KYBERNETIKA — VOLUME 40 (2004), NUMBER 4, PAGES 477–489

ANTI-PERIODIC SOLUTIONS TO A PARABOLIC HEMIVARIATIONAL INEQUALITY[1]

JONG YEOUL PARK, HYUN MIN KIM AND SUN HYE PARK

In this paper we deal with the anti-periodic boundary value problems with nonlinearity of the form $b(u)$, where $b \in L^{\infty}_{\text{loc}}(\mathbb{R})$. Extending b to be multivalued we obtain the existence of solutions to hemivariational inequality and variational-hemivariational inequality.

Keywords: hemivariational inequality, variational-hemivariational inequality, anti-periodic boundary value problems

AMS Subject Classification: 35K50, 35K55, 47J20, 35B10

1. INTRODUCTION

The purpose of this paper is two-fold. First, we discuss the existence of solutions to the discontinuous nonlinear nonmonotone parabolic anti-periodic boundary value problem, i. e. a parabolic hemivariational inequality (P) :

$$u'(t) + Au(t) + \Xi(t) = f(t) \quad \text{a.e. in } (0,T), \tag{1.1}$$

$$\Xi(t,x) \in \hat{b}(u(t,x)) \text{ a.e. } (t,x) \in Q = (0,T) \times \Omega, \tag{1.2}$$

$$u(T) = -u(0). \tag{1.3}$$

The nonlinearity and the discontinuity is assumed to be in the lower order term $\hat{b}$ and whereas the operator A is linear and continuous. Secondly, we shall consider a parabolic variational-hemivariational inequality $(P)_c$:

$$f(t) - u'(t) - Au(t) - \Xi(t) \in \partial\Psi(u(t)) \text{ a.e. in } (0,T), \tag{1.4}$$

$$\Xi(t,x) \in \hat{b}(u(t,x)) \text{ a.e. } (t,x) \in Q = (0,T) \times \Omega, \tag{1.5}$$

$$u(T) = -u(0), \tag{1.6}$$

where Ψ is a lower semicontinuous convex functional defined on a real Hilbert space H. The precise hypotheses on the above two systems will be given in the next section. The background of these problems are in physics, especially in solid mechanics, where

[1]This work was supported by grant No. (R01-2002-000-00491-0) from the Korea Science and Engineering Foundation.

nonmonotone, multivalued constitutive laws lead to hemivariational inequalities. The concept of a hemivariational inequality is introduced by Panagiotopoulos in [10]. Recently, anti-periodic boundary value problems to the various systems have been studied in a series of papers [1, 2, 3, 7, 8] after Okochi's pioneering work [9]. An important advantage of anti-periodicity is that one can handle non-coercive evolution equations which generally cannot be shown to admit classical periodic solutions. It is also worth noting that anti-periodic solutions arise naturally in the mathematical modelling of a variety of physical processes. There are some papers [5, 6] dealing with these kinds of problems concerning the initial value problem, that is, (1.1) - (1.2) or (1.4) - (1.5) together with $u(0) = u_0$. M. Miettinen [5] proved the existence results to the system (1.1) - (1.2) with such a given initial value. However, in this paper, we prove the existence of anti-periodic solutions for (1.1) - (1.2) and (1.4) - (1.5) with anti-periodic boundary condition (1.3) and (1.4), respectively. Our technique employs some ideas of [3] and [11]. The plan of this paper is as follows. In Section 2, the assumptions and the problems are formulated. In Section 3, the existence of a solution to the problem (P) is proved by using the Galerkin method. The paper concludes with a discussion of existence of a solution to the problem $(P)_c$ in Section 4.

2. FORMULATION OF THE MAIN PROBLEM

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega, 0 < T < \infty$ and $Q = (0,T) \times \Omega$. Let us denote by H the real Hilbert space $L^2(\Omega)$ and by $|\cdot|$ the norm and $(\cdot,\cdot)$ the inner product of $L^2(\Omega)$. Let V be a real Hilbert space with the norm $\|\cdot\|_V$ such that $V \hookrightarrow H^1(\Omega)$. V^* denotes the dual space of V with the norm $\|\cdot\|_{V^*}$ and $\langle\cdot,\cdot\rangle$ is the corresponding duality. Assume that the imbedding $V \hookrightarrow H$ is dense, continuous and compact. Let $X = L^2(0,T;V), X^* = L^2(0,T;V^*)$ and their norms $\|\cdot\|_X, \|\cdot\|_{X^*}$ and the duality $\langle\cdot,\cdot\rangle_X$. It is well known that space $W(V) = \{u \in X : u' \in X^*\}$ forms a real Hilbert space with the norm $\|u\|_W = \|u\|_X + \|u'\|_{X^*}$ and is continuously imbedded in $C([0,T];H)$.

