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HAAR WAVELETS METHOD FOR SOLVING POCKLINGTON'S INTEGRAL EQUATION

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A simple and effective method based on Haar wavelets is proposed for the solution of Pocklington's integral equation. The properties of Haar wavelets are first given. These wavelets are utilized to reduce the solution of Pocklington's integral equation to the solution of algebraic equations. In order to save memory and computation time, we apply a threshold procedure to obtain sparse algebraic equations. Through numerical examples, performance of the present method is investigated concerning the convergence and the sparseness of resulted matrix equation.

Keywords: Pocklington integral equation, numerical solutions, Haar wavelets

AMS Subject Classification: 65R20, 65T60, 78M25

1. INTRODUCTION

In recent years wavelets have gained a lot of interest in many application fields, such as signal processing [3], and solving differential and integral equations [2]. Different variations of wavelet bases (orthogonal, biorthogonal, multiwavelets) have been presented and the design of the corresponding wavelet and scaling functions has been addressed [4, 7]. Wavelets permit the accurate representation of variety of functions and operators. Moreover wavelets establish a connection with fast numerical algorithms [1].

The use of wavelets in integral equations is a subject recently studied by many authors. It was discovered in [1] that the representation of an integral operator by compactly supported orthonormal wavelets produces numerically sparse matrices to some degree of precision.

In the theory of linear antenna and scatters, the current distribution along the wire satisfies the so-called Pocklington's equation, first applied to dipoles by H. C. Pocklington in 1897 [8]. This equation, which is in the form of a Fredholm integral equation of first kind, is basically an integrodifferential equation. Due to the presence of derivatives in the integral as well as the singular nature of the kernel, its numerical evaluation requires a special treatment. In the past, the method of moments has been used to solve Pocklington's equation and several approaches are introduced to eliminate the differential operator [5, 6]. These may be categorized as follows:

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1) writing the integrodifferential equation in the form of harmonic differential equation whose solution is used to form a single integral equation of Hallen's type which is then solved for the unknown function;

2) applying piecewise linear functions or sinusoids for basis and/or testing functions in the moments methods and using integration by parts twice to eliminate the second derivative which results in a difference equation;

3) replacing the second derivative in the equation by finite difference approximation and thus obtain an integrodifference equation.

While the above approaches are aimed at removing the differential operator from Pocklington's equation, very little work has been done to treat the singularity of the integral, which may cause serious difficulties in numerical computations.

In this paper we apply the Haar wavelet bases to solve Pocklington's integral equation. The method consists in reducing the Pocklington's equation to a set of algebraic equations by expanding the current as Haar wavelets with unknown coefficients. The properties of Haar wavelets are then utilized to evaluate the unknown coefficients.

The paper is organized as follows: In Section 2 we describe the basic formulation of the Haar wavelets required for our subsequent development. Section 3 is devoted to the formulation of the Pocklington's integral equation. In Section 4 the proposed method is used to approximate the Pocklington's equation. In Section 5, we report our numerical finding and demonstrate the accuracy of the proposed numerical scheme by considering numerical examples.

2. HAAR WAVELETS

The Haar wavelets are the simplest orthonormal wavelets. The Haar scaling function is defined as

$$\phi(x) = \begin{cases} 1, & 0 \le x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

The Haar wavelet function is defined as

$$\psi(x) = \begin{cases} 1, & 0 \le x < 1/2 \\ -1, & 1/2 \le x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore $\phi(x)$ and $\psi(x)$ satisfy the following two-scale difference equations,

$$\phi(x) = \phi(2x) + \phi(2x-1), \tag{1}$$

$$\psi(x) = \phi(2x) - \phi(2x-1). \tag{2}$$

Let $\phi_{jl}(x)$ and $\psi_{jl}(x)$, $j = 0, 1, \cdots$ and $l = 0, \cdots, 2^j - 1$, be obtained from $\phi(x)$ and $\psi(x)$ by dilation and translation,

