $\pi_{1,\Delta}(\mathcal{S})$ and increasing with respect to $\pi_{2,\Delta}(\mathcal{S})$. Evidently, independence of Y_t's in Theorem 3 can be replaced by the weaker condition, e.g. the process $(Y_t)_{t=-\infty}^{\infty}$ is strictly stationary. You can interpret control chart design task as a problem of the extreme point of a continuous function on a closed subset of the compact space $\mathbf{X}_k$. The most prevailing problem is to find

$$\max\{ARL_{\Delta_a}(\mathcal{S}); ARL_{\Delta_r}(\mathcal{S}) = l_r, \mathcal{S} \in \mathbf{X}_k\} \tag{9}$$

or

$$\min\{ARL_{\Delta_r}(\mathcal{S}); ARL_{\Delta_a}(\mathcal{S}) = l_a, \mathcal{S} \in \mathbf{X}_k\}. \tag{10}$$

For example, the sample characteristic Y_t is the sample mean of quality indicator with normal distribution. We need to find such control chart frame of span 2 that ARL is 370 (corresponding to the Shewhart diagram with 3σ limits) if no shift in the process mean is present and such that ARL is minimal if the shift is the 1.5 multiple of the standard deviation of sample characteristic. There is the optimal solution here. We have not identified exactly the optimal frame for the task of

$$\min\{ARL_{1.5}(\mathcal{S}); ARL_0(\mathcal{S}) = 370 \& \mathcal{S} \in \mathbf{X}_2 \text{ is symmetrical}\}$$

but we have found approximate boundary of the optimal solution in various functional classes. They are shaped like that in Figure 3. In this case is $ARL_{1.5}(\mathcal{S}_{opt})$ approximately 7.238, less than any classical system of span 2. The greater span k of control chart system the better performance we can expect because of natural embedding of $\mathbf{X}_r$ into $\mathbf{X}_s$ for $r < s$.

5. CONCLUSION

An existence of the optimal frame of a common control chart problem of span k over a process of continuously distributed sample characteristics has been proved here. Its delimitation will be chiefly a case of numerical analysis. Moreover, this note offers an aid for measurement of distance of control chart systems interpreted as points of a metric space. There are troubles in an exact specification of the criterion ARL in charts which depend on all previous values. One eventuality is an application of the optimal solution in the space $\mathbf{X}_t$ for every time t. On the other hand, the longer span we consider the better performance is guaranteed.

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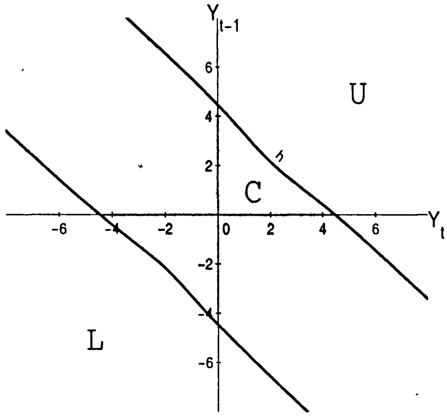

Fig. 3. Optimal system frame of span 2.

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Jaroslav Skřivánek, Department of Mathematics, Technical University of Košice, B. Nemcovej 32, 042 00 Košice. Slovak Republic.
e-mail: Jaroslav.Skrivanek@tuke.sk