We formulate the following assumptions

(HA) $A : V \to V^*$ is linear, continuous, symmetric and coercive, i.e.

$$\exists\, C_1 \geq 0 : \|Av\|_{V^*} \leq C_1\|v\|_V,\ \forall\, v \in V,$$
$$\exists\, C_2 > 0, C_3 \geq 0 : \langle Av, v\rangle \geq C_2\|v\|_V^2 - C_3|v|^2,\ \forall\, v \in V.$$

(HB) (1) $b \in L^\infty_{\text{loc}}(\mathbb{R})$,

(2) $\exists\, s_0 \geq 0 : 0 \leq \text{ess}\ \inf_{s_0<t<\infty} b(t)$,

(3) $b(-s) = -b(s)$ a.e. $s \geq s_0$.

(HF) $f \in L^2(0,T;H),\ f(t+T) = -f(t)\ \forall\, t \geq 0$.

The multi-valued function $\hat{b} : \mathbb{R} \to \mathbb{R}$ is obtained by filling in jumps of a function

$b : \mathbb{R} \to \mathbb{R}$ by means of the functions $\underline{b}_\epsilon, \bar{b}_\epsilon, \underline{b}, \bar{b} : \mathbb{R} \to \mathbb{R}$ as follows

$$\underline{b}_\epsilon(t) = \text{ess inf}_{|s-t|\le\epsilon} b(s),\ \bar{b}_\epsilon(t) = \text{ess sup}_{|s-t|\le\epsilon} b(s);$$
$$\underline{b}(t) = \lim_{\epsilon\to0^+} \underline{b}_\epsilon(t),\ \bar{b}(t) = \lim_{\epsilon\to0^+} \bar{b}_\epsilon(t);$$
$$\hat{b}(t) = [\underline{b}(t), \bar{b}(t)].$$

We shall need a regularization of b defined by

$$b^n(t) = n\int_{-\infty}^{\infty} b(t-\tau)\rho(n\tau)\,\mathrm{d}\tau,$$

where $\rho \in C_0^\infty((-1,1)), \rho \ge 0$ and $\int_{-1}^1 \rho(\tau)\,\mathrm{d}\tau = 1$. It is easy to show that b^n is continuous and odd for all $n \in \mathbb{N}$. Moreover (HB) implies that there exist positive constants S_0 and ν such that for all $n \in \mathbb{N}$,

$$tb^n(t) \ge 0 \text{ for } |t| > S_0, \tag{2.1}$$
$$|b^n(t)| \le \nu \text{ for } |t| \le S_0. \tag{2.2}$$

Since we shall use the Galerkin method for proving the existence of a solution of the problem (P), we need a family of finite-dimensional subspaces $V_n \subset C^\infty(\bar{\Omega}) \cap V$ such that $\cup_{n=1}^\infty V_n$ is dense in $\tilde{V} = V \cap C(\bar{\Omega})$ in the following sense:

$$\forall\, v \in \tilde{V}; \exists\, v_n \in V_n \text{ such that } v_n \to v \text{ in } V \cap C(\bar{\Omega}). \tag{2.3}$$

Further assume that $V \cap C(\bar{\Omega})$ is dense in V. Let us formulate a regularized Galerkin equation $(P)_n$:

Find $u_n \in W(V_n) = \{u \in L^2(0,T;V_n) : u' \in L^2(0,T;V_n)\}$ such that

$$\langle u_n'(t) + Au_n(t) + b^n(u_n(t)), v_n\rangle = \langle f(t), v_n\rangle,\ \forall\, v_n \in V_n,\ \text{a.e. } t \in (0,T), \tag{2.4}$$
$$u_n(0) = -u_n(T). \tag{2.5}$$

Substituting of $u_n = \sum_{j=1}^n c_{jn}(t)\varphi_{jn}$, where $\{\varphi_{jn}\}_{j=1}^n$ is a basis of V_n, to $(P)_n$ gives a first-order system of ordinary differential equations for the real functions $t \to c_{jn}(t),\ j = 1, 2, \ldots, n$. The solvability of the problem $(P)_n$ is guaranteed by the Caratheodory theorem and a priori estimates.

3. HEMIVARIATIONAL PROBLEMS

Definition 3.1. A function $u \in W(V)$ is a solution of the problem (P) if there exists $\Xi \in L^1(Q) \cap X^*$ such that

(1) $\displaystyle\int_0^T \langle u'(t) + Au(t) + \Xi(t), \Phi(t)\rangle\,\mathrm{d}t = \int_0^T \langle f(t), \Phi(t)\rangle\,\mathrm{d}t,\ \forall\, \Phi \in X$

(2) $u(0) = -u(T)$,

(3) $\Xi(t,x) \in \hat{b}(u(t,x))$ a.e. $(t,x) \in Q$.

Theorem 3.1. Assume that (HA), (HB) and (HF) hold. Then the problem (P) has at least one solution.

Proof. We divide the existence proof into three steps.