$$\phi_{jl}(x) = 2^{j/2}\phi(2^j x - l),$$

$$\psi_{jl}(x) = 2^{j/2}\psi(2^j x - l).$$

For any fixed nonnegative integer J, a function $f(x)$ defined over $[0,1)$ can be approximated by the scaling functions as

$$f(x) \simeq \sum_{l=0}^{2^J-1} c_{Jl}\phi_{Jl}(x) = C^T\Phi(x), \tag{3}$$

where

$$C = \left[c_{J0}, \cdots, c_{J,2^J-1}\right]^T,$$
$$\Phi(x) = \left[\phi_{J0}(x), \cdots, \phi_{J,2^J-1}(x)\right]^T,$$

and the coefficients c_{Jl} are computed by

$$c_{Jl} = \int_0^1 f(x)\phi_{Jl}(x)\,\mathrm{d}x = 2^{J/2}\int_{\bar{\tau}_{Jl}}^{\bar{\tau}_{J,l+1}} f(x)\,\mathrm{d}x,$$

where

$$\bar{\tau}_{Jl} = \frac{l}{2^J}, \quad J = 0, 1, \cdots, \quad l = 0, \cdots, 2^J - 1.$$

The decomposition of $f(x)$ in the Haar wavelets is given by

$$f(x) \simeq c_{00}\phi_{00}(x) + \sum_{j=0}^{J-1}\sum_{l=0}^{2^j-1} d_{jl}\psi_{jl}(x) = D^T\Psi(x), \tag{4}$$

where

$$D = [\ c_{00} \mid d_{00} \mid d_{10},\ d_{11} \mid \cdots \mid d_{J0}, \cdots, d_{J,2^J-1}\ ]^T,$$
$$\Psi(x) = [\ \phi_{00}(x) \mid \psi_{00}(x) \mid \psi_{10}(x),\ \psi_{11}(x) \mid \cdots \mid \psi_{J-1,0}(x), \cdots, \psi_{J-1,2^{J-1}-1}(x)\ ]^T,$$

and the coefficients d_{jl} are computed by

$$d_{jl} = \int_0^1 f(x)\psi_{jl}(x)\,\mathrm{d}x.$$

In practice one will first calculate coefficients c_{Jl}, for fixed J in equation (3), and then the wavelets coefficients in equation (4) are computed by applying the following relations recursively for $j = J-1, \cdots, 0$ and $l = 0, \cdots, 2^j - 1$ [1,3],

$$c_{jl} = \frac{1}{\sqrt{2}}\left(c_{j+1,2l} + c_{j+1,2l+1}\right), \tag{5}$$
$$d_{jl} = \frac{1}{\sqrt{2}}\left(c_{j+1,2l} - c_{j+1,2l+1}\right). \tag{6}$$

These relations follow from equations (1) and (2) and are linear mappings from R^{2^J} to R^{2^J} that convert the vector C to D.

Now, let $\kappa(x, y)$ be a function of two independent variables defined for $x, y \in [0, 1)$. then κ can be expanded in the Haar wavelets as

$$\kappa(x,y) \simeq \sum_{m=1}^{2^J} \sum_{n=1}^{2^J} U_{mn} \Psi_m(x) \Psi_n(y) = \Psi^T(x) U \Psi(y), \tag{7}$$

where

$$U_{mn} = \int_0^1 \int_0^1 \kappa(x,y) \Psi_m(x) \Psi_n(y) \, \mathrm{d}x \mathrm{d}y.$$

Similarly one may expand $\kappa(x, y)$ by the scaling functions as

$$\kappa(x,y) \simeq \sum_{m=1}^{2^J} \sum_{n=1}^{2^J} V_{mn} \Phi_m(x) \Phi_n(y) = \Phi^T(x) V \Phi(y), \tag{8}$$

where

$$V_{mn} = \int_0^1 \int_0^1 \kappa(x,y) \Phi_m(x) \Phi_n(y) \, \mathrm{d}x \mathrm{d}y = 2^J \int_{\bar{\tau}_{Jm}}^{\bar{\tau}_{J,m+1}} \int_{\bar{\tau}_{Jn}}^{\bar{\tau}_{J,n+1}} \kappa(x,y) \, \mathrm{d}x \mathrm{d}y.$$

To obtain matrix U in equation (7) in practice we first calculate matrix V in equation (8), and then by using equations (5) and (6), we evaluate U.

3. POCKLINGTON'S INTEGRAL EQUATION

Pocklington's integral equation for the thin-wire cylindrical antenna of length L and radius a is given in [8] as

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \beta^2\right) \int_0^L I_z(z') \frac{e^{-j\beta R}}{R} \, \mathrm{d}z' = -j4\pi\omega\epsilon_0 E_z^i(z), \tag{9}$$

where

$$R = \sqrt{(z - z')^2 + a^2}.$$

In equation (9) $\beta = 2/\lambda$ (λ=wavelength) is the wavenumber, ω is the angular frequency and ϵ_0 is the permitivity of the medium. This equation relates the unknown total axial current I_z to the axial component of known incident electric field E_z^i on the surface of the cylinder. Richmond [9] has derived a convenient form of (9) by interchanging the order of integration and differentiation. The result is

$$\int_0^L I_z(z') K(z, z') \, \mathrm{d}z' = \theta(z), \tag{10}$$

where

$$K(z, z') = \frac{e^{-j\beta R}}{R^5} \left[(1 + j\beta R)(2R^2 - 3a^2) + (\beta a R)^2\right], \tag{11}$$

and

$$\theta(z) = -j4\pi\omega\epsilon_0 E_z^i(z).$$

4. DISCRETIZATION OF POCKLINGTON'S INTEGRAL EQUATION

In this section we discretize the Pocklington integral equation (10) by using Haar wavelets. For this propose, for any fixed nonnegative integer J, we use equation (4), to approximate the current $I_z(z')$ as

$$I_z(z') = \alpha^T \Psi(z'), \tag{12}$$

where α is an unknown vector. We also expand $\theta(z)$ and $K(z,z')$ as

$$\theta(z) = F^T \Psi(z), \tag{13}$$

$$K(z,z') = \Psi^T(z) B \Psi(z'), \tag{14}$$

where F and B are calculated similarly to equations (4) and (7) respectively. By using equations (12) – (14) and the orthonormal property of $\Psi(z')$, the integral equation (3) is thereby approximated by the system

$$B\alpha = F, \tag{15}$$

which is a system of 2^J equations in 2^J unknowns. Equation (15) may be solved numerically for α and hence yield an approximate solution to equation (12).

It should be pointed out that because of the local supports and vanishing moment property of the Haar wavelets, many of the elements of matrix B are very small compared to the largest element, and can be dropped without significantly affecting the solution [1]. This is referred to as "thresholding", which set those elements of the matrix to zero that are smaller (in magnitude) than some positive number ε $(0 \leq \varepsilon < 1)$, known as the threshold parameter, multiplied by the largest elements of the matrix [6]. The value of threshold parameter ε needs to be well chosen so as to balance the computational efficiency and accuracy of the approximation solutions. After this process, a sparse matrix is obtained and the sparse matrix can be very efficiently solved by using a sparse matrix solver such as the conjugate gradient method.