Step 1: A priori estimates.
Let u_n be a solution of the $(P)_n$. Choose u_n' as a test function in $(P)_n$ and integrate the resulting equation over $(0,T)$ to obtain

$$\|u_n'\|^2_{L^2(0,T;H)} + \int_0^T \langle Au_n(t), u_n'(t)\rangle \,\mathrm{d}t + \int_0^T (b^n(u_n(t)), u_n'(t))\,\mathrm{d}t \tag{3.1}$$
$$= \int_0^T (f(t), u_n'(t))\,\mathrm{d}t.$$

By using (HA) and (2.5), we have

$$\int_0^T \langle Au_n(t), u_n'(t)\rangle\,\mathrm{d}t = \frac{1}{2}\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\langle Au_n(t), u_n(t)\rangle\,\mathrm{d}t = 0.$$

Next, let us rewrite $\int_0^T (b^n(u_n(t)), u_n'(t))\,\mathrm{d}t$ in the more useful form

$$\int_0^T \int_\Omega b^n(u_n(t,x))u_n'(t,x)\,\mathrm{d}x\,\mathrm{d}t \tag{3.2}$$
$$= \int_0^T \int_\Omega \left(\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{u_n(t,x)} b^n(\tau)\mathrm{d}\tau\right)\mathrm{d}x\,\mathrm{d}t$$
$$= \int_\Omega \left(\int_0^{u_n(T,x)} b^n(\tau)\mathrm{d}\tau - \int_0^{u_n(0,x)} b^n(\tau)\,\mathrm{d}\tau\right)\mathrm{d}x.$$

Using the anti-periodicity $u_n(T,x) = -u_n(0,x)$ and the oddness of b^n, we observe from (3.2) that

$$\int_0^T (b^n(u_n(t)), u_n'(t))\,\mathrm{d}t = 0.$$

Hence, from (3.1), we have

$$\|u_n'\|_{L^2(0,T;H)} \le \|f\|_{L^2(0,T;H)}. \tag{3.3}$$

This inequality and Poincare's inequality for anti-periodic functions (see e. g. [3]) yields,

$$\|u_n\|_{L^\infty(0,T;H)} \le \frac{1}{2}\sqrt{T}\|f\|_{L^2(0,T;H)}. \tag{3.4}$$

Substitute v_n by $u_n(t)$ to (2.4) and integrate over $(0,T)$. Using (HA) and (2.5), we arrive at

$$C_2\|u_n\|_X^2 + \int_0^T \int_\Omega b^n(u_n(t,x))u_n(t,x)\,\mathrm{d}x\,\mathrm{d}t \tag{3.5}$$
$$\le C_3\|u_n\|^2_{L^2(0,T;H)} + \int_0^T (f(t), u_n(t))\,\mathrm{d}t.$$

Define $Q_1 = \{(t,x) \in Q : |u_n(t,x)| > S_0\}$ and $Q_2 = \{(t,x) \in Q : |u_n(t,x)| \leq S_0\}$. Observe that

$$\int_0^T \int_\Omega b^n(u_n(t,x))u_n(t,x)\,\mathrm{d}x\,\mathrm{d}t \tag{3.6}$$
$$= \int_{Q_1} b^n(u_n(t,x))u_n(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_2} b^n(u_n(t,x))u_n(t,x)\,\mathrm{d}x\,\mathrm{d}t,$$

where the integral over Q_1 is nonnegative (see (2.1)). Combining (3.5) and (3.6), we get

$$\begin{aligned} C_2\|u_n\|_X^2 \quad \leq \quad & C_3\|u_n\|^2_{L^2(0,T;H)} + \int_0^T (f(t), u_n(t))\,\mathrm{d}t \\ & - \int_{Q_2} b^n(u_n(t,x))u_n(t,x)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.7}$$

In what follows, we use C to denote a generic positive constant independent of n. On account of (2.2), we conclude that

$$\int_{Q_2} |b^n(u_n(t,x))u_n(t,x)|\,\mathrm{d}x\,\mathrm{d}t \leq C. \tag{3.8}$$

Now, using (3.4), (3.7) and (3.8), we infer that

$$\|u_n\|_X^2 \leq C. \tag{3.9}$$

Returning to (3.5) we deduce by (3.4) that

$$\int_0^T \int_\Omega b^n(u_n(t,x))u_n(t,x)\,\mathrm{d}x\,\mathrm{d}t \leq C. \tag{3.10}$$