To obtain matrix B in equation (14), we first calculate matrix A where

$$K(z,z') = \Psi^T(z) B \Psi(z') = \Phi^T(z) A \Phi(z'),$$

and then using equations (5) and (6) we evaluate B. For $m, n = 1, \cdots, 2^J$ we get

$$A_{mn} = 2^J \int_{\bar{\tau}_{Jm}}^{\bar{\tau}_{J,m+1}} \int_{\bar{\tau}_{Jn}}^{\bar{\tau}_{J,n+1}} K(z,z')\,\mathrm{d}z\mathrm{d}z'. \tag{16}$$

For $m \neq n$ the entry A_{mn} in equation (16) of the system matrix A can be evaluated by using the numerical quadrature rule. When $m = n$ the integrand of equation (16) is very sharply peaked, particularly for small value of a. Therefore, from the computational point of view, it would be advantageous to isolate and extract the singularity from equation (11). This may be accomplished by writhing $K(z,z')$ as

$$K(z,z') = K^{(n)}(z,z') + K^{(s)}(z,z'), \tag{17}$$

where $K^{(n)}$ and $K^{(s)}$ denote the nonsingular and singular parts of kernel K and are given in [10] as

$$\begin{aligned} K^{(n)}(z,z') = & \frac{[e^{-j\beta R}+j\beta R-1][(1+j\beta R)(2R^2-3a^2)+(3aR)^2]}{R^5} \\ & + \frac{[R^2/2][(2+(3a/2)^2)(3R)^2-(3+j2\beta R)(3a)^2]}{R^5}, \end{aligned} \tag{18}$$

and

$$K^{(s)}(z,z') = \beta^2\left[1-\frac{1}{8}(\beta a)^2\right]\frac{1}{R}+2\left[1-\frac{1}{4}(\beta a)^2\right]\frac{1}{R^3}-3a^2\frac{1}{R^5}. \tag{19}$$

By using equation (17), we can express equation (16) as

$$A_{mn} = A^{(n)}_{mn} + A^{(s)}_{mn},$$

where

$$A^{(n)}_{mn} = 2^J\int_{\bar{\tau}_{Jm}}^{\bar{\tau}_{J,m+1}}\int_{\bar{\tau}_{Jn}}^{\bar{\tau}_{J,n+1}} K^{(n)}(z,z')\,\mathrm{d}z\mathrm{d}z', \tag{20}$$

and

$$A^{(s)}_{mn} = 2^J\int_{\bar{\tau}_{Jm}}^{\bar{\tau}_{J,m+1}}\int_{\bar{\tau}_{Jn}}^{\bar{\tau}_{J,n+1}} K^{(s)}(z,z')\,\mathrm{d}z\mathrm{d}z'. \tag{21}$$

The integrand of the integral in equation (20) is extremely well behaved and, as a consequence, may be efficiently and accurately evaluated numerically. The integrand of the integral in equation (21) contains a singularity and can be evaluated analytically as follows,

$$A^{(s)}_{mn} = h(\bar{\tau}_{J,l+1},\bar{\tau}_{J,l'+1}) - h(\bar{\tau}_{J,l+1},\bar{\tau}_{J,l'}) - h(\bar{\tau}_{J,l},\bar{\tau}_{J,l'+1}) + h(\bar{\tau}_{J,l},\bar{\tau}_{J,l'}),$$

where

$$\frac{\partial^2 h(z,z')}{\partial z\partial z'} = K^{(s)}(z,z'). \tag{22}$$

By using equations (19) and (21) we can get $h(z,z')$ as

$$\begin{aligned} h(z,z') = & \left(-\frac{1}{8}\beta^4a^2+\frac{3}{2}\beta^2\right)R-\frac{1}{R} \\ & + \left(\beta^2-\frac{1}{8}\beta^4a^2\right)\left(z\ln\left(-z+z'+R\right)+z'\ln\left(z-z'+R\right)\right). \end{aligned}$$

5. ILLUSTRATIVE EXAMPLES

In this section, two numerical examples are presented to illustrate the validity and the merits of this technique. In both examples data are given for two selected wire lengths so that they include special cases of practical interest, e. g., $L = \lambda/2$ and $L = \lambda$. In Example 1, the convergence of our method is reported. As mentioned before one main merit of this technique is the generation of a sparse matrix from an integral operator. This advantage is illustrated in Example 2. For this propose for thresholding parameter ε the matrix sparsity (S_ε), is defined by [6]

$$S_\varepsilon = \frac{N_0 - N_\varepsilon}{N_0} \times 100\,\%$$

where N_0 is the total number of elements and N_ε is the number of elements remaining after thresholding. The relative error caused by thresholding is defined by

$$e_\varepsilon = \frac{||I_0 - I\varepsilon||_2}{||I_0||_2} \times 100\,\%.$$

The symbol $||\cdot||_2$ denotes the L^2 norm and I_0 and I_ε represent the solution obtained from equation (15) without and with thresholding the matrix elements.