Next, we show the weak precompactness of the subsequence $\{b^n(u_n(t,x))\}$ in $L^1(Q)$. Using (3.8) and (3.10), we get

$$\int_Q |b^n(u_n(t,x))u_n(t,x)|\,\mathrm{d}x\,\mathrm{d}t \tag{3.11}$$
$$\begin{aligned} &= \int_{Q_1} b^n(u_n(t,x))u_n(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_2} |b^n(u_n(t,x))u_n(t,x)|\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_Q b^n(u_n(t,x))u_n(t,x)\,\mathrm{d}x\,\mathrm{d}t - \int_{Q_2} b^n(u_n(t,x))u_n(t,x)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_{Q_2} |b^n(u_n(t,x))u_n(t,x)|\,\mathrm{d}x\,\mathrm{d}t \\ &\leq \int_Q b^n(u_n(t,x))u_n(t,x)\,\mathrm{d}x\,\mathrm{d}t + 2\int_{Q_2} |b^n(u_n(t,x))u_n(t,x)|\,\mathrm{d}x\,\mathrm{d}t \\ &\leq C. \end{aligned}$$

Using (3.11) we can show that for each $\epsilon > 0$, there exist K so large and $\delta(\epsilon) > 0$ such that

$$\frac{1}{K}\int_Q |b^n(u_n(t,x))u_n(t,x)|\,\mathrm{d}x\,\mathrm{d}t < \frac{\epsilon}{2} \text{ and}$$

$$\delta(\epsilon)\,\mathrm{ess\,sup}_{|s|\le K+1}\ |b^n(s)| < \frac{\epsilon}{2}.$$

If $\omega \subset Q$ with $\mathrm{meas}(\omega) < \delta(\epsilon)$, then

$$\begin{aligned}
&\int_\omega |b^n(u_n(t,x))|\,\mathrm{d}x\,\mathrm{d}t \\
&\le \int_\omega \frac{1}{K}|b^n(u_n(t,x))u_n(t,x)|\,\mathrm{d}x\,\mathrm{d}t + \int_\omega \sup_{|u_n(t,x)|\le K+1} |b^n(u_n(t,x))|\,\mathrm{d}x\,\mathrm{d}t \\
&\le \int_Q \frac{1}{K}|b^n(u_n(t,x))u_n(t,x)|\,\mathrm{d}x\,\mathrm{d}t + \mathrm{ess\,sup}_{|u_n(t,x)|\le K+1}\ |b^n(u_n(t,x))|\cdot \mathrm{meas}(\omega) \\
&< \epsilon,\ \forall n \in \mathbb{N},
\end{aligned}$$

where we used the estimate $|b^n(s)| \le \frac{1}{k}|sb^n(s)| + sup_{|s|\le k+1}|b^n(s)|,\ \forall k > 0$. Thus, applying the Dunford–Pettis criterion, we conclude that $\{b^n(u_n)\}$ is weakly precompact in $L^1(Q)$.

Step 2: Convergences of subsequences.
By using the priori estimates (3.3), (3.4) and (3.9), the compactness of the imbedding of V into H and Arzela Ascoli's theorem, we have subsequences (in the sequel we denote subsequences by the same symbols as original sequences) such that

$$u_n \to u \text{ weakly in } X \cap W(V) \text{ and stronly in } C([0,T];H), \tag{3.12}$$

$$u_n' \to u' \text{ weakly in } L^2(0,T;H), \tag{3.13}$$

$$b^n(u_n) \to \Xi \text{ weakly in } L^1(Q), \tag{3.14}$$

$$Au_n \to Au \text{ weakly in } X^*. \tag{3.15}$$

Step 3: (u, Ξ) is a solution of (P).
From $(P)_n$ we get

$$\begin{aligned}
&\int_0^T \langle u_n'(t) + Au_n(t) + b^n(u_n(t)), v(t)\rangle\,\mathrm{d}t \\
&= \int_0^T \langle f(t), v(t)\rangle\,\mathrm{d}t,\ \forall v \in C([0,T];V_n).
\end{aligned} \tag{3.16}$$

Letting n tend to infinity in (3.16) and taking into account (3.12) – (3.15), (2.3) gives

$$\begin{aligned}
&\int_0^T \langle u'(t) + Au(t) + \Xi(t), v(t)\rangle\,\mathrm{d}t \\
&= \int_0^T \langle f(t), v(t)\rangle\,\mathrm{d}t,\ \forall v \in C([0,T];\tilde{V}).
\end{aligned} \tag{3.17}$$

Since $C([0,T];\tilde{V})$ is dense in X (see e.g. [6]), (3.17) holds for all $\Phi \in X$ and hence $\Xi \in X^*$. Because $u_n \to u$ strongly in $C([0,T];H)$ and $u_n(T) = -u_n(0)$, it is clear that $u(T) = -u(0)$. It remains to show that $\Xi(t,x) \in \hat{b}(u(t,x))$ *a.e.* in Q. Since $u_n \to u$ strongly in $L^2(Q)$, we can assume that

$$u_n(t,x) \to u(t,x) \text{ a.e. in } Q \text{ as } n \to \infty.$$

Let $\eta > 0$. Using the theorems of Lusin and Egoroff, we can choose a subset $\omega \subset Q$ such that $meas(\omega) < \eta$, $u \in L^\infty(Q \setminus \omega)$ and $u_n \to u$ uniformly on $Q \setminus \omega$. Thus, for each $\epsilon > 0$, there is an $N > \frac{2}{\epsilon}$ such that

$$|u_n(t,x) - u(t,x)| < \frac{\epsilon}{2}, \; \forall (t,x) \in Q \setminus \omega.$$

Then, if $|u_n(t,x) - s| < \frac{1}{n}$, we have $|u(t,x) - s| < \epsilon$ for all $n > N$ and $(t,x) \in Q \setminus \omega$. Therefore we have

$$\underline{b}_\epsilon(u(t,x)) \le b^n(u_n(t,x)) \le \bar{b}_\epsilon(u(t,x)), \; \forall n > N, (t,x) \in Q \setminus \omega.$$

Let $\phi \in L^\infty(Q), \phi \ge 0$. Then

$$\begin{aligned} \int_{Q\setminus\omega} \underline{b}_\epsilon(u(t,x))\phi(t,x)\,\mathrm{d}x\,\mathrm{d}t &\le \int_{Q\setminus\omega} b^n(u_n(t,x))\phi(t,x)\,\mathrm{d}x\,\mathrm{d}t \\ &\le \int_{Q\setminus\omega} \bar{b}_\epsilon(u(t,x))\phi(t,x)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.18}$$

Letting $n \to \infty$ in (3.18) and using (3.14), we obtain

$$\begin{aligned} \int_{Q\setminus\omega} \underline{b}_\epsilon(u(t,x))\phi(t,x)\,\mathrm{d}x\,\mathrm{d}t &\le \int_{Q\setminus\omega} \Xi(t,x)\phi(t,x)\,\mathrm{d}x\,\mathrm{d}t \\ &\le \int_{Q\setminus\omega} \bar{b}_\epsilon(u(t,x))\phi(t,x)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.19}$$

Letting $\epsilon \to 0^+$ in (3.19), we infer that

$$\Xi(t,x) \in \hat{b}(u(t,x)) \text{ a.e. in } Q \setminus \omega,$$

and letting $\eta \to 0^+$ we get

$$\Xi(t,x) \in \hat{b}(u(t,x)) \text{ a.e. in } Q.$$

Remark 3.1. If we impose the following additional linear growth condition on b:

(HB*) $\qquad \exists c > 0 : |b(t)| \le c(1 + |t|) \text{ a.e. in } \mathbb{R},$

we can obtain stronger results on $b^n(u_n)$, i.e. $b^n(u_n) \to \Xi$ weakly in $L^2(Q)$.

Indeed, from (HB*), it follows that

$$\begin{aligned} \|b^n(u_n(\cdot))\|^2_{L^2(Q)} &= \int_0^T \int_\Omega |b^n(u_n(t,x))|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\le \int_0^T \int_\Omega (c(1+|u_n(t,x)|))^2 \mathrm{d}x\mathrm{d}t \\ &\le \int_0^T \int_\Omega (c_1(1+|u_n(t,x)|^2))\,\mathrm{d}x\,\mathrm{d}t \\ &\le c_2(1+\|u_n\|^2_{L^2(Q)}). \end{aligned} \tag{3.20}$$

Since it is shown that $\{u_n\}$ is bounded in $L^2(Q), \{b^n(u_n)\}$ is bounded in $L^2(Q)$. Thus we may assume that $b^n(u_n) \to \Xi$ weakly in $L^2(Q)$.

Remark 3.2. If b satisfies the condition (HB*), it is easily shown that $\underline{b}, \bar{b}, \underline{b}_\epsilon, \bar{b}_\epsilon, b^n$ defined in Section 2 satisfy the same condition (HB*) with a possibly different constant c.

4. VARIATIONAL–HEMIVARIATIONAL PROBLEMS

This section is devoted to study the variational-hemivariational problem $(P)_c$:

$$f(t) - u'(t) - Au(t) - \Xi(t) \in \partial\Psi(u(t)) \quad \text{a.e. in } (0,T), \tag{4.1}$$

$$u(0) = -u(T), \tag{4.2}$$

$$\Xi(t,x) \in \hat{b}(u(t,x)) \quad \text{a.e. } (t,x) \in Q, \tag{4.3}$$

where Ψ is a proper convex, lower semicontinuous and even functional from H to $\mathbb{R} \cup \{+\infty\}$ and $\partial\Psi$ is a subdifferential of Ψ defined by

$$\partial\Psi(u) = \{z \in H : \Psi(v) - \Psi(u) \ge \langle z, v-u\rangle, \ \forall\, v \in H\}.$$