Example 1. In this example we consider the Pocklington integral equation (10) for a thin wire with radius $a = .001\lambda$, $\beta = 2\pi/\lambda$, $\omega = 6\pi/\lambda \times 10^8$ and $\epsilon_0 = 8.854 \times 10^{-10}$ for the following cases

a) $L = \lambda/2$, $\lambda = 2$,

b) $L = \lambda$, $\lambda = 1$.

In both cases we consider a rectangular pulse for $E_z^i(z)$,

$$E_z^i(z) = \begin{cases} 1/2\Delta, & |z - L/2| < \Delta \\ 0, & \text{otherwise}, \end{cases}$$

where $\Delta = 0.01\lambda$.

By applying the technique described in preceding section with $J = 7, 8, 9$ we approximate current $I_z(z)$ by using equation (12). The magnitudes of normalized currents $(I_z(z')/||I_z(z')||_2)$ are shown in Figures 1 and 2 for cases (a) and (b) respectively.

As can be seen in Figures 1 and 2 the solution converge rapidly with increasing the values of J.

Example 2. In this example, our goal is to see how the thresholding affects the sparsity of the matrix and precision of the solution. We use the same antenna as in Example 1.

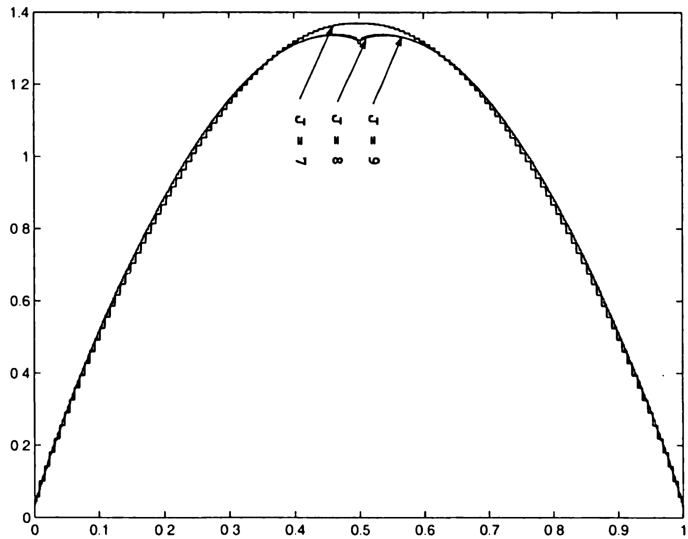


Fig. 1. The magnitude of normalized current for case $L = \lambda/2$.