We show that under the linear growth condition (HB*) on b the above problem has a solution. The proof is based on the following approach (see [6]): We approximate Ψ by a sequence of even and convex Gateaux differentiable functions $\Psi_\epsilon, \epsilon > 0$, on H and prove the solvability of $(P)_{c\epsilon}$ defined by $(P)_c$ in which Ψ is replaced by Ψ_ϵ. Then we show that $(P)_{c\epsilon}$ tends to $(P)_c$ as $\epsilon \to 0^+$. We denote by Ψ'_ϵ the Gateaux derivative of Ψ_ϵ, $\epsilon > 0$. First we need impose some conditions for the approximating sequence $\{\Psi_\epsilon\}$:

(HΨ) (1) $\displaystyle\int_0^T \Psi_\epsilon(v(t))\,\mathrm{d}t \to \int_0^T \Psi(v(t))\,\mathrm{d}t, \ \forall\, v \in L^2(0,T;H)$ as $\epsilon \to 0^+$.

(2) $\Psi'_\epsilon(0) = 0, \ \epsilon > 0$.

(3) If $v_\epsilon \to v$ weakly in $L^2(0,T;H)$, $v'_\epsilon \to v'$ weakly in $L^2(0,T;H)$ and $\displaystyle\int_0^T \Psi_\epsilon(v_\epsilon(t))\,\mathrm{d}t \le M$, where M is a constant independent of ϵ, then

$$\underline{\lim}_{\epsilon\to 0^+} \int_0^T \Psi_\epsilon(v_\epsilon(t))\,\mathrm{d}t \ge \int_0^T \Psi(v(t))\,\mathrm{d}t.$$

(4) The family $\{\Psi_\epsilon\}$ is uniformly proper, i. e. there exists an element $g \in H$ and a constant $C > 0$ such that

$$\Psi_\epsilon(v) \geq \langle g, v\rangle - C, \ \forall\, v \in H, \epsilon > 0.$$

Let $\varphi : H \to (-\infty, +\infty]$ be a proper, convex, lower semicontinuous function with effective domain $D(\varphi) = \{x \in H : \varphi(x) < +\infty\}$. Then $\varphi_\epsilon : H \to \mathbb{R}$ defined by

$$\varphi_\epsilon(x) = \inf_{y\in H} \left\{\varphi(y) + \frac{|y-x|^2}{2\epsilon}\right\}, \ x \in H, \epsilon > 0,$$

is convex and Gateaux differentiable on H. Once φ is taken to be even, then it is easy to check that $0 \in \partial\varphi(0), \varphi_\epsilon$ is even and φ'_ϵ and $\partial\varphi$ are odd. Moreover, φ_ϵ satisfies all the conditions stated in (HΨ) (see [4] for details). Thus our assumptions (HΨ) make sense. Also, we need the following assumption:

(HA*) A is the operator stated in (HA) with $C_3 = 0$.

Definition 4.1. A function $u \in W(V)$ is a solution of the problem $(P)_c$ if there exists $\Xi \in L^2(Q) \cap X^*$ such that

(1) $$\int_0^T \Psi(v(t))\,\mathrm{d}t - \int_0^T \Psi(u(t))\,\mathrm{d}t \geq \int_0^T \langle f(t) - u'(t) - Au(t), v(t) - u(t)\rangle\,\mathrm{d}t - \int_0^T (\Xi(t), v(t) - u(t))\,\mathrm{d}t, \ \forall\, v \in X,$$

(2) $\Xi(t,x) \in \hat{b}(u(t,x))$ a. e. $(t,x) \in Q$,

and

(3) $u(0) = -u(T)$.

Theorem 4.1. Assume that (HA*), (HF), (HB) and (HB*) hold. Then the problem $(P)_c$ has at least one solution.

Since $\Psi_\epsilon, \epsilon > 0$, is Gateaux differentiable, we can consider the following approximating problem $(P)_{c\epsilon}$:

$$\int_0^T \langle u'_\epsilon(t) + Au_\epsilon(t) + \Xi_\epsilon(t) + \Psi'_\epsilon(u_\epsilon(t)), v(t)\rangle\,\mathrm{d}t = \int_0^T \langle f(t), v(t)\rangle\,\mathrm{d}t, \ \forall\, v \in X,$$

$$\Xi_\epsilon(t,x) \in \hat{b}(u_\epsilon(t,x)) \text{ a. e. } (t,x) \in Q.$$

4.1. Existence of solutions of the differentiable problem $(P)_{c\epsilon}$

The regularized Galerkin problem $(P)^n_{c\epsilon}$ is formulated as follows:

Find a solution $u_{\epsilon n} \in W(V_n)$ such that
$\langle u'_{\epsilon n}(t), v\rangle + \langle Au_{\epsilon n}(t), v\rangle + (b^n(u_{\epsilon n}(t), v) + \langle \Psi'_\epsilon(u_{\epsilon n}(t)), v\rangle$
$= \langle f(t), v\rangle,\ \forall v \in V_n,$ a.e. $t \in (0,T)$.
and $u_{\epsilon n}(0) = -u_{\epsilon n}(T)$.

The solvability of this problems is guaranteed by the same arguments as for $(P)_n$. By using the evenness of Ψ_ϵ and $u_{\epsilon n}(0) = -u_{\epsilon n}(T)$, we have

$$\begin{aligned}\int_0^T \langle \Psi'_\epsilon(u_{\epsilon n}(t)), u'_{\epsilon n}(t)\rangle\, dt &= \int_0^T \frac{d}{dt}(\Psi_\epsilon(u_{\epsilon n}(t)))\, dt \\ &= \Psi_\epsilon(u_{\epsilon n}(T)) - \Psi_\epsilon(u_{\epsilon n}(0)) \\ &= 0.\end{aligned} \tag{4.4}$$

Due to the monotonicity of Ψ'_ϵ and (HΨ)(2), we have

$$\int_0^T \langle \Psi'_\epsilon(u_{\epsilon n}(t)), u_{\epsilon n}(t)\rangle dt \geq 0. \tag{4.5}$$

Equations (4.4) and (4.5) enable to accomplish similar estimate as in Section 3 and Remark 3.1. Thus we get the convergence results:

$$u_{\epsilon n} \to u_\epsilon \text{ weakly in } X \cap W(V), \tag{4.6}$$
$$u_{\epsilon n} \to u_\epsilon \text{ strongly in } C([0,T]; H), \tag{4.7}$$
$$u'_{\epsilon n} \to u'_\epsilon \text{ weakly in } L^2(0,T; H), \tag{4.8}$$
$$b^n(u_{\epsilon n}) \to \Xi_\epsilon \text{ weakly in } L^2(Q), \tag{4.9}$$
$$Au_{\epsilon n} \to Au_\epsilon \text{ weakly in } X^*. \tag{4.10}$$

Since $\{u_{\epsilon n}\}$ is bounded in $L^\infty(0,T;H)$, $\{\Psi'_\epsilon(u_{\epsilon n})\}$ is bounded in $L^\infty(0,T;H)$. Thus we can extract a subsequence $\{u_{\epsilon n}\}$ such that

$$\Psi'_\epsilon(u_{\epsilon n}) \to Z \text{ weak-star in } L^\infty(0,T;H). \tag{4.11}$$

Now, we shall show that u_ϵ is a solution of $(P)_{c\epsilon}$. The only thing we need to prove is that $Z = \Psi'_\epsilon(u_\epsilon)$. Indeed, the monotonicity of Ψ'_ϵ implies that

$$X_n = \int_0^T \langle \Psi'_\epsilon(u_{\epsilon n}) - \Psi'_\epsilon(v), u_{\epsilon n} - v\rangle\, dt \geq 0,\ \forall v \in X. \tag{4.12}$$

On the other hand, by (HA*), we get

$$\begin{aligned}&-\langle Au_{\epsilon n}(t), u_\epsilon(t)\rangle - \langle Au_\epsilon(t), u_{\epsilon n}(t)\rangle + \langle Au_\epsilon(t), u_\epsilon(t)\rangle \\ &\geq -\langle Au_{\epsilon n}(t), u_{\epsilon n}(t)\rangle.\end{aligned} \tag{4.13}$$

Letting $n \to \infty$ in (4.13) and using the above convergence results (4.6) and (4.10) and Fatou's lemma, we get

$$\overline{\lim}_{n\to\infty} \int_0^T \langle -Au_{\epsilon n}(t), u_{\epsilon n}(t)\rangle \,\mathrm{d}t \le \int_0^T \langle -Au_\epsilon(t), u_\epsilon(t)\rangle \,\mathrm{d}t. \tag{4.14}$$

Applying $(P)_{c\epsilon}$, we have

$$\begin{aligned} X_n &= \int_0^T \langle -Au_{\epsilon n} + f, u_{\epsilon n}\rangle \,\mathrm{d}t - \int_0^T (b^n(u_{\epsilon n}), u_{\epsilon n}) \,\mathrm{d}t \\ &\quad - \int_0^T \langle \Psi'_\epsilon(u_{\epsilon n}), v\rangle \,\mathrm{d}t - \int_0^T \langle \Psi'_\epsilon(v), u_{\epsilon n} - v\rangle \,\mathrm{d}t \ge 0. \end{aligned} \tag{4.15}$$

Using the convergence results (4.6) – (4.10), (4.12) and (4.15), we conclude that

$$\begin{aligned} &\int_0^T \langle -Au_\epsilon(t) + f(t), u_\epsilon(t)\rangle \,\mathrm{d}t - \int_0^T (\Xi_\epsilon(t), u_\epsilon(t)) \,\mathrm{d}t \\ &\ge \int_0^T \langle Z(t), v(t)\rangle \,\mathrm{d}t + \int_0^T \langle \Psi'_\epsilon(v(t)), u_\epsilon(t) - v(t)\rangle \,\mathrm{d}t, \ \forall v \in X. \end{aligned} \tag{4.16}$$