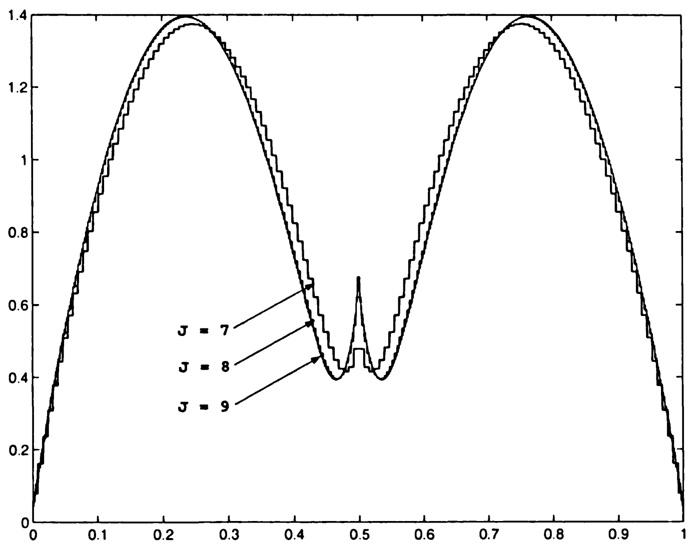
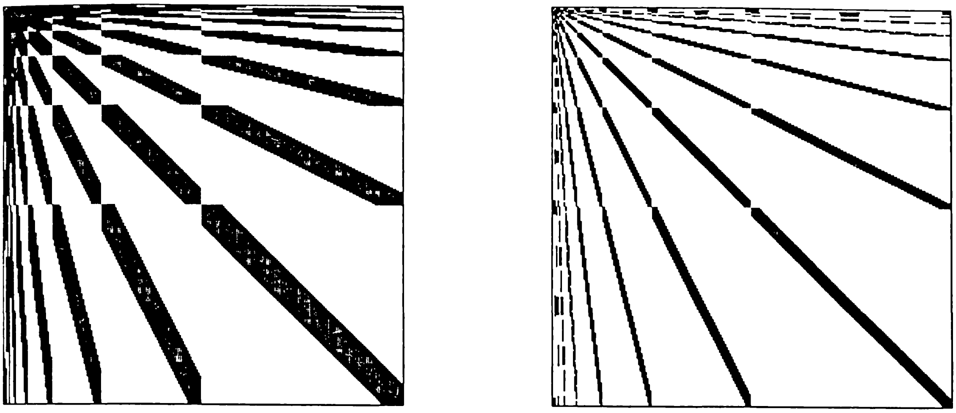


Fig. 2. The magnitude of normalized current for case $L = \lambda$.



$\varepsilon = 10^{-7}$ $\varepsilon = 10^{-5}$

Fig. 3. The remaining nonzero elements in B after thresholding with ε where $\varepsilon = 10^{-7}$, 10^{-5}, $L = \lambda$ and $J = 8$.

Table. Sparsity(S_ε) and Relative error(e_ε) for Haar wavelets. as a function of Threshold parameter and J.

		$L = \lambda/2$		$L = \lambda$	
	Threshold Parameter (ϵ)	Sparsity (S_ϵ)	Relative Error (e_ϵ)	Sparsity (S_ϵ)	Relative Error (e_ϵ)
	10^{-7}	44 %	0.0053 %	47 %	0.0004 %
$J = 7$	10^{-6}	60 %	0.1363 %	65 %	0.0143 %
	10^{-5}	73 %	2.1627 %	77 %	0.5076 %
	10^{-7}	61 %	0.0086 %	67 %	0.0018 %
$J = 8$	10^{-6}	74 %	0.1453 %	78 %	0.0549 %
	10^{-5}	83 %	4.2817 %	85 %	0.6813 %
	10^{-7}	73 %	0.0097 %	79 %	0.0027 %
$J = 9$	10^{-6}	82 %	0.1477 %	86 %	0.0770 %
	10^{-5}	88 %	4.1797 %	91 %	0.6483 %

In table we report sparsity and relative error of matrix B in equation (15), as a function of threshold parameter and J. This table show that even with 80 % sparsity the result is reasonably accurate. Figure 3 illustrate the sparseness structures of the matrix elements B in equation (15) for $\varepsilon = 10^{-7}$ and $\varepsilon = 10^{-5}$ for cases (b) in Example 1. The results are given for $J = 8$. In this figure the solid boxes indicate the remaining nonzero elements.

6. CONCLUSION

The Haar wavelets are used to solve Pocklington's integral equation for a thin-wire. Some properties of Haar wavelets are presented and are utilized to reduce the computation of Pocklington's integral equation to some sparse matrix equation. The method is computationally attractive and applications are demonstrated through illustrative examples.

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