On the other hand, by the convergence results (4.6) – (4.10), it is easy to see also that

$$\begin{aligned} &\int_0^T \langle u'_\epsilon(t) + Au_\epsilon(t) + Z(t), v(t)\rangle \,\mathrm{d}t + \int_0^T (\Xi_\epsilon(t), v(t)) \,\mathrm{d}t \\ &= \int_0^T \langle f(t), v(t)\rangle \,\mathrm{d}t, \ \forall v \in X. \end{aligned} \tag{4.17}$$

Substituting v by u_ϵ in (4.17) and combining the result with (4.16), we have

$$\int_0^T \langle Z(t) - \Psi'_\epsilon(v(t)), u_\epsilon(t) - v(t)\rangle \,\mathrm{d}t \ge 0, \ \forall v \in X. \tag{4.18}$$

Note that we have used the fact that

$$\int_0^T \langle u'_\epsilon(t), u_\epsilon(t)\rangle \,\mathrm{d}t = \int_0^T \frac{1}{2}|u_\epsilon(t)|^2 \,\mathrm{d}t = 0$$

in deriving (4.18). Thus, from Minty's monotonicity argument (see [12]), we have $Z = \Psi'_\epsilon(u_\epsilon)$.

4.2. Existence of solutions of the problem $(P)_c$

As a result of the previous subsection for each $\epsilon > 0$, we have

$$\begin{aligned} &\int_0^T \langle u'_\epsilon(t) + Au_\epsilon(t) + \Xi_\epsilon(t) + \Psi'_\epsilon(u_\epsilon(t)), v(t)\rangle \,\mathrm{d}t \\ &= \int_0^T \langle f(t), v(t)\rangle \,\mathrm{d}t, \ \forall v \in X, \end{aligned}$$

or equivalently

$$\int_0^T \Psi_\epsilon(v(t))\,\mathrm{d}t - \int_0^T \Psi_\epsilon(u_\epsilon(t))\,\mathrm{d}t \geq \int_0^T \langle f(t) - u_\epsilon'(t) - Au_\epsilon(t) - \Xi_\epsilon(t), v(t) - u_\epsilon(t)\rangle\,\mathrm{d}t, \ \forall v \in X. \tag{4.19}$$

Because of the same arguments as Step 1 in previous subsection (see, e. g. (4.4), (4.5)), we can easily get a priori estimates and the convergence results

$$u_\epsilon \to u \ \text{ weakly in } X \cap W(V), \tag{4.20}$$

$$u_\epsilon \to u \ \text{ strongly in } C([0,T]:H), \tag{4.21}$$

$$u_\epsilon' \to u' \ \text{ weakly in } L^2(0,T;H), \tag{4.22}$$

$$Au_\epsilon \to Au \ \text{ weakly in } X^*, \tag{4.23}$$

$$\Xi_\epsilon \to \Xi \ \text{ weakly in } L^2(Q). \tag{4.24}$$

Substituting $v(t) \equiv u_0 \in D(\Psi)$ in (4.19), we get

$$\begin{aligned}&\int_0^T \Psi_\epsilon(u_\epsilon(t))\,\mathrm{d}t \\ &\leq \int_0^T \langle u_\epsilon'(t) + Au_\epsilon(t) + \Xi_\epsilon(t) - f(t), u_0 - u_\epsilon(t)\rangle\,\mathrm{d}t + \Psi_\epsilon(u_0)T \\ &\leq C, \quad \forall \epsilon > 0.\end{aligned}$$

Thus, all the assumptions of the condition (HΨ)(3) are satisfied and we have

$$\int_0^T \Psi(u(t))\,\mathrm{d}t \leq \underline{\lim}_{\epsilon\to 0^+} \int_0^T \Psi_\epsilon(u_\epsilon)\,\mathrm{d}t. \tag{4.25}$$

Taking into account (HΨ)(1) and equations (4.20) – (4.24) and (4.25), the limit $\epsilon \to 0^+$ of equation (4.19) yields

$$\begin{aligned}&\int_0^T \Psi(v(t))\,\mathrm{d}t - \int_0^T \Psi(u(t))\,\mathrm{d}t \\ &\geq \int_0^T \langle f(t) - u'(t) - Au(t), v(t) - u(t)\rangle\,\mathrm{d}t - \int_0^T (\Xi(t), v(t) - u(t))\,\mathrm{d}t, \\ &\quad \forall v \in X.\end{aligned} \tag{4.26}$$

This completes the proof of Theorem 4.1. □

(Received March 4, 2003.)

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Jong Yeoul Park, Hyun Min Kim and Sun Hye Park, Department of Mathematics, Pusan National University, Busan 609-735. Korea.
e-mail: jyepark@pusan.ac.